Title: Lecture - Causal Inference, PHYS 777

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Subject: Quantum Foundations

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How to define causation

Empiricist approaches to causation and their problems

First candidate definition of A causes B

$$P_{B|A} \neq P_B$$

Equivalent to:

$$P_{A|B} \neq P_A$$

$$P_{AB} \neq P_A P_B$$

Problem: the condition is symmetric between A and B

$P(\text{crows}|\text{sunrise}) \neq P(\text{crows})$

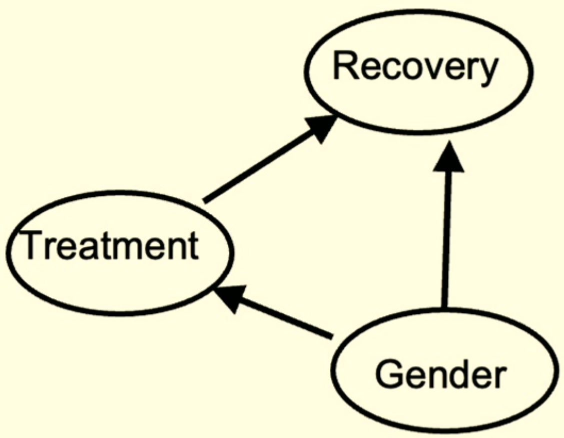
$P(\text{sunrise}|\text{crows}) \neq P(\text{sunrise})$

correlation is a symmetric relation

P(recovery | drug) > P(recovery | no drug)

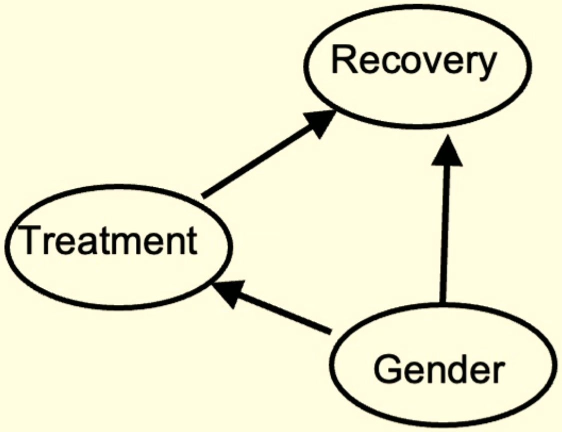
P(recovery | drug, male) < P(recovery | no drug, male) ✓

P(recovery | drug, female) < P(recovery | no drug, female) ✓

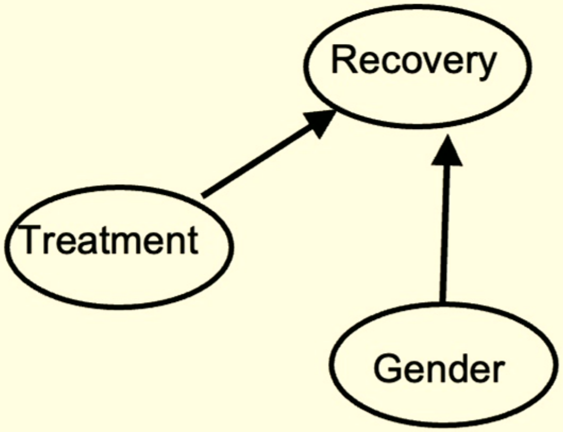


Therefore: stratify the data by the common cause

Actual world



Counterfactual



Standard conditional

$$P_{R|T} = \sum_G P_{R|TG} P_{G|T}$$

"Do conditional"

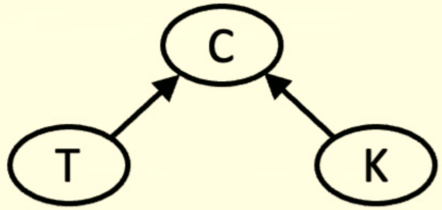
$$P_{R|\mathrm{do}T} = \sum_G P_{R|TG} P_G$$

Third candidate definition of A causes B

$$P_{B|\mathrm{do}A} \neq P_B$$

Vernam cypher

C = cyphertext
T = plaintext
K = key



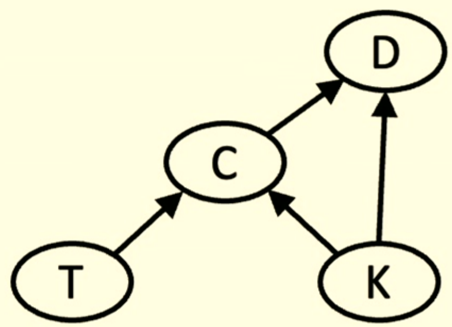
$C = (T + K)\mathrm{mod}2$

$P_K = ½[0]_K + ½[1]_K$

$$P_{C|\mathrm{do}\,T} = ½[0]_C + ½[1]_C = P_C$$

yet T causes C!

If not, we could not decode T from C using K



D = decoded text

$D = (C + K)\mathrm{mod}2$

Here, we take:

causal relationships between systems to be facts about the world (ontic)

probabilities to be the degrees of belief of a rational agent (epistemic)

The key dichotomy:

Causal
vs.
Inferential

Inferential theory

Definition: Probability Distribution on variable A

Let A denote the variable and its set of values

$$P_A : A \to \mathbb{R} :: a \mapsto P_A(a)$$

$$\forall a \in A : 0 \leq P_A(a) \leq 1$$

$$\sum_{a \in A} P_A(a) = 1$$

Let $\mathcal{P}_A$ denote the set of distributions on A.

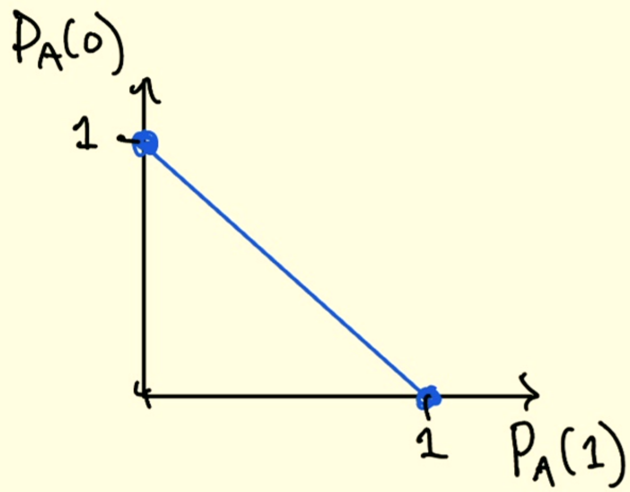
Some notation:

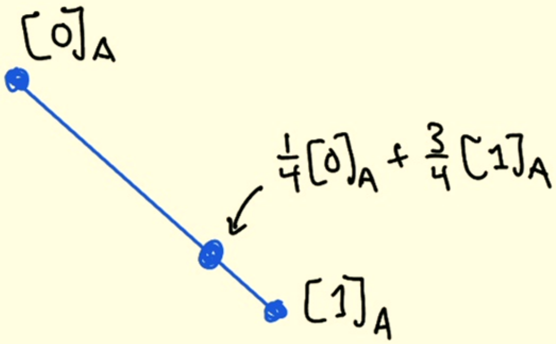
$P_A = [0]_A$ shorthand for $P_A = \delta_{A,0}$

A distribution defines a vector on the reals

$$\vec{P}_A \in \mathbb{R}^{|A|}$$

$$\vec{P}_A = (P_A(0), P_A(1), \ldots, P_A(|A|))$$





Definition: Marginal Distribution on A of a joint distribution on AB

$$\forall a \in A : P_A(a) := \sum_{b \in B} P_{AB}(ab)$$

Definition: Conditional Probability Distribution for A given B

$$P_{A|B} : A \times B \to \mathbb{R} :: (a, b) \mapsto P_{A|B}(a|b)$$

$$\forall (a, b) \in A \times B : 0 \leq P_{A|B}(a|b) \leq 1$$

$$\forall b \in B : \sum_{a \in A} P_{A|B}(a|b) = 1$$

The conditional that is defined by a joint distribution

$$P_{A|B} = P_{AB} P_B^{-1}$$ Note: defined only for $b : P_B(b) > 0$

Check:

$$P_{A|B}(a|b) = \frac{P_{AB}(ab)}{P_B(b)} = \frac{P_{AB}(ab)}{\sum_a P_{AB}(ab)}$$

$$0 \leq P_{A|B}(a|b) \leq 1$$

$$\sum_a P_{A|B}(a|b) = 1$$

For any ordering of variables, one can write:

$$P_{ABCDE} = P_{A|BCDE} P_{B|CDE} P_{C|DE} P_{D|E} P_E$$

Proof:

$$P_{ABCDE} = P_{A|BCDE} P_{BCDE}$$
$$P_{BCDE} = P_{B|CDE} P_{CDE}$$
$$P_{CDE} = P_{C|DE} P_{DE}$$
$$P_{DE} = P_{D|E} P_E$$

Def'n: A and B are marginally independent

$$P_{AB} = P_A P_B$$
$$P_{B|A} = P_B$$
$$P_{A|B} = P_A$$

Denote this
$$(A \perp B)$$

Expressed as a set of polynomial equality constraints on P_{AB}:

$$\forall a, b : P_{AB}(ab) = \left(\sum_{b'} P_{AB}(ab') \right) \left(\sum_{a'} P_{AB}(a'b) \right)$$

Def'n: A and B are conditionally independent given C

$$P_{AB|C} = P_{A|C} P_{B|C}$$
$$P_{B|AC} = P_{B|C}$$
$$P_{A|BC} = P_{A|C}$$

Denote this
$(A \perp B|C)$

Expressed as a set of polynomial equality constraints on P_{ABC}

$$\forall a, b, c: P_{ABC}(abc) \left(\sum_{a'b'} P_{ABC}(a'b'c) \right) = \left(\sum_{b'} P_{ABC}(ab'c) \right) \left(\sum_{a'} P_{ABC}(a'bc) \right)$$

Semi-graphoid axioms

$$\text{Symmetry:} \quad (X \perp Y \,|\, Z) \Leftrightarrow (Y \perp X \,|\, Z)$$

$$\text{Decomposition:} \quad (X \perp YW \,|\, Z) \Rightarrow (X \perp Y | Z)$$

$$\text{Weak Union:} \quad (X \perp YW \,|\, Z) \Rightarrow (X \perp Y \,|\, ZW)$$

$$\text{Contraction:} \quad (X \perp Y \,|\, Z) \text{ and } (X \perp W \,|\, ZY) \Rightarrow (X \perp YW \,|\, Z)$$

Marginal problem

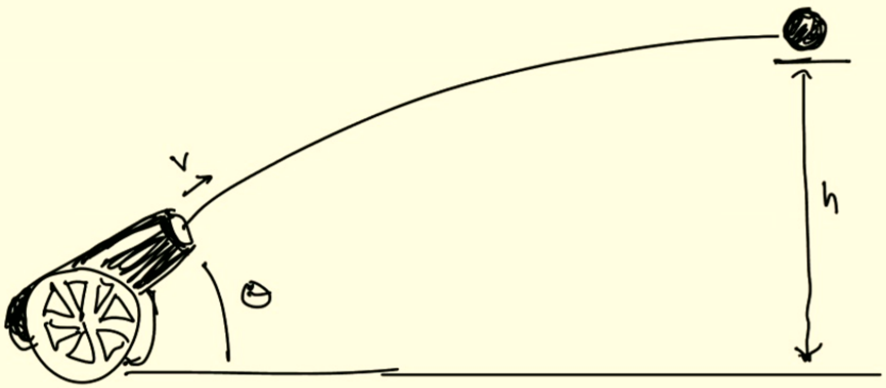
Is there a distribution on A, B, C that has the following marginals?

$$P_{AB} = \frac{1}{2}[00] + \frac{1}{2}[11]$$
$$P_{AC} = \frac{1}{2}[00] + \frac{1}{2}[11]$$
$$P_{BC} = \frac{1}{2}[00] + \frac{1}{2}[11]$$

Yes! $P_{ABC} = \frac{1}{2}[000] + \frac{1}{2}[111]$

Causal theory

The law of projectile motion



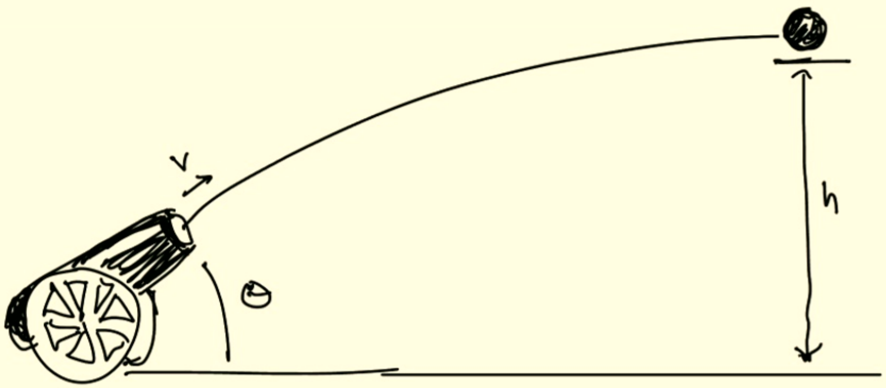
$$v = \frac{\sqrt{2gh}}{\sin\theta}$$

$$h = \frac{v^2 \sin^2\theta}{2g}$$

$$\theta = \arcsin\left(\frac{\sqrt{2gh}}{v}\right)$$

inferential relations

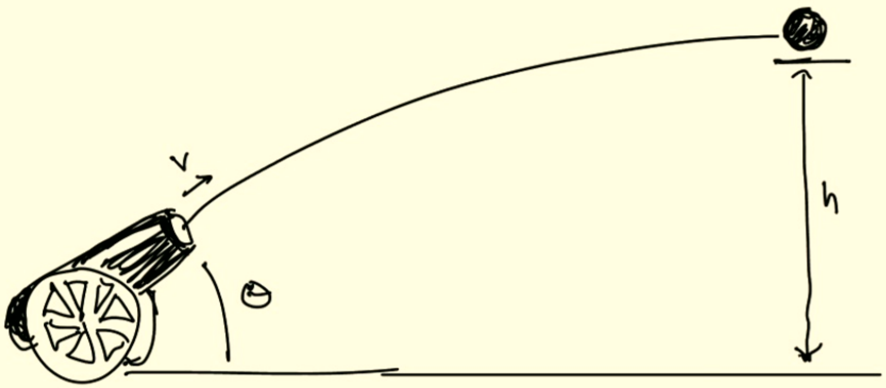
The law of projectile motion



$$h = \frac{v^2 \sin^2 \theta}{2g}$$

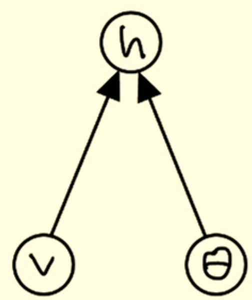
causal relation

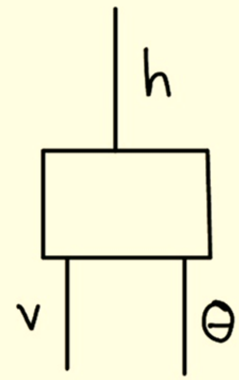
The law of projectile motion



$$h := \frac{v^2 \sin^2 \theta}{2g}$$

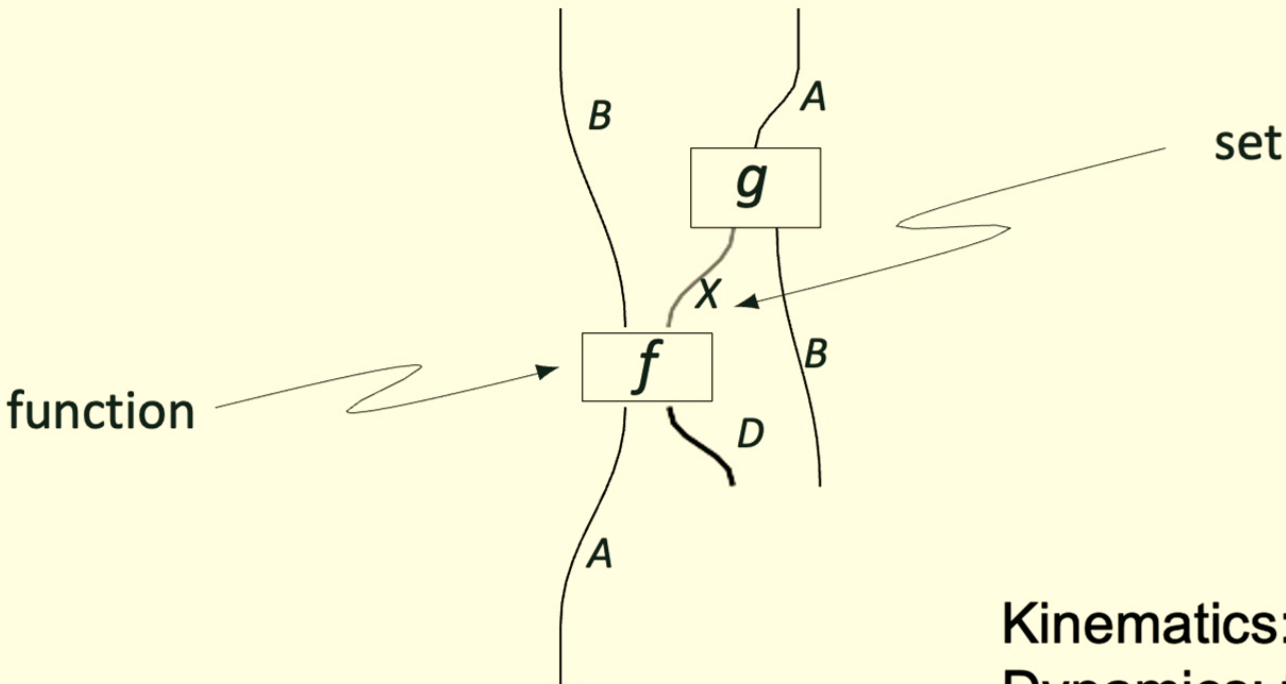
causal relation





Causal laws warrant inferences about counterfactuals

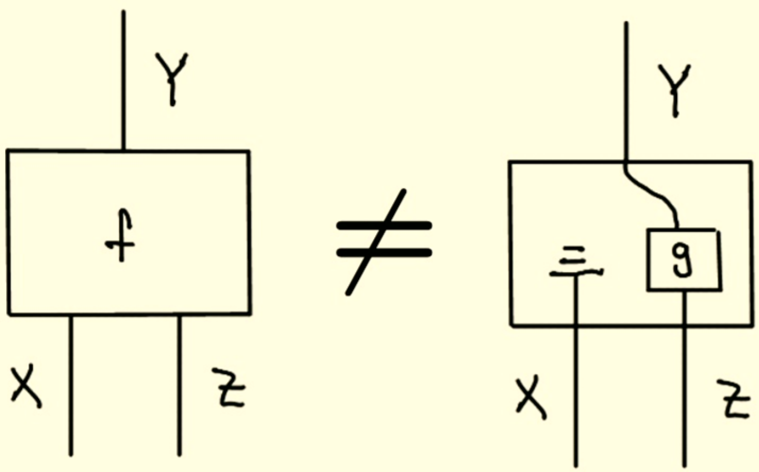
Causal theory



Kinematics: sets
Dynamics: functions

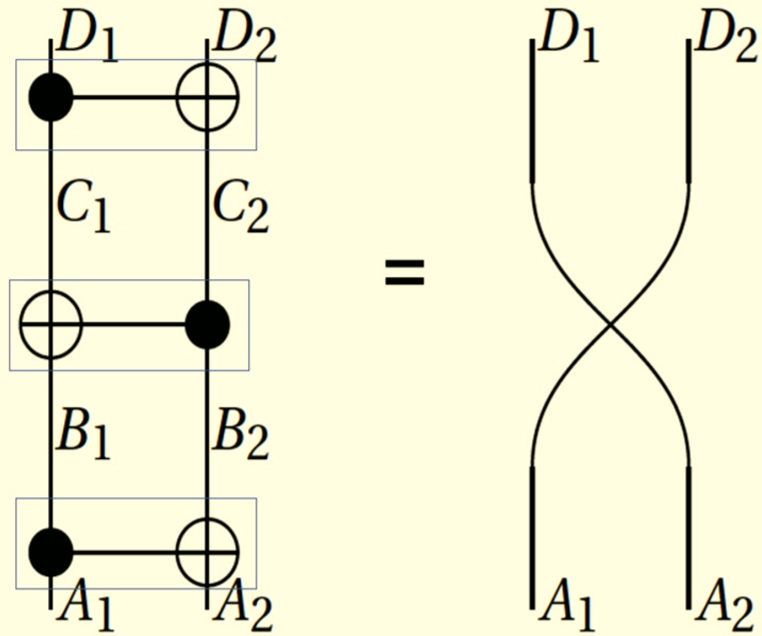
Definition: X has a nontrivial influence on Y

The function f that determines Y from its causal antecedents has a nontrivial dependence on X

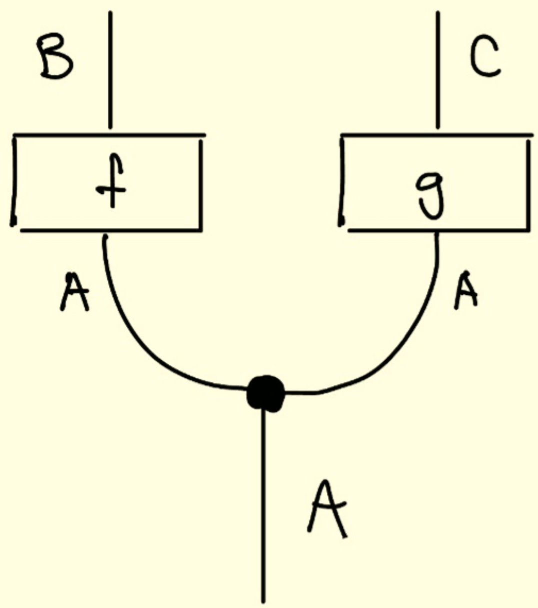


$$Y = f(X, Z) \neq g(Z)$$

Effective causal influences

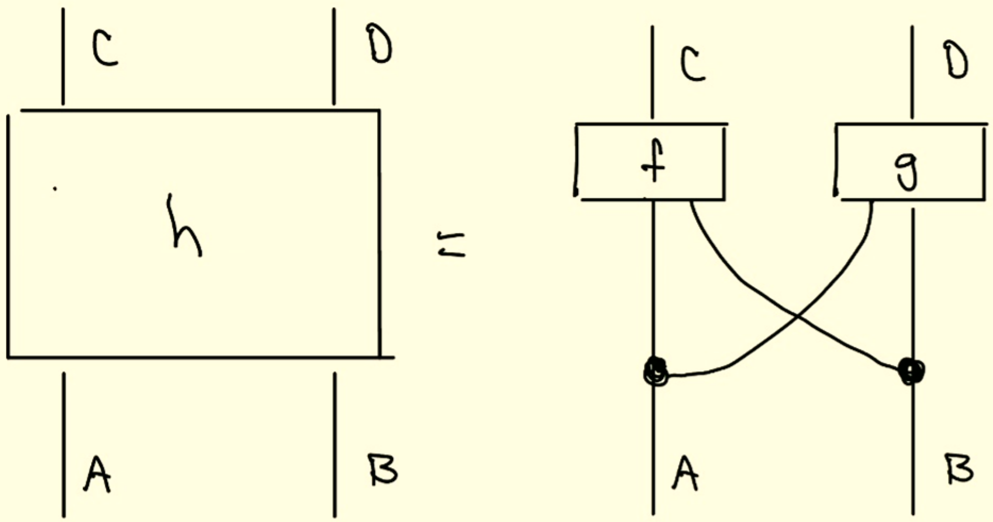


How to represent a variable that influences more than one other?



 $: A \to A \times A :: a \mapsto (a, a)$

Causal autonomy

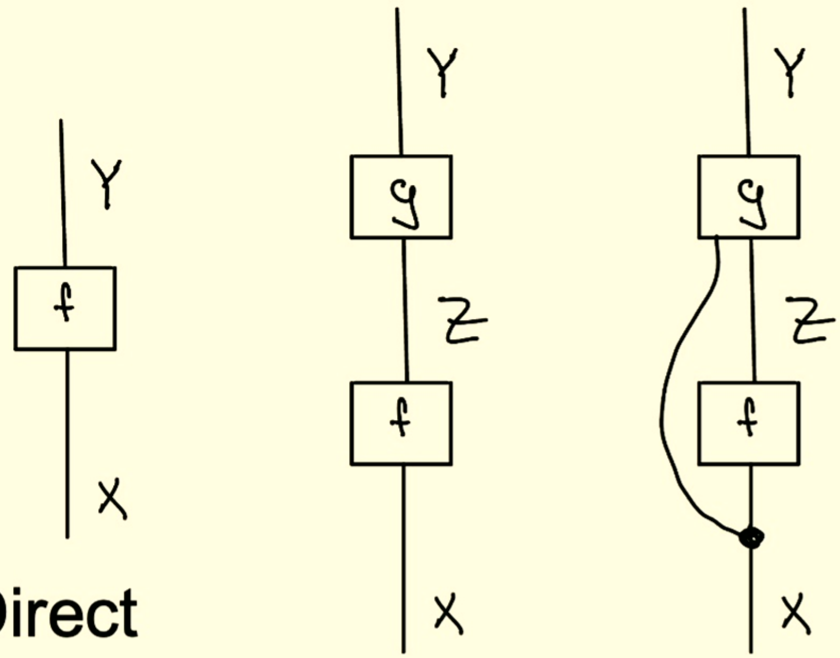


where f and g can be specified independently

Causal autonomy fails in many circumstances:

- Symplectic dynamics
- Conservation laws

Direct versus indirect causal influence

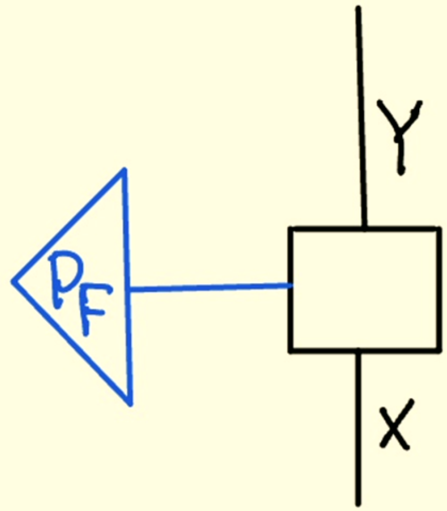


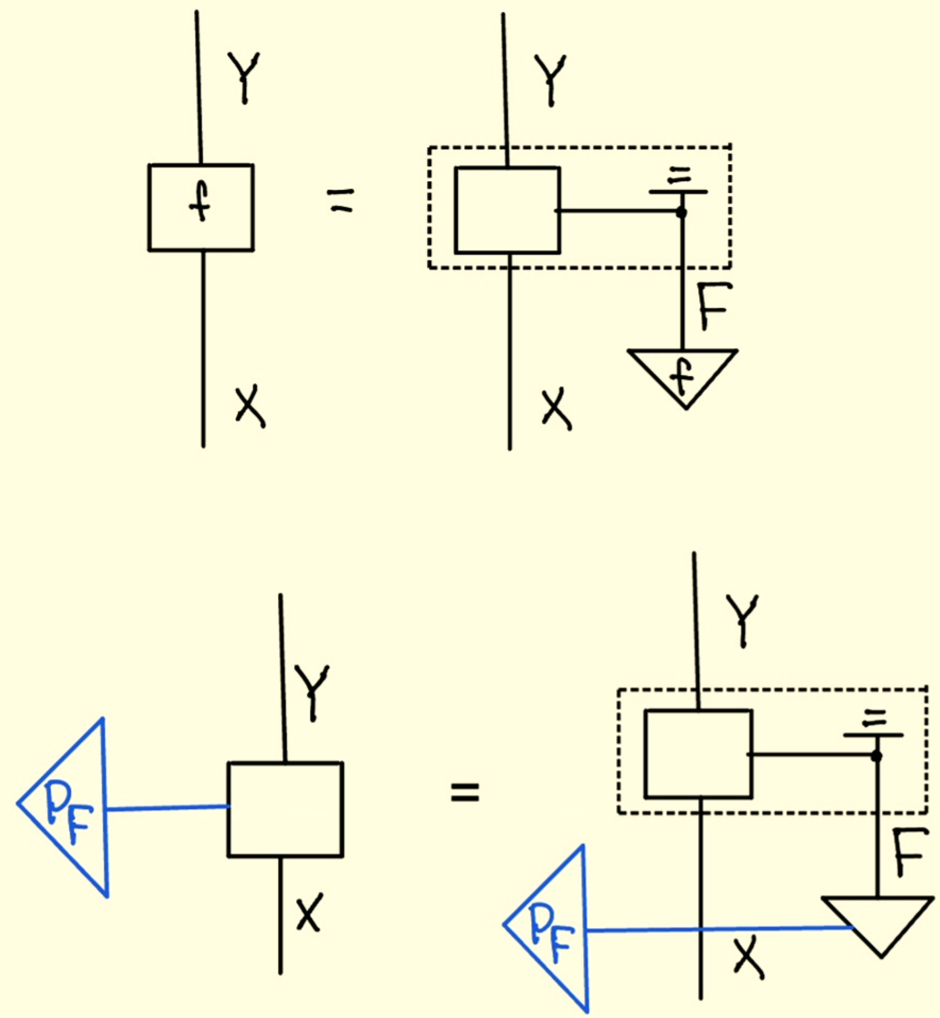
Direct

Indirect

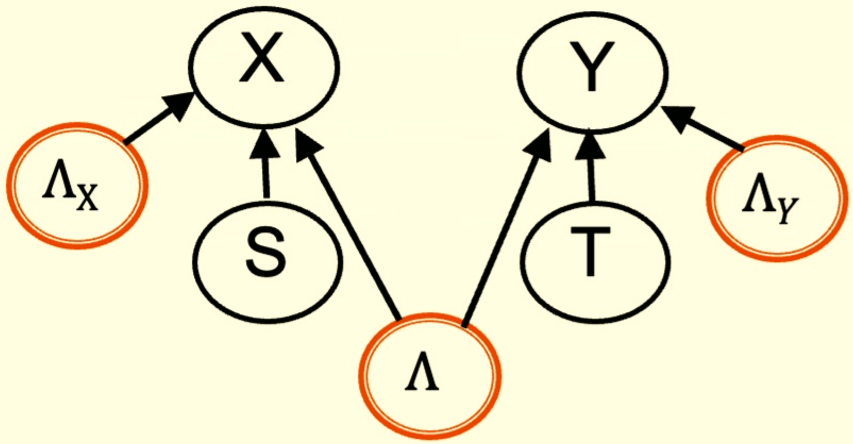
Both direct and indirect

Uncertainty about causal mechanisms


Y
P_F
X


Y
f
=
F
X
P_F
=

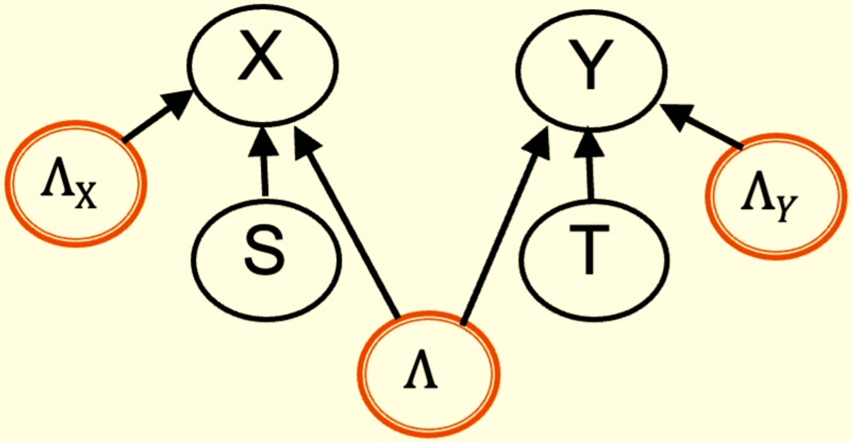
Bell scenario



$$X = f_X(S, \Lambda, \Lambda_X)$$
$$Y = f_Y(T, \Lambda, \Lambda_Y)$$
$$P_\Lambda$$
$$P_{\Lambda_X}$$
$$P_{\Lambda_Y}$$

$$P_{XY|ST} = \sum_{\Lambda,\Lambda_X,\Lambda_Y} \delta_{X,f_X(S,\Lambda,\Lambda_X)} \delta_{Y,f_Y(T,\Lambda,\Lambda_Y)} P_\Lambda P_{\Lambda_X} P_{\Lambda_Y}$$

Bell scenario

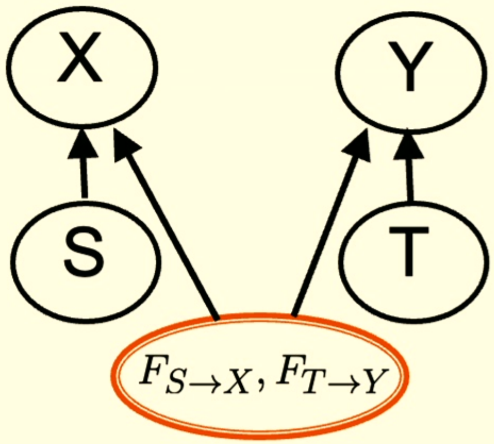


$$X = f_X(S, \Lambda, \Lambda_X)$$
$$Y = f_Y(T, \Lambda, \Lambda_Y)$$
$$P_\Lambda$$
$$P_{\Lambda_X}$$
$$P_{\Lambda_Y}$$

$$P_{XY|ST} = \sum_{\Lambda,\Lambda_X,\Lambda_Y} \delta_{X,f_X(S,\Lambda,\Lambda_X)} \delta_{Y,f_Y(T,\Lambda,\Lambda_Y)} P_\Lambda P_{\Lambda_X} P_{\Lambda_Y}$$

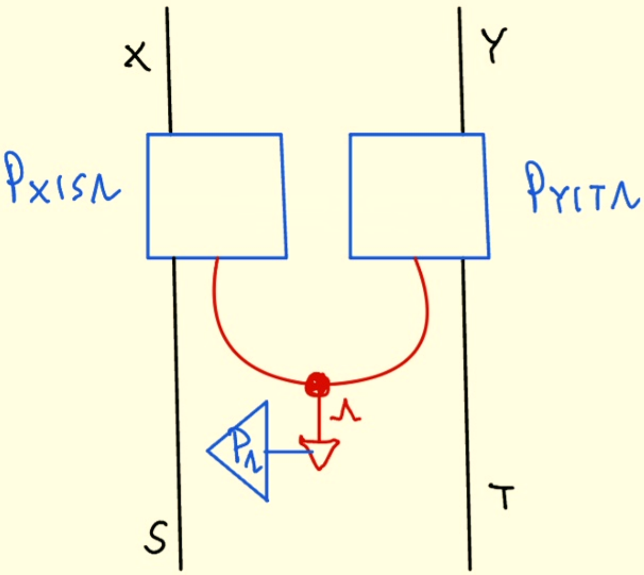
$$P_{X|S\Lambda} = \sum_{\Lambda_X} \delta_{X,f_X(S,\Lambda,\Lambda_X)} P_{\Lambda_X}$$
$$P_{Y|T\Lambda} = \sum_{\Lambda_Y} \delta_{Y,f_Y(T,\Lambda,\Lambda_Y)} P_{\Lambda_Y}$$

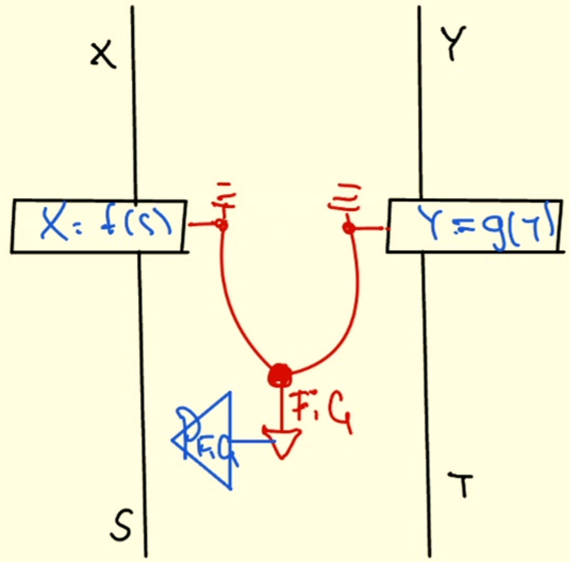
Bell scenario



$P_{F_{S\to X},F_{T\to Y}}$

$$P_{XY|ST} = \sum_{f,f'} \delta_{X,f(S)} \delta_{Y,f'(T)} P_{F_{S\to X},F_{T\to Y}}(f,f')$$


X
Y
$P_{X|S\Lambda}$
$P_{Y|T\Lambda}$
Λ
P_Λ
S
T


X
Y
X = f(S)
Y = g(T)
F, G
$P_{F,G}$
S
T

One's uncertainty about the causal mechanism is captured by P_F

What is often of interest is the do-conditional $P_{Y|doX}$

$$P_{Y|\text{do}X} = \sum_f \delta_{Y,f(X)} P_F(f)$$

But there are many P_F consistent with a given $P_{Y|doX}$