Title: Lecture - AdS/CFT, PHYS 777

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Subject: Quantum Fields and Strings, Quantum Gravity

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Holographic renormalization: scalar field

… a systematic method for dealing with near boundary divergences

- Using **Fefferman-Graham** coordinates

$$ds^2 = g_{ab}dx^a dx^b = \ell^2\Big(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\delta_{\mu\nu}dx^\mu dx^\nu\Big)$$

consider **scalar field**

$$\begin{aligned} S_0 &= \frac{C}{2}\int d\rho d^d x\sqrt{g}\Big(g^{ab}\partial_a\phi\partial_b\phi + m^2\phi^2\Big) \\ (\Box - m^2)\phi &= 0\,, \quad \Box\phi = \frac{1}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}\partial_b\phi)\,. \end{aligned}$$

Holographic renormalization: scalar field

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- Theorem: Solution of EOM can be found in the form:

$$\phi(\rho, x) = \rho^{(d-\Delta)/2}\big(\underbrace{\phi_0(x) + \rho\phi_2(x) + \rho^2\phi_4(x) + \dots}_{\varphi(\rho,x)}\big)$$

Holographic renormalization: scalar field

- Namely, plugging to EOM we get

$$\phi(\rho,x)=\rho^{(d-\Delta)/2}\Big(\underbrace{\phi_0(x)+\rho\phi_2(x)+\rho^2\phi_4(x)+\dots}_{\varphi(\rho,x)}\Big)$$

$$[\Delta(\Delta-d)-m^2\ell^2]\varphi+\rho\Box_0\varphi+2(d-2\Delta+2)\rho\partial_\rho\varphi+4\rho^2\partial_\rho^2\varphi=0$$

- Solution order by order in ρ:

$$m^2\ell^2=\Delta(\Delta-d)\,,\qquad \phi_{(2n)}=\frac{1}{2n(2\Delta-d-2n)}\Box_0\phi_{(2n-2)}$$

… **all determined** via phi_0 $\quad \Box_0=\delta^{\mu\nu}\partial_\mu\partial_\nu$

If the denominator zero, we have to add $\rho^k\log(\rho)\chi_{2k}$

$$\chi_{2k}=-\frac{1}{2^{2k}\Gamma(k)\Gamma(k+1)}(\Box_0)^k\phi_0$$

ϕ_{2k} no longer determined

Holographic renormalization: scalar field

- The on-shell action needs to be regularized $\rho = \epsilon$

$$S_r = -\frac{C}{2}\int d^d x\sqrt{g}g^{\rho\rho}\phi\partial_\rho\phi\Big|_{\rho=\epsilon}.$$

and reads

$$S_r = CL^{d-1}\int d^d x\Big(\epsilon^{-\Delta+\frac{d}{2}}a_0 + \epsilon^{-\Delta+\frac{d}{2}+1}a_2 + \cdots - \log\epsilon\, a_{2\Delta-d}\Big)$$

$$\begin{aligned} a_0 &= -\frac{1}{2}(d-\Delta)\phi_0^2\,, \quad a_2 = -\frac{d-\Delta+1}{2(2\Delta-d-2)}\phi_0\Box_0\phi_0 \\ a_{2\Delta-d} &= -\frac{d}{2^{2k+1}\Gamma(k)\Gamma(k+1)}\phi_0(\Box_0)^k\phi_0\,. \end{aligned}$$

- Since $\Delta > d/2$, it diverges and needs to be **renormalized** by introducing **counterterms** – covariantly expressed in terms of **induced boundary metric** and **boundary field**

$$\gamma_{\mu\nu} = \frac{\ell^2}{\epsilon}\delta_{\mu\nu}\,, \quad \Box_\gamma = \gamma^{\mu\nu}\partial_\mu\partial_\nu \qquad \phi_b(x) = \phi(\epsilon, x).$$

Holographic renormalization: scalar field

- To do so, need to invert

$$\phi(\rho,x) = \rho^{(d-\Delta)/2}\Big(\underbrace{\phi_0(x) + \rho\phi_2(x) + \rho^2\phi_4(x) + \dots}_{\varphi(\rho,x)}\Big)$$

and write ϕ_{2n} in terms of $\phi_b(x) = \phi(\epsilon, x)$

- To 2nd order we have

$$\begin{aligned}\phi_0 &= \epsilon^{-(d-\Delta)/2}\Big(\phi(\epsilon,x) - \frac{1}{2(2\Delta-d-2)}\Box_\gamma\phi(\epsilon,x)\Big)\\ \phi_2 &= \epsilon^{-(d-\Delta)/2-1}\frac{1}{2(2\Delta-d-2)}\Box_\gamma\phi(\epsilon,x)\,.\end{aligned}$$

- To cancel divergencies one by one we add counterterms

$$S_{ct} = \frac{C}{\ell}\int d^dx\sqrt{\gamma}\Big(\frac{d-\Delta}{2}\phi_b^2(x) + \frac{1}{2(2\Delta-d-2)}\phi_b(x)\Box_\gamma\phi_b(x) + \dots\Big)\,.$$

Holographic renormalization: scalar field

- Total action $S = S_0 + S_{ct}$

… is finite and yields finite correlation functions

- In particular, since ϕ_0 is the source, we have

$$\langle O(x)\rangle = -\frac{\delta S}{\delta\phi_0(x)} = \lim_{\epsilon\to 0}\Big(\frac{\ell^d}{\epsilon^{\Delta/2}}\frac{1}{\sqrt{\gamma}}\frac{\delta S}{\delta\phi(\epsilon,x)}\Big)$$

and

$$\langle O(x)O(y) = -\frac{\delta^2 S}{\delta\phi_0(x)\delta\phi_0(y)}$$

gives $\propto \frac{1}{|x-y|^{2\Delta}}$

Holographic renormalization: gravity

- Similar to scalar field we start from the Einstein–Hilbert action (supplemented by the York–Gibbons–Hawking term):

$$S_0 = -\frac{1}{16\pi G}\int d^{d+1}x\sqrt{g}\Big(R + \frac{d(d-1)}{\ell^2}\Big) - \frac{1}{8\pi G}\int d^d x\sqrt{\gamma}\mathcal{K}\,, \qquad (3.47)$$

and perform the Fefferman–Graham expansion of the metric near the boundary:

$$ds^2 = \ell^2\Big(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\Big[\underbrace{g_{0\mu\nu}(x) + \rho g_{2\mu\nu}(x) + \rho^2 g_{4\mu\nu}(x)\ldots}_{g_{\mu\nu}(\rho,x)}\Big]dx^\mu dx^\nu\Big)\,, \qquad (3.48)$$

considering the asymptotically AdS manifolds. (If the boundary is even-dimensional, additional logarithmic term appears: $\rho^{d/2}\log\rho h_{d\mu\nu}$.)

- Inserting this ansatz into Einstein equations then determines g_d in terms of g_0. For example, we have

$$g_{2\mu\nu} = \frac{\ell^2}{d-2}\Big(R_{\mu\nu} - \frac{1}{2(d-1)}Rg_{0\mu\nu}\Big)\,. \qquad (3.49)$$

- We next plug these back to the Einstein–Hilbert action and identify the divergent terms,

$$S_{\rm reg} = -\frac{1}{16\pi G}\int d^d x\sqrt{\det g^0}(\epsilon^{-d/2}a_0 + \epsilon^{d/2+1}a_2 + \cdots - \log\epsilon a_d) + \text{finite}\,, \quad (3.50)$$

where a's are expressed in terms of g^0.

Holographic renormalization: gravity

- Writing these in terms of the boundary metric $\gamma_{\mu\nu} = \frac{\ell^2}{\rho} g_{0\mu\nu}$ (and boundary curvature invariants $\mathcal{R}, \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}, \ldots$), and choosing the counterterms to cancel these divergencies, we find

$$S_{ct} = \frac{1}{8\pi G_{d+1}} \int d^d x \sqrt{\gamma} \Big(\underbrace{\frac{d-1}{\ell}}_{1st} + \underbrace{\frac{\ell}{2(d-2)}\mathcal{R}}_{2nd} + \underbrace{\frac{\ell^3}{2(d-2)^2}\Big(\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} - \frac{1}{d-1}\mathcal{R}^2\Big)}_{3rd} + \ldots \Big), \tag{3.51}$$

where the 3rd counter cancels the logarithmic divergence present in $d = 4$ dimensions; we have seen the 1st and 2nd conterterms before for $d = 3$. The total action $S = S_0 + S_{ct}$ is then finite.

- Since $g_{0\mu\nu}$ is the source for the quantum operator $T_{\mu\nu}(x)$, we have

$$\langle T_{\mu\nu}(x) \rangle = -\frac{2}{\sqrt{\det g_0}} \frac{\delta S}{\delta g_0^{\mu\nu}(x)}. \tag{3.52}$$

However, since

$$\gamma_{\mu\nu}(x) = \lim_{\epsilon \to 0} \frac{l^2}{\epsilon} g_{0\mu\nu}, \tag{3.53}$$

we have

$$\boxed{\langle T_{\mu\nu}(x) \rangle = \lim_{\epsilon \to 0} \Big(\frac{\ell^{d-2}}{\epsilon^{d/2-1}} \tau_{\mu\nu} \Big),} \tag{3.54}$$

where $\tau_{\mu\nu}$ is the boundary stress tensor discussed previously:

$$\tau_{\mu\nu} = -\frac{2}{\sqrt{\gamma}} \frac{\delta S}{\delta \gamma^{\mu\nu}} = \frac{1}{8\pi} \Big(\mathcal{K} h_{\mu\nu} - \mathcal{K}_{\mu\nu} + \ell \mathcal{G}_{\mu\nu} - \frac{2}{\ell}\gamma_{\mu\nu} \Big). \tag{3.55}$$

It is the $\langle T_{\mu\nu}(x) \rangle$ that is finite and can be used to calculate properties of the CFT/bulk spacetime.

HOLOGRAPHIC RENORMALIZATION

= A SYSTEMATIC WAY FOR ADDING COUNTERTERMS TO THE ACTION TO DEAL WITH BOUNDARY DIVERGENCES.

SPEC: FOR GRAVITY.

$$S_{TOT} = -\frac{1}{16\pi G_{d+1}} \int d^{d+1}x \sqrt{g} \left(R + \frac{d(d-1)}{\ell^2} \right) - \frac{1}{8\pi G_{d+1}} \int d^d x \sqrt{\gamma}\, K$$

$$+ \frac{1}{8\pi G_{d+1}} \int d^d x \sqrt{\gamma} \left(\underbrace{\frac{d-1}{\ell}}_{1st} + \underbrace{\frac{\ell}{2(d-2)} R_\gamma}_{2ND} + \underbrace{\frac{\ell^3}{2(d-2)^2} \left(R^\gamma_{\mu\nu} R_\gamma^{\mu\nu} - \frac{1}{d-1} R_\gamma^2 \right)}_{3RD} + \dots \right)$$

NEW COUNTERMS ALWAYS APPEAR IN

d = EVEN DIMENSIONS (AS LOG DIVERGENT TERMS)

$d = 2, 4, \ldots$

3RD

CES.

$\int d^dx \sqrt{g}\, K$

$R_{\mu\nu} R^{\mu\nu} - \frac{1}{d-1} R^2) + \ldots$

3RD

1st 2ND

DERIVED IN F-G COORDINATES

$$g = \ell^2\left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(g_0 + \rho g_2 + \dots\right)\right)$$

SOURCE FOR CFT

BOUNDARY METRIC $\boxed{\gamma = \frac{\ell^2}{\rho} g_0}$

① VACUUM ENERGY: 5D SCHW-AdS BH

$(d=4)$

$$\ldots(d-2)} R_{gr} + \frac{\ell^2}{2(d-2)^2}\left(R_{\mu\nu}R_{gr}^{\mu\nu} - \frac{1}{d-1}R_{gr}^2\right) + \ldots$$

2ND 3RD

$+ \varphi g_2 + \ldots)$

SOURCE FOR CFT

SCHW-AdS BH

$$F = \frac{S_{TOT}}{\beta} = M - TS + M_0$$

"CONSTANT SHIFT IN ENERGY"

"ZERO POINT ENERGY"

M_0

0

MORE CONCRETELY:

EXACT CALC.

$$M_0 = \frac{3\pi\ell^2}{32 G_5} = \left|\frac{\ell^3}{G_5} = \frac{2N_c^2}{\pi}\right| = \frac{3N_c^2}{16\ell}$$

$$G_5 \sim \frac{G_{10}}{VOL(S_5)} \sim g_s^2 \frac{\ell_s^8}{\ell^8}\frac{\ell^8}{\ell^5} \sim \frac{\ell^3}{N_c^2}$$

$\frac{\ell^2}{N_c^2}$ $\sim \frac{1}{\lambda^2}$ VOL(S5)

• FREE FIELDS ON $S^3 \times \mathbb{R}$ WITH RADIUS ℓ
HAVE CASIMIR ENERGY:

$$E_{CASIMIR} = \frac{1}{960\ell}\left(4\, n_0 + 17\, n_{1/2} + 88\, n_1\right)$$

FIELD SPECIES (ALREADY TAKES INTO ACCOUNT THE SPIN)

SPEC: $\mathcal{N}=4$ $SU(N_c)$ SYM

REAL SCALARS: $n_0 = 6(N_c^2-1)$
WEYL FERMIONS: $n_{1/2} = 4(N_c^2-1)$
VE

• FREE FIELDS ON $S^3 \times \mathbb{R}$ WITH RADIUS ℓ

HAVE CASIMIR ENERGY:

$$E_{CASIMIR} = \frac{1}{960\ell}\left(4\, n_0 + 17\, n_{1/2} + 88\, n_1\right) = 3\ldots$$

FIELD SPECIES (ALREADY TAKES INTO ACCOUNT THE SPI…

SPEC: $\mathcal{N}=4$ $SU(N_c)$ SYM

REAL SCALARS: $n_0 = 6(N_c^2-1)$

WEYL FERMIONS: $n_{1/2} = 4(N_c^2-1)$

VECTORS: $n_1 = (N_c^2-1)$

$(d=4)$ $\qquad = \dfrac{\quad}{16\ell}$ $\qquad$ WHAT IS THIS

• FREE FIELDS ON $S^3 \times \mathbb{R}$ WITH RADIUS ℓ HAVE CASIMIR ENERGY.

$$E_{\text{CASIMIR}} = \frac{1}{960\,\ell}\left(4\,n_0 + 17\,n_{1/2} + 88\,n_1\right) = \frac{3(N_c^2-1)}{16\ell} \to \underline{\underline{n_0}}$$

FIELD SPECIES (ALREADY TAKES INTO ACCOUNT THE SPIN)

SPEC: $\mathcal{N}=4$ $SU(N_c)$ SYM

REAL SCALARS: $n_0 = 6(N_c^2-1)$

WEYL FERMIONS: $n_{1/2} = 4(N_c^2-1)$

VECTORS: $n_1 = (N_c^2-1)$

$$\frac{3(N_c^2-1)}{16\ell} \to \underline{\underline{M_0}}$$

(2) CONFORMAL ANOMALY

$$\boxed{\langle T_{\mu\nu}(x)\rangle = -\frac{2}{\sqrt{g_0}}\frac{\delta S_{TOT}}{\delta g_0^{\mu\nu}(x)}}$$

↑ CFT ENERGY-MOM. TENSOR

- CLASICALLY: CFT $T^{\mu}_{\mu} = 0$
- CONFORMAL ANOMALY: WHEN $T_{\mu\nu}$ DOES NOT REMAIN TRACELESS UNDER QUANTUM CORRECTIONS

$$\langle T^{\mu}_{\mu}\rangle \neq 0$$

E.G: 2D: $\boxed{\langle T^{\mu}{}_{\mu}(x) \rangle = \frac{c}{24\pi} R_0}$

CENTRAL CHARGE

TOP. DENSITY IN 2D

SPEC: $\mathcal{N}=$

E.G: 2D: $\boxed{\langle T^{\mu}{}_{\mu}(x)\rangle = \frac{c}{24\pi} R_0}$

CENTRAL CHARGE

TOP. DENSITY IN 2D

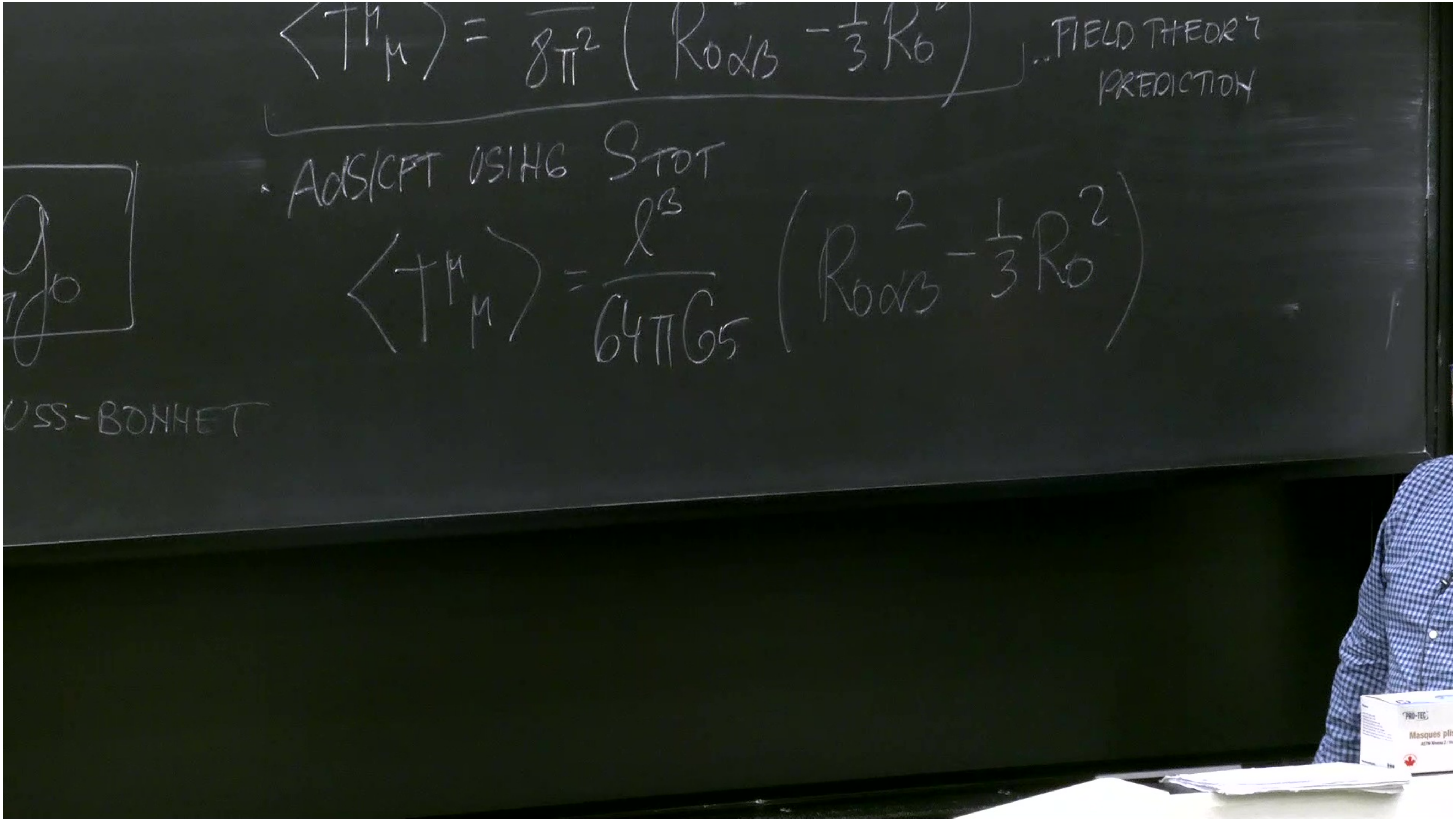
$$\int d^2x \sqrt{g_0}\, R_0 = 2\pi \chi(M)$$

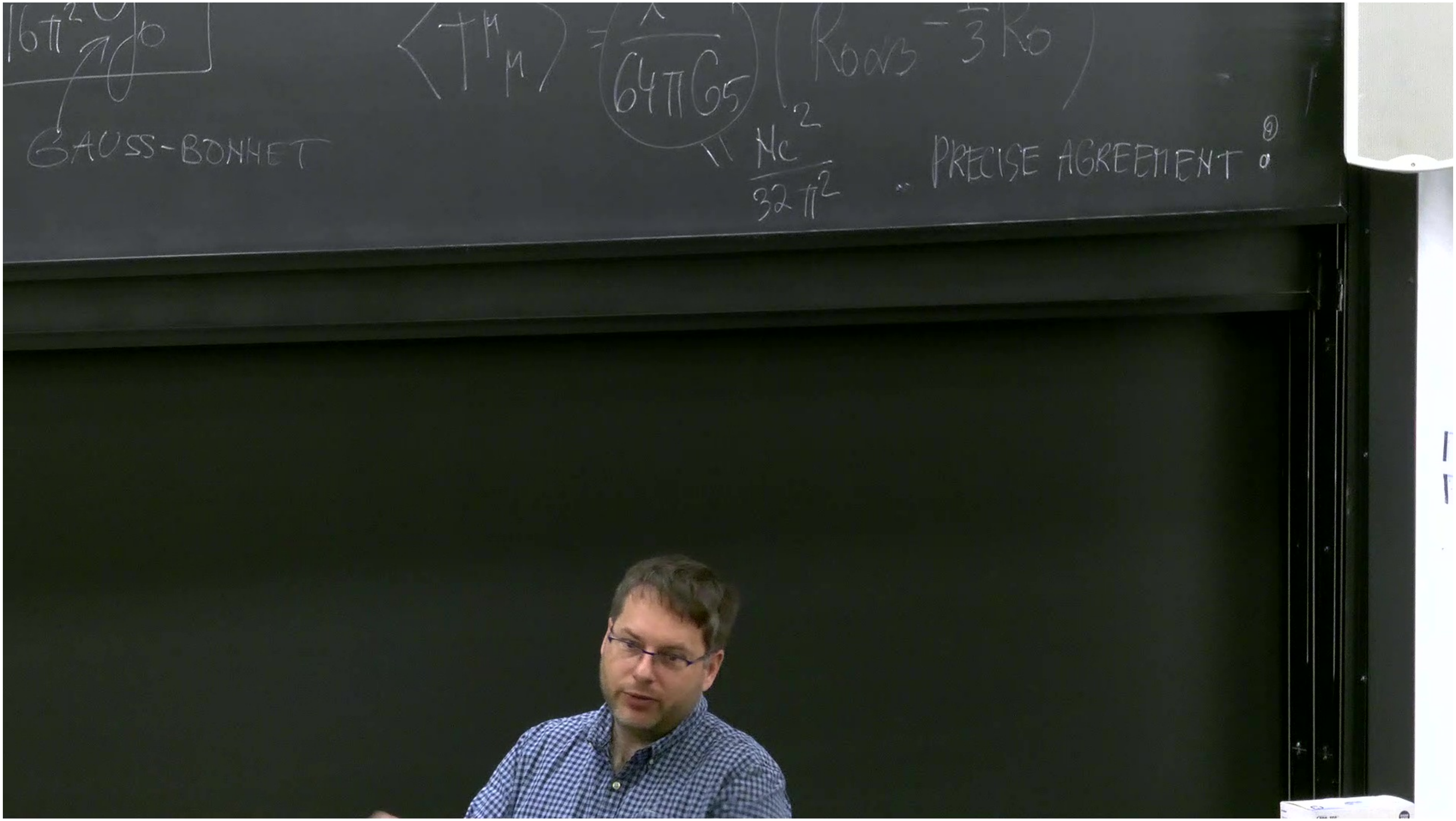
4D: $\boxed{\langle T^{\mu}{}_{\mu}(x)\rangle = \frac{c}{16\pi^2} C_{0\,\mu\nu\varkappa\lambda} C_0^{\mu\nu\varkappa\lambda} - \frac{a}{16\pi^2} \mathcal{G}_0}$

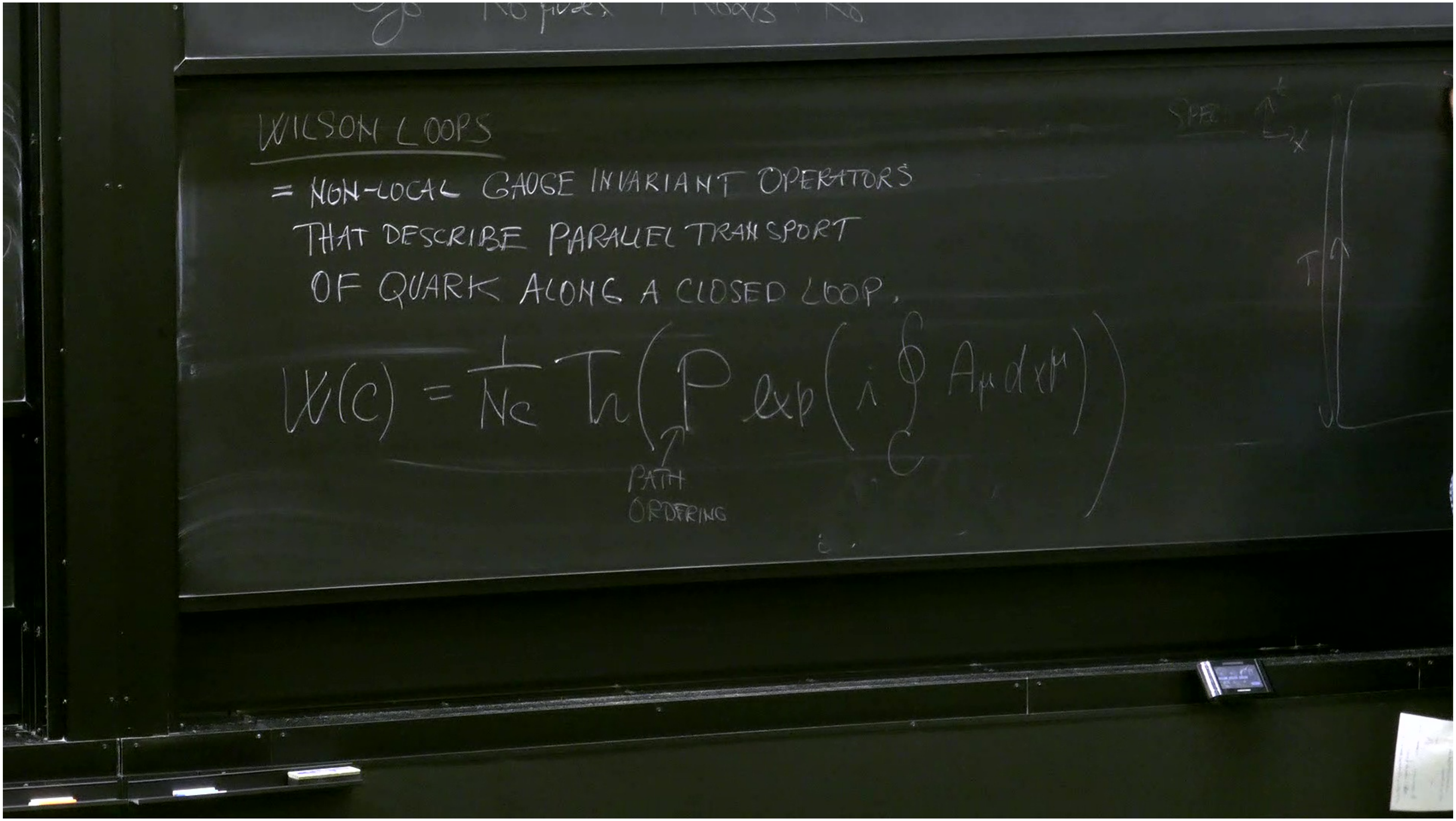
WEYL

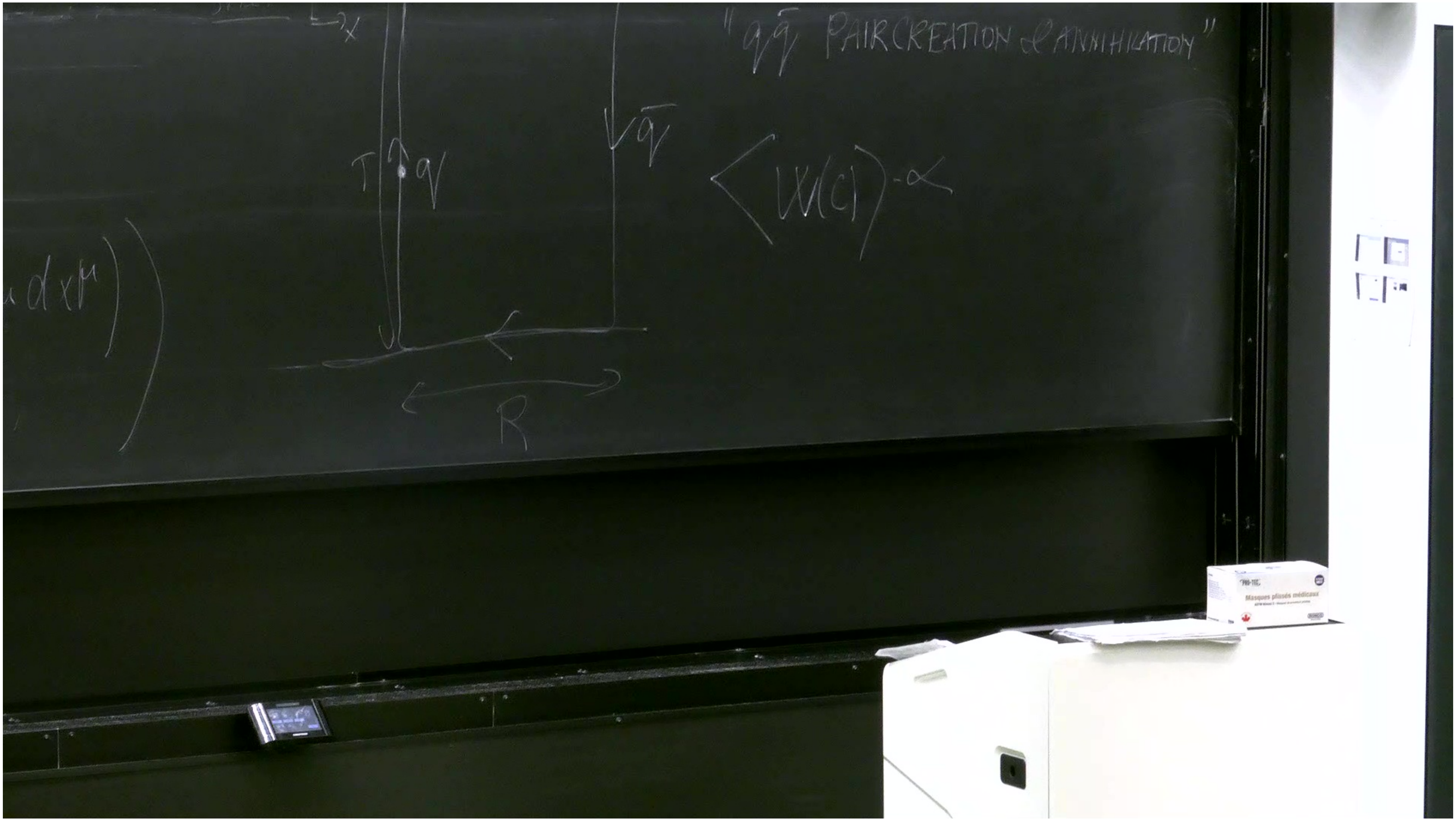
GAUSS-BONNET

$$\mathcal{G}_0 = R_{0\,\mu\nu\varkappa\lambda}^2 - 4 R_{0\,\alpha\beta}^2 + R_0^2$$


$\langle T^\mu{}_\mu \rangle = \frac{}{8\pi^2}\left(R_{0\alpha\beta} - \frac{1}{3}R_0\right)$
...FIELD THEORY PREDICTION
. AdS/CFT USING S_{TOT}
$\langle T^\mu{}_\mu \rangle = \frac{\ell^3}{64\pi G_5}\left(R_{0\alpha\beta}^2 - \frac{1}{3}R_0^2\right)$
USS-BONNET


GAUSS-BONNET
PRECISE AGREEMENT


WILSON LOOPS
= NON-LOCAL GAUGE INVARIANT OPERATORS
THAT DESCRIBE PARALLEL TRANSPORT
OF QUARK ALONG A CLOSED LOOP.
$W(C) = \frac{1}{N_c} \mathrm{Tr}\left(P \exp\left(i \oint_C A_\mu dx^\mu\right)\right)$
PATH ORDERING
T

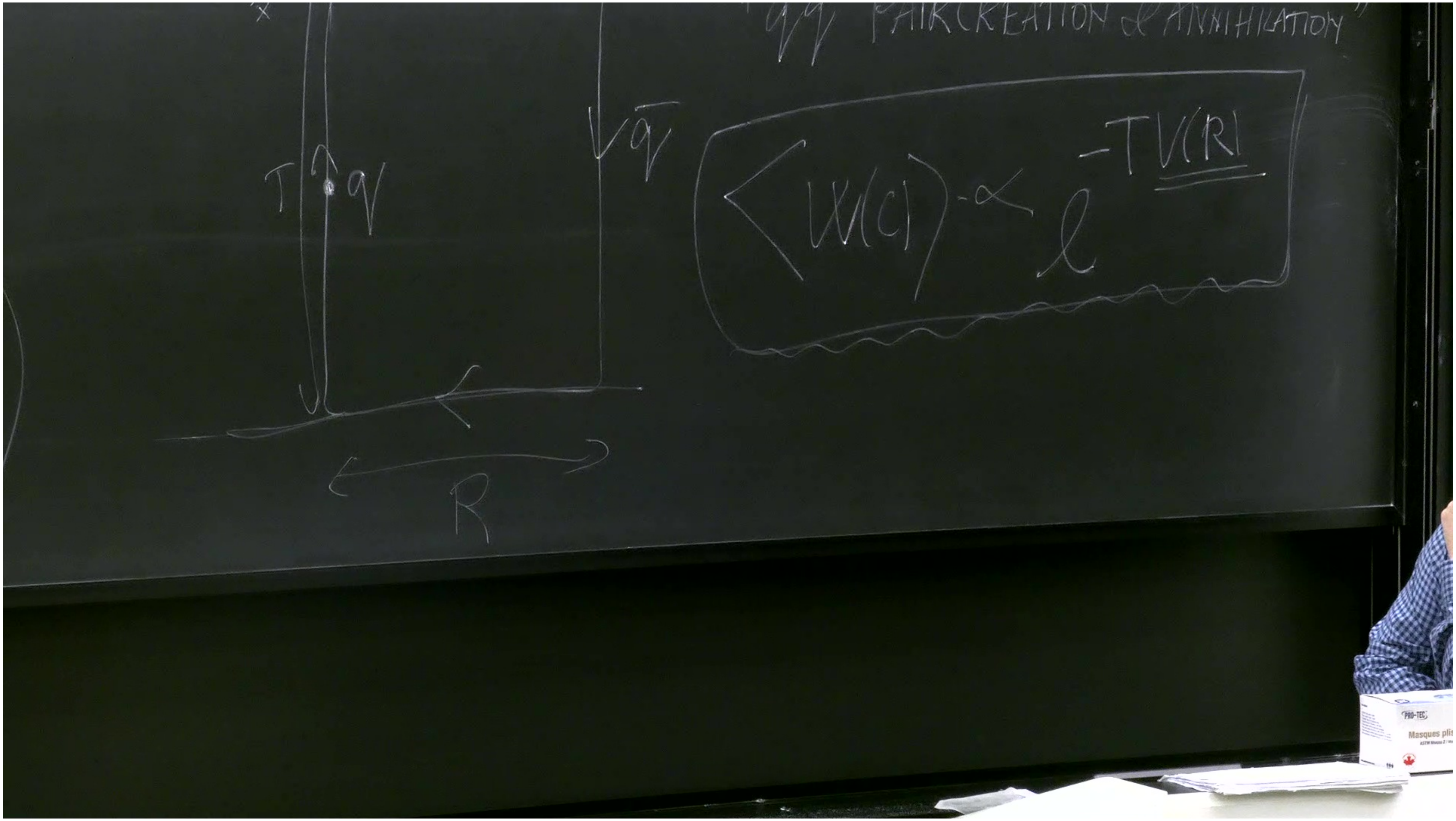

"$q\bar{q}$ PAIRCREATION & ANNIHILATION"
$\langle W(C) \rangle^{-\alpha}$
R
Masques plissés médicaux

THAT DESCRIBE PARALLEL TRANSPORT
OF QUARK ALONG A CLOSED LOOP.

$$W(C) = \frac{1}{N_c} \mathrm{Tr}\left(P \exp\left(i \oint_C A_\mu \, dx^\mu \right) \right)$$

PATH
ORDERING

$$A^\mu = (V, 0)$$


PAIRCREATION & ANNIHILATION
T
q
q̄
⟨W(C)⟩ ∝ e^{-TV(R)}
R
PRO-TEC
Masques