Title: Lecture - AdS/CFT, PHYS 777

Speakers: David Kubiznak

Collection/Series: AdS/CFT (Elective), PHYS 777, March 31 - May 2, 2025

Subject: Quantum Fields and Strings, Quantum Gravity

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AdS|CFT: BASIC DICTIONARY

$$Z^{\lambda\to\infty,\ N_c\to\infty}_{\mathcal{N}=4\ SO(N_c)\ SYM\ \text{in}\ (3+1)}[J] = \int DO\, e^{-S_E + \int d^d x\, O(x) J(x)} \equiv e^{-W[J]}$$

$$\parallel \text{ AdS/CFT}$$

$$Z_{\text{IIB SUGRA on } AdS_5\times S_5}[J] \simeq e^{-S_{\text{IIB SUGRA on } AdS_5\times S_5}[J]}$$

$$\boxed{W[J] = S_{SUGRA}[J]}$$

NOTE: $x \to \lambda x \Rightarrow O_\Delta \sim \lambda^{-\Delta}\ ,\ d^d x \sim \lambda^d$

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STATE-OPERATOR CORRESPONDENCE: USEFUL TABLE (AdS_{d+1}/CFT_d)

OPERATOR (CFT)	FIELD (AdS)	m VS. Δ
SCALAR O_Δ	SCALAR ϕ	$m^2 l^2 = \Delta(\Delta - d)$
CURRENT J_μ	VECTOR A^a	$m^2 l^2 = (\Delta - 1)(\Delta + 1 - d)$
ENERGY-MOM TENSOR $T_{\mu\nu}$	TENSOR h_{ab}	$\Delta = d$ $m^2 l^2 = 0$

TOY EXAMPLE: CONSIDER MASSIVE SCALAR FIELD IN AdS_{d+1}

AdS_{d+1}: $$ds^2 = \frac{\ell^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right) \quad \ldots \text{ POINCARÉ}$$

($z=0$ .. BOUNDARY, $z \to \infty$ POINCARÉ HORIZON)

$$S = -\frac{c}{2}\int dz\, d^d x \sqrt{g}\left(g^{ab}\partial_a\phi\,\partial_b\phi + m^2\phi^2\right)$$

$\delta\phi$: $$\Box\phi = \frac{1}{\sqrt{g}}\left(\sqrt{g}\, g^{ab}\partial_a\phi\right)_{,b} = m^2\phi$$

$T_{\mu\nu}$ h_{ab} $m^2\ell^2=0$

MORE EXPLICITLY:

$$\Box\phi = \frac{1}{\ell^2}\left(z^2\partial_z^2-(d-1)z\partial_z+z^2\eta^{\mu\nu}\partial_\mu\partial_\nu\right)\phi = m^2\phi$$

$$\phi(z,x) = \int\frac{d^dk}{(2\pi)^d}\, e^{ik\cdot x}\,\phi(z,k)$$

$$\boxed{z^2\partial_z^2\phi(z,k)-(d-1)z\partial_z\phi(z,k)-(\underline{m^2\ell^2}+\underline{k^2z^2})\phi(z,k)=0} \quad (*)$$

- FIRST LOOK CLOSE TO $z=0$ $\rightarrow$ CAN NEGLECT k^2z^2 TERM

SOL CAN BE FOUND $\phi(z,k)=z^{\Delta}$

$$\Rightarrow \quad z^2 \cancel{z^{\Delta-2}}\,\Delta(\Delta-1) - (d-1)\, z\, \cancel{z^{\Delta-1}}\,\Delta - m^2\ell^2 \cancel{z^{\Delta}} = 0$$

$$\Delta^2 - \Delta + (-d+1)\Delta - m^2\ell^2 = 0$$

$$\Delta^2 - d\Delta - m^2\ell^2 = 0 \iff \boxed{\Delta(\Delta-d) = m^2\ell^2}$$

$$\boxed{\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2}}$$

DEEP MATH: $\boxed{\Delta_- = d - \Delta_+}$

ASYMPTOTICALLY:

$$\phi(z,x) \sim \phi_0 \, \underbrace{z^{\Delta_-}}_{\text{NON-RENORMALIZABLE}} + \phi_+ \, z^{\Delta_+} \ldots$$

RENORMALIZABLE

LET'S IDENTIFY: $\boxed{\Delta = \Delta_+}$

$\langle O_\Delta \rangle$ ... EXPECT. VALUE OF O_Δ

$\phi_0 \, z^{d-\Delta}$

SOURCE FOR OPERATOR O_Δ.

$$\boxed{\phi_0 = \lim_{z \to 0} \phi(z,x) \, z^{\Delta - d}}$$

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USE THIS TO CALCULATE CORRELATION FUNCTIONS ; RECIPE

1. DETERMINE BULK FIELD ϕ DUAL TO $\mathcal{O}$
2. SOLVE (KK REDUCED) SUGRA EOM FOR ϕ
   SUBJECT TO BCS: $\phi(z,x) \sim z^{d-\Delta}\phi_0(x)$ AS $z \to 0$
3. PLUG BACK TO THE ACTION (OBTAIN "HAMILTONS FUNCTION")
4. $W[\phi_0] = S_{SUGRA}[\phi]\Big/_{\lim_{z\to 0} \phi(z,x) z^{\Delta-d} = \phi_0}$

(5) $$\langle O(x_1)\dots O(x_m)\rangle = -\left.\frac{\delta^m W(\phi_0)}{\delta\phi_0(x_1)\dots\delta\phi_0(x_m)}\right|$$

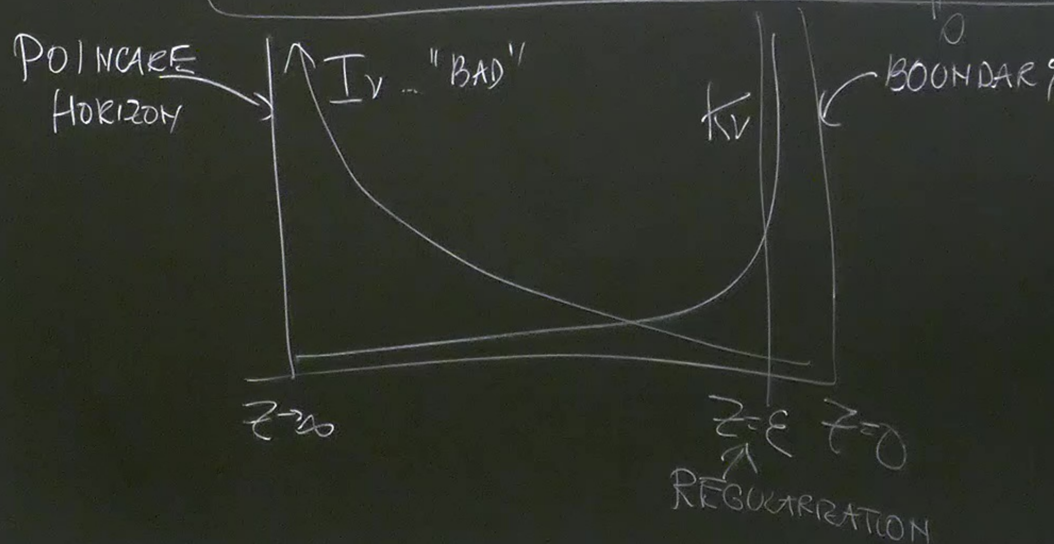
LET'S CONTINUE THE TOY EXAMPLE & CALCULATE 2-POINT FUNCTION

WE WANT $\langle O_\Delta(x) O_\Delta(y)\rangle \overset{\text{SPOILER}}{\propto} \dfrac{1}{|x-y|^{2\Delta}}$ (1 LINE CALC.)

(1) ✓

(2) EOM's NEED THE FULL SOLUTION OF (*)

SOL: $\underbrace{\phi(z,k) = A_k\, z^{d/2} K_\nu(z|k|) + \underset{=0}{B_k}\, z^{d/2} \cancel{I_\nu}(z|k|)}$, $\nu = \Delta - d/2 = \sqrt{\frac{d^2}{4} + m^2 \ell^2}$



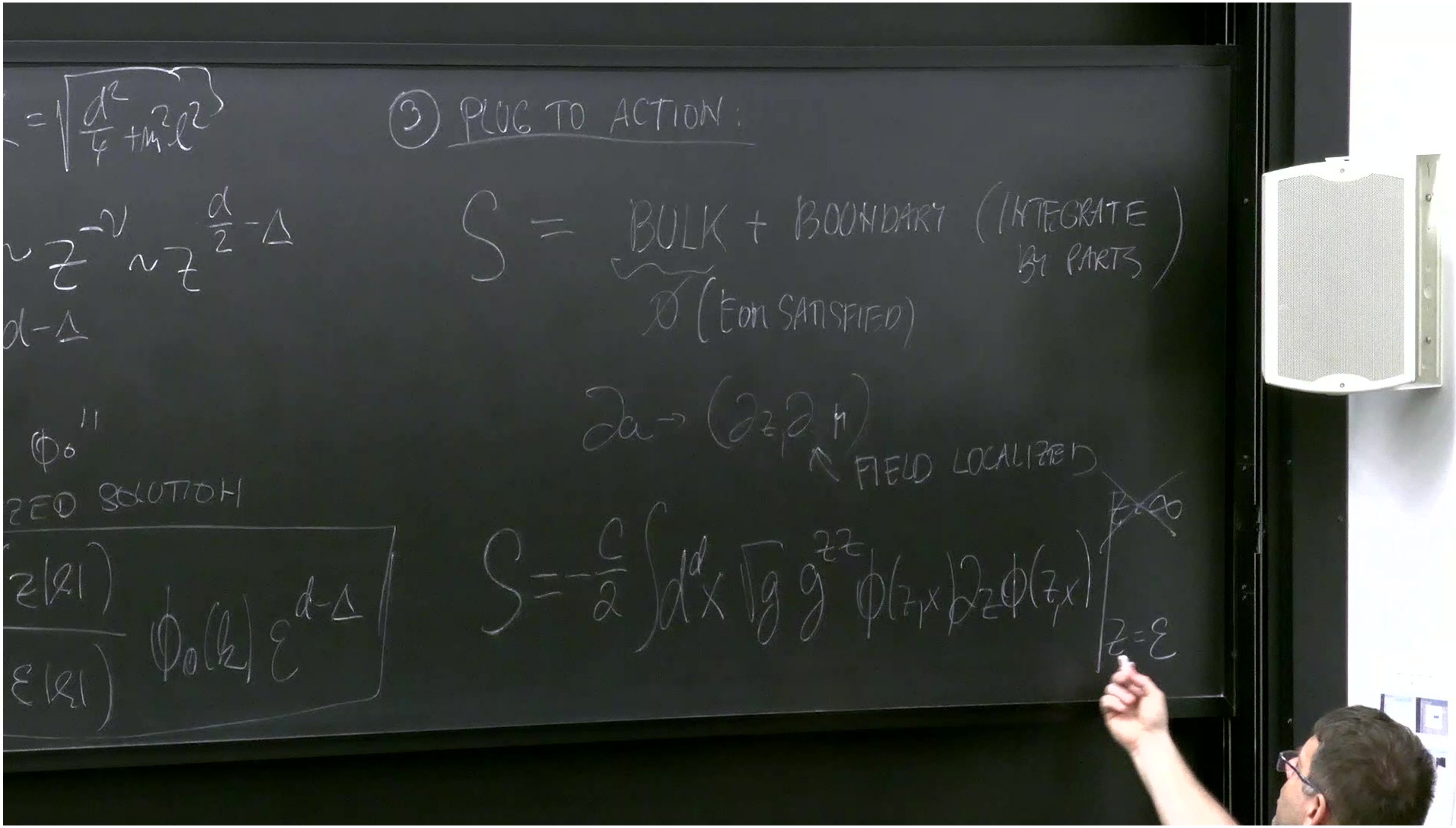
• AS $z \to 0$ $K_\nu(z) \sim z^{-\nu} \sim z^{\frac{d}{2} - \Delta}$

$\phi(z,k) \sim A_k\, z^{d-\Delta}$

"SOURCE ϕ_0"

WE WRITE THE NORMALIZED SOLUTION

$$\boxed{\phi(z,k) = \frac{z^{d/2} K_\nu(z|k|)}{\varepsilon^{d/2} K_\nu(\varepsilon|k|)} \phi_0(k)\, \varepsilon^{d-\Delta}}$$


③ PLUG TO ACTION:
S = BULK + BOUNDARY (INTEGRATE BY PARTS)
∅ (EOM SATISFIED)
∂a → (∂z, ∂μ)
FIELD LOCALIZED
S = −c/2 ∫ d^d x √g g^zz φ(z,x) ∂z φ(z,x) |z=ε
ZED SOLUTION

$$S[\phi_0] = -\frac{C\,\ell^{d-1}}{2\,\varepsilon^{d-1}} \int \frac{d^d k}{(2\pi)^d}\,\frac{d^d p}{(2\pi)^d}\,(2\pi)^d\,\delta(k+p)\,\phi(z,k)\,\partial_z\,\phi(z,p)\Big|_{z=\varepsilon}$$

GOING FROM X-SPACE TO p-SPACE

PERCIVAL: $\displaystyle\int d^d x\, O(x)\,\phi_0(x) = \int \frac{d^d k}{(2\pi)^d}\, O(k)\,\phi_0(-k)$

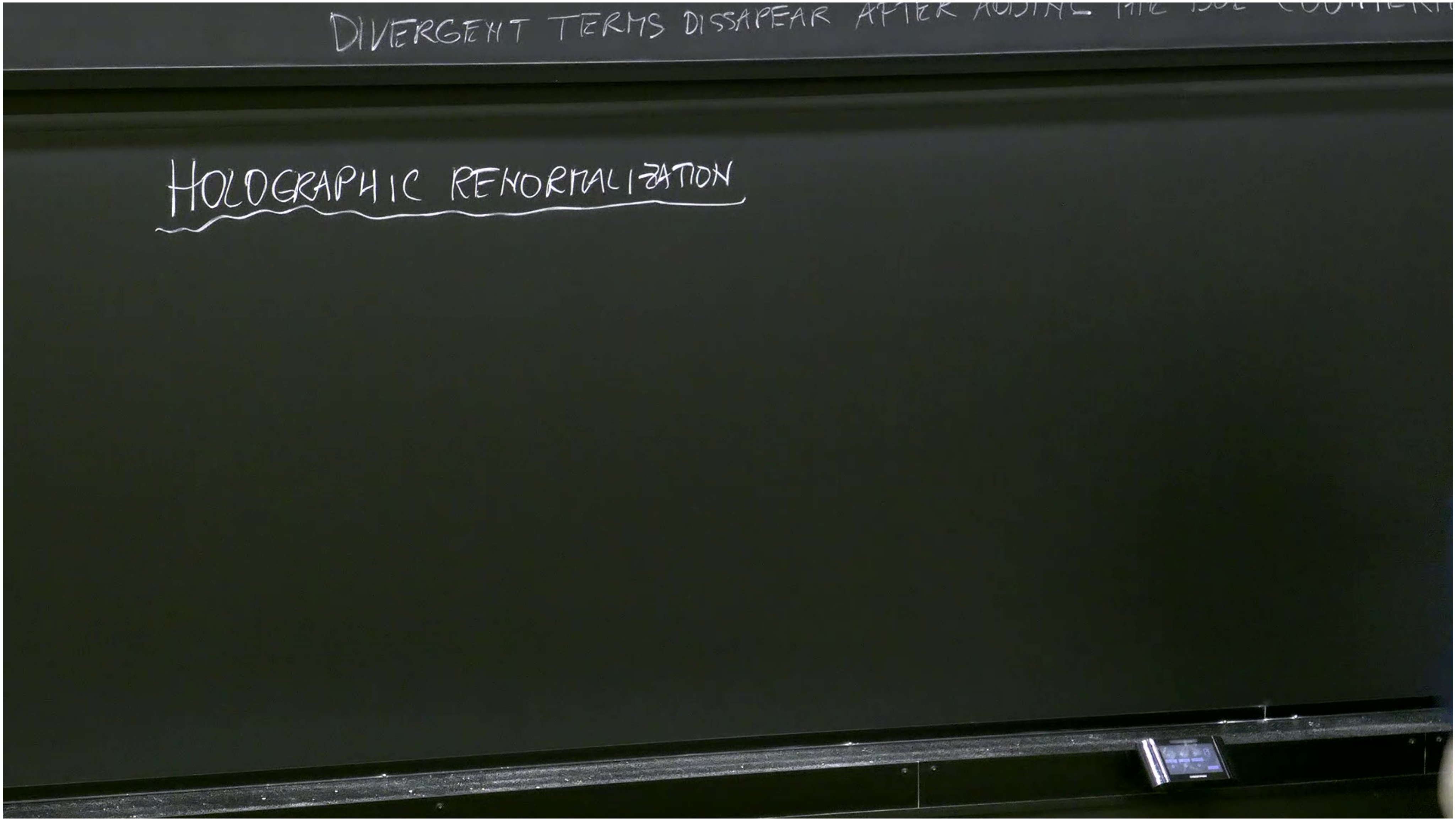
$$\langle O(k)\, O(p)\rangle = -(2\pi)^d\,\frac{\delta^2 S[\phi_0]}{\delta\phi_0(-k)\,\delta\phi_0(-p)} = \ldots \propto \frac{C}{\varepsilon^{2\Delta-d}}\left(\frac{d}{2} + \frac{\varepsilon|k|\,K_\nu'(\varepsilon|k|)}{K_\nu(\varepsilon|k|)}\right)\delta(k+p)$$

$$\int e^{i(k+p)x}\, d^d x \sim \delta(k+p)$$

NEED $\epsilon \to 0$:

$$\delta(k+p)\left(\frac{\#}{\epsilon^{2\Delta-d}} + \frac{\#}{\epsilon^{2\Delta-d-2}} + \cdots \# \log\epsilon \right.$$
$$\left. \underbrace{\# |k|^{2\nu} \log|k|}_{\text{RESULT}} + \underbrace{O(\epsilon^2)}_{\cancel{0}} \right)$$

$$\left. \frac{\epsilon|k|\, K_\nu'(\epsilon|k|)}{K_\nu(\epsilon|k|)} \right) \delta(k+p) =$$


DIVERGENT TERMS DISSAPEAR AFTER
HOLOGRAPHIC RENORMALIZATION