Title: Lecture - AdS/CFT, PHYS 777

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Collection/Series: AdS/CFT (Elective), PHYS 777, March 31 - May 2, 2025

Subject: Quantum Fields and Strings, Quantum Gravity

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AdS/CFT conjecture

Conjecture (Maldacena 1997). Type IIB superstring theory on $AdS_5 \times S^5$ is *dual* to $\mathcal{N} = 4$ $SU(N_c)$ SYM in $d = (3+1)$ dimensions.

String theory

Dual

QFT

$$g_s\,, \qquad L/l_s \quad \Longleftrightarrow \quad N_c\,, \qquad \lambda = g_{YM}^2 N_c$$

- A theory of **QG**
- A **QFT** without gravity

$$2\pi g_s = g_{YM}^2 = \frac{\lambda}{N_c}\,, \qquad \frac{L^4}{l_s^4} = 4\pi g_s N_c = 2\lambda$$

Useful weaker version

$$\boxed{2\pi g_s = g_{YM}^2 = \frac{\lambda}{N_c}, \quad \frac{L^4}{l_s^4} = 4\pi g_s N_c = 2\lambda}$$

Classical (super)gravity as a limit of ST:

- Strings almost point-like – standard geometry valid

$$L \gg l_s \quad \Rightarrow \quad \lambda \gg 1$$

- Quantum corrections (loops) are small $1 \gg g_s \sim \dfrac{\lambda}{N_c}$

Weak (weakly interacting) gravity is dual to strongly coupled QFT

$$\boxed{1 \ll \lambda \ll N_c}$$ (**strong-weak duality**)

Amazing tool to study strongly coupled QFTs (by simple gravitational calculations).

What is AdS?

= maximally symmetric solution of EE with negative cosmological constant

Embedding perspective: "hyperboloid in $\mathbb{R}^{2,d-1}$"

$$
\begin{aligned}
ds^2 &= -(dY^{-1})^2 - (dY^0)^2 + (dY^1)^2 + \cdots + (dY^{d-1})^2 \equiv \eta^{2,d-1}_{AB} dY^A dY^B \,, \\
-\ell^2 &= -(Y^{-1})^2 - (Y^0)^2 + (Y^1)^2 + \cdots + (Y^{d-1})^2 = \eta^{2,d-1}_{AB} Y^A Y^B \,.
\end{aligned}
$$

- Eliminating Y^{-1} from the constraint we get:

$$\boxed{g_{ab} = \eta^{1,d-1}_{ab} - \frac{Y_a Y_b}{\ell^2 + Y^a Y_a}} \qquad Y_a = \eta^{1,d-1}_{ab} Y^b$$

- Yields:

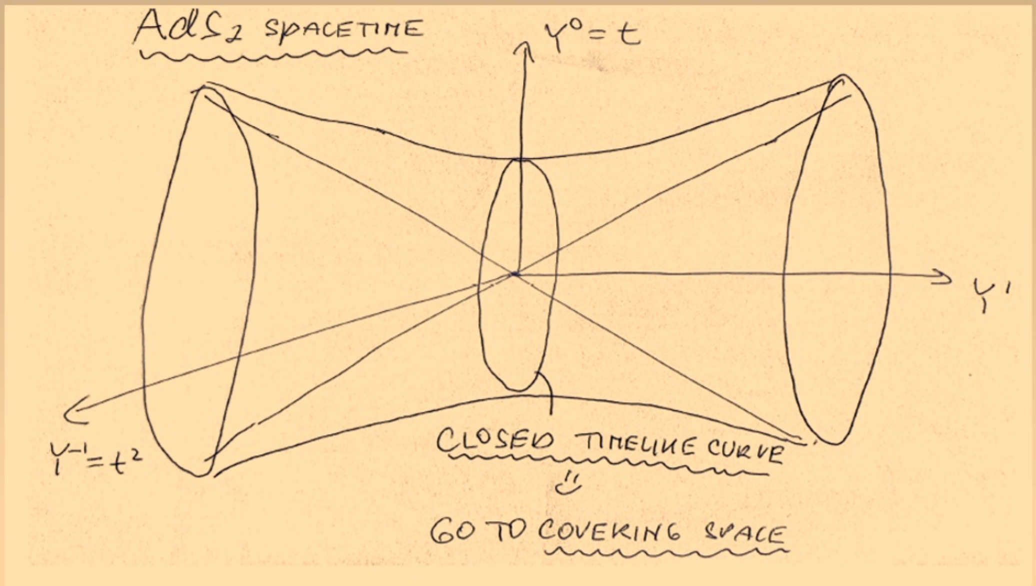
$$R_{abcd} = -\frac{1}{\ell^2}(g_{ac}g_{bd} - g_{ad}g_{bc})$$

$$G_{ab} + \Lambda g_{ab} = 0$$

$$\Lambda = -\frac{(d-1)(d-2)}{2\ell^2}$$

What is AdS?

Embedding perspective: "hyperboloid in $\mathbb{R}^{2,d-1}$"



Maximally symmetric: invariant under $\tilde{x}^A = \Lambda^A{}_B x^B \quad \text{where} \quad \eta^{2,d-1}_{AB} = \eta^{2,d-1}_{CD} \Lambda^C{}_A \Lambda^D{}_B$

… O(d-1,2) symmetry

AdS global coordinates

- Single out Y^{-1} and Y^{0} and "move" the constraint

$$Y^{-1} = \ell \cosh\tilde{\rho} \cos\tilde{t}\,, \quad Y^0 = \ell \cosh\tilde{\rho} \sin\tilde{t}\,, \quad Y^i = \ell \sinh\tilde{\rho}\Omega_i$$

$$\Omega_i^2 \;=\; 1$$ (angular coordinates on the sphere)

$$ds^2 = \ell^2\Big(-\cosh^2\tilde{\rho}d\tilde{t}^2 + d\tilde{\rho}^2 + \sinh^2\tilde{\rho}d\Omega_{d-2}^2\Big)$$

- Let's go to the **universal covering space**:

$$\tilde{t} \sim \tilde{t} + 2\pi \quad \Longrightarrow \quad \tilde{t} \in (-\infty, \infty)$$

… no longer have closed timelike curves (CTCs).

AdS global coordinates

- Can further compactify "radial coordinate": $\tan\theta = \sinh\tilde{\rho}$,

$$ds^2 = \frac{\ell^2}{\cos^2\theta}\left(-d\tilde{t}^2 + d\theta^2 + \sin^2\theta d\Omega_{d-2}^2\right)$$

$$\theta \in [0, \pi/2)$$

… we have
(non Euclidean) **cylinder**

- **Conformal boundary** at $\theta = \pi/2$

$$ds^2|_{\partial\Omega} = -d\tilde{t}^2 + d\Omega_{d-2}^2$$

… **Einstein static Universe**: $\mathbb{R} \times S^{d-2}$

This is where the field theory lives!

AdS global coordinates

- Single out Y^{-1} and Y^{0} and "move" the constraint

$$Y^{-1} = \ell \cosh\tilde{\rho}\cos\tilde{t}\,, \quad Y^0 = \ell\cosh\tilde{\rho}\sin\tilde{t}\,, \quad Y^i = \ell\sinh\tilde{\rho}\Omega_i$$

$$\Omega_i^2 = 1$$ (angular coordinates on the sphere)

$$ds^2 = \ell^2\left(-\cosh^2\tilde{\rho}d\tilde{t}^2 + d\tilde{\rho}^2 + \sinh^2\tilde{\rho}d\Omega_{d-2}^2\right)$$

- Let's go to the **universal covering space**:

$$\tilde{t} \sim \tilde{t} + 2\pi \quad \Longrightarrow \quad \tilde{t} \in (-\infty, \infty)$$

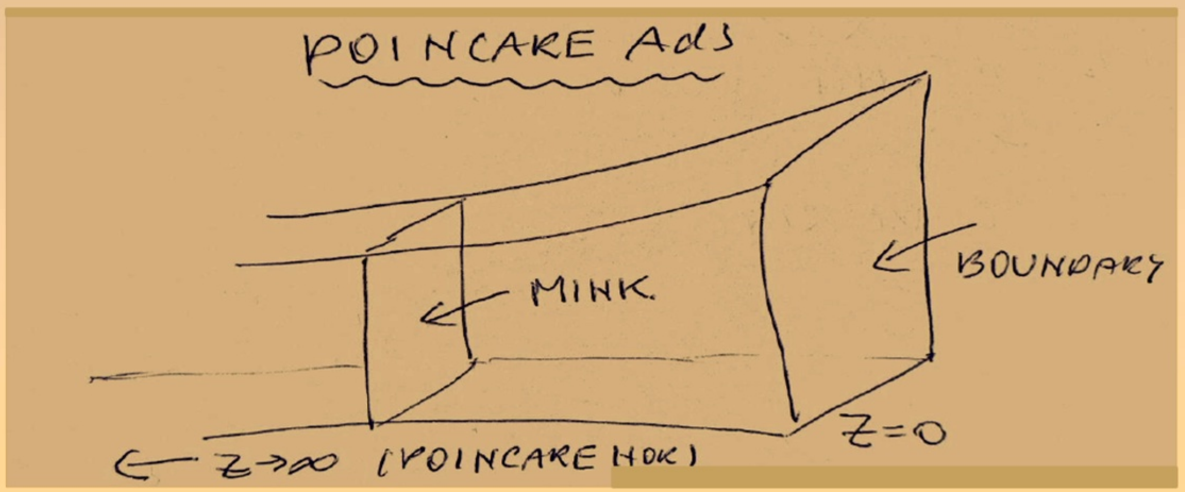
… no longer have closed timelike curves (CTCs).

Poincare AdS coordinates

- Single out Y^{-1} and Y^{d-1} and remove the constraint

$$Y^{-1} = \frac{1 + z^2 + \eta_{\mu\nu} x^\mu x^\nu}{2z}, \quad Y^\mu = \frac{x^\mu}{z}, \quad Y^{d-1} = \frac{1 - z^2 - \eta_{\mu\nu} x^\mu x^\nu}{2z}$$

$$ds^2 = \frac{\ell^2}{z^2}\left(\underbrace{-dt^2 + d\vec{x}^2}_{\eta_{\mu\nu} dx^\mu dx^\nu} + dz^2\right)$$



Poincare AdS = volume filling slices of Minkowski:

Poincare AdS coordinates

$$ds^2 = \frac{\ell^2}{z^2}\Big(\underbrace{-dt^2 + d\vec{x}^2}_{\eta_{\mu\nu}dx^\mu dx^\nu} + dz^2\Big)$$

- Isometry: $(t, \vec{x}, z) \;\to\; \lambda(t, \vec{x}, z)$
- **Conformal completion**: multiply by $\Omega^2(z, x^\mu)$

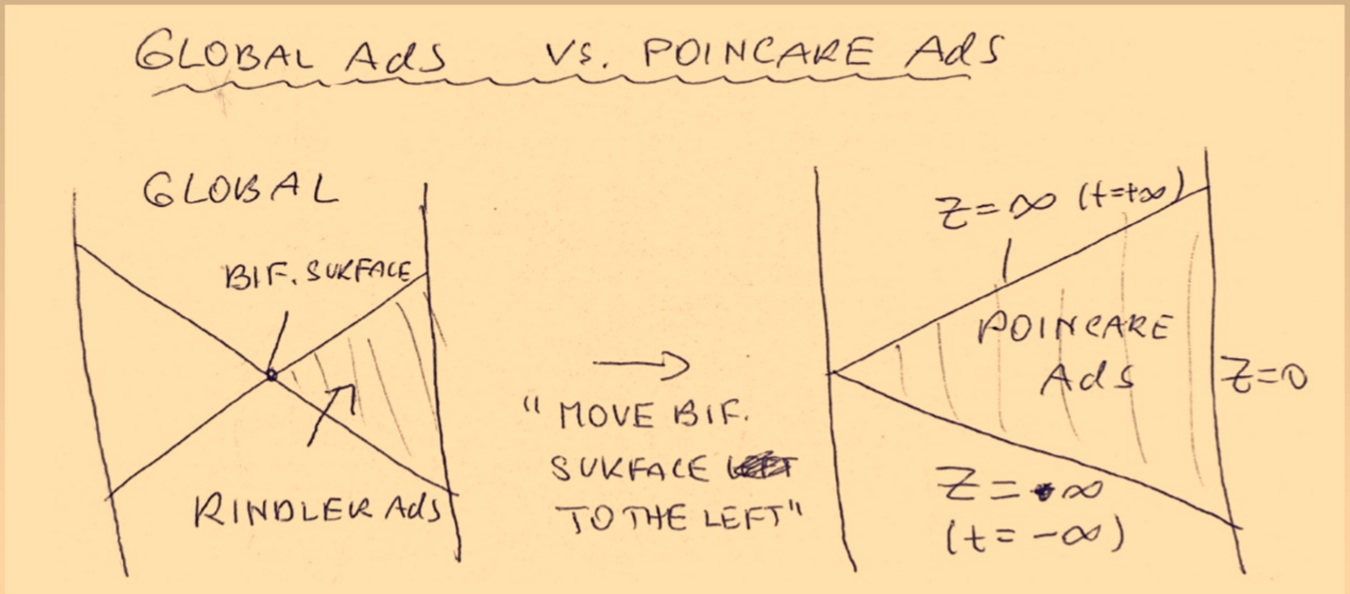
$$\Omega^2 = \frac{z^2}{\ell^2}\omega^2(x^\mu) \quad \Rightarrow \quad ds^2|_{\partial AdS} = \omega^2(x^\mu)\eta_{\mu\nu}dx^\mu dx^\nu$$

… **class** of boundary metrics related by conformal transformations (**conformal infinity**). (Moreover, isometries of AdS act on boundary as conformal group.)

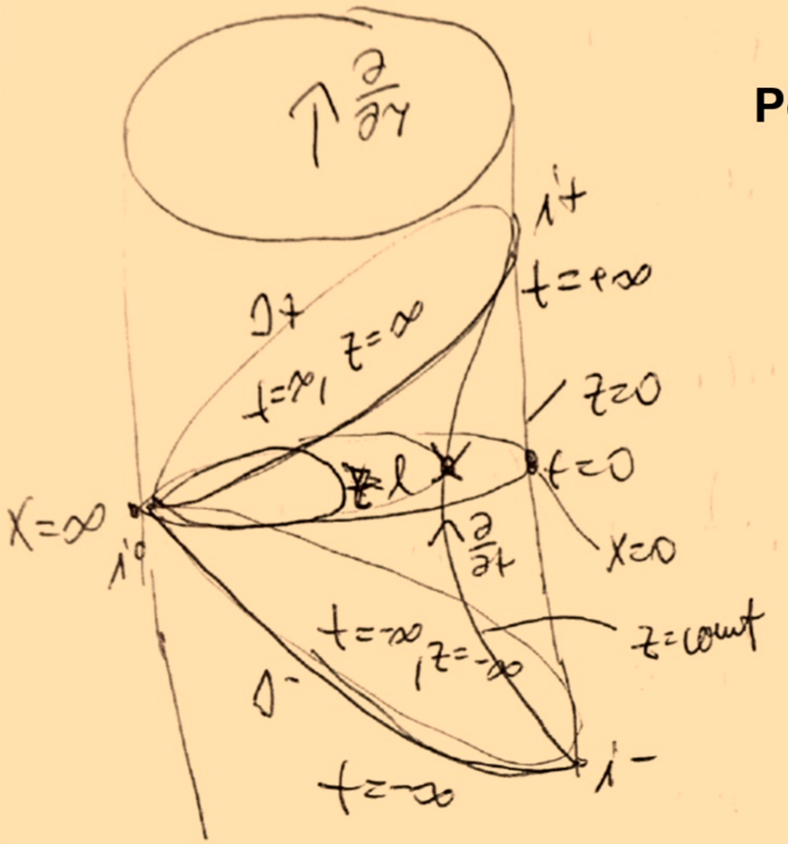
This is why boundary theory has to be a **CFT!**

Poincare AdS coordinates

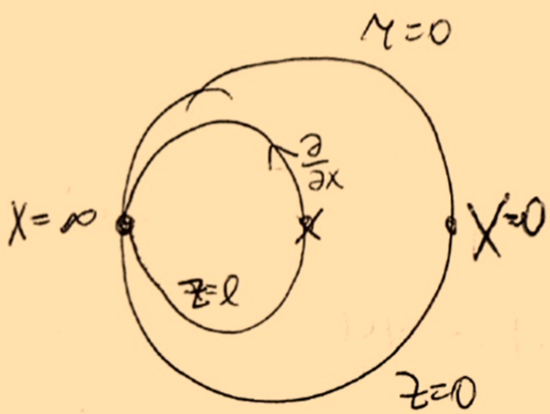
- Poincare patch is a bit like Rindler frame:



Poincare vs. global coordinates



Poincare disc: (τ=0 slice)



Two more coordinate systems

- The following coordinate system is used in brane world scenarios:

$$ds^2 = dr^2 + \ell^2 e^{2r/\ell} \eta_{\mu\nu} dx^\mu dx^\nu \,,$$

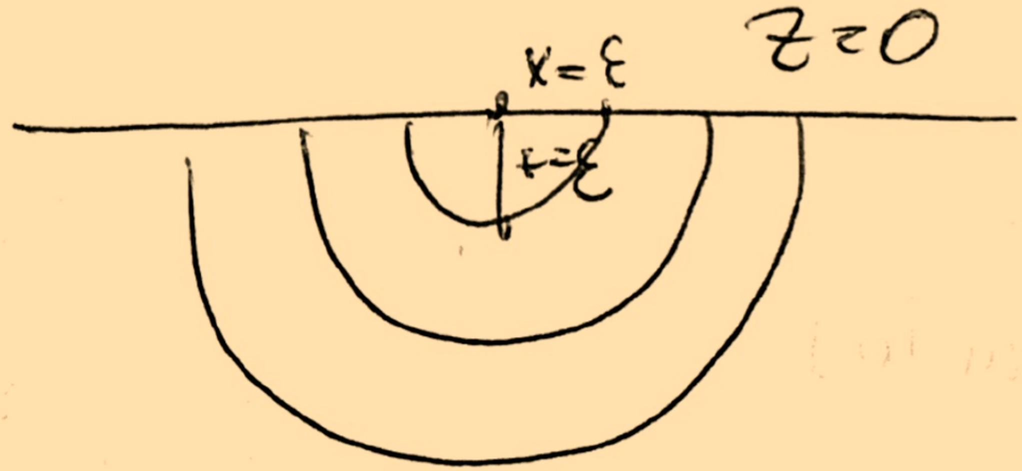
which is obtained from Poincare by $z = \exp(-r/\ell)$. Here the conformal boundary is at $r \to \infty$ and the horizon at $r \to -\infty$.

- Fefferman–Graham metric is obtained by setting $z^2 = \rho$, to get

$$\boxed{ds^2 = \ell^2 \Big(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} \eta_{\mu\nu} dx^\mu dx^\nu \Big) \,.}$$

This will play a role for the holographic renormalization, and calculation of the holographic stress tensor.

UV/IR correspondence



- **UV cutoff in CFT** corresponds to **IR cutoff** (large distances from origin) **in AdS.**

  (UV behavior of CFT is not sensitive to what happens deep in the bulk)

3) BASIC AdS/CFT DICTIONARY

$$Z_{\substack{\mathcal{N}=4\ SU(N_c)\ SYM \\ \text{IN } d=3+1}}[J] = Z_{\substack{\text{TYPE IIB ST} \\ \text{ON } AdS_5 \times S^5}}[J]$$

J ↑ SOURCE

J ↑ WHAT IS THIS ?

$$S_0 \to S_0 - \int d^dx \, O(x) J(x)$$

$J(x)$ ↑ SOURCE

WEAKER VERSION:

$$Z^{\lambda\to\infty,\, N_c\to\infty}_{\substack{N=4\ SO(N_c)\ SYM \\ \text{IN } d=3+1}}[J] \overset{WKB}{\simeq} \exp\left(-S_{\substack{\text{TYPE IIB SUGRA} \\ \text{ON } AdS_5\times S^5}}[J]\right) + O\left(\frac{1}{\lambda^{3/2}},\, \frac{1}{N_c^2}\right) \simeq$$

"ANOTHER EXPANSION"

3) BASIC AdS/CFT DICTIONARY

$$Z_{\mathcal{N}=4\ SU(N_c)\ SYM}[J] = Z_{\text{TYPE IIB ST}}[J]$$

IN $d=3+1$ — SOURCE — ON $AdS_5 \times S_5$ — WHAT IS THIS?

$$S_0 \to S_0 - \int d^dx\, O(x)\, J(x)$$

SOURCE

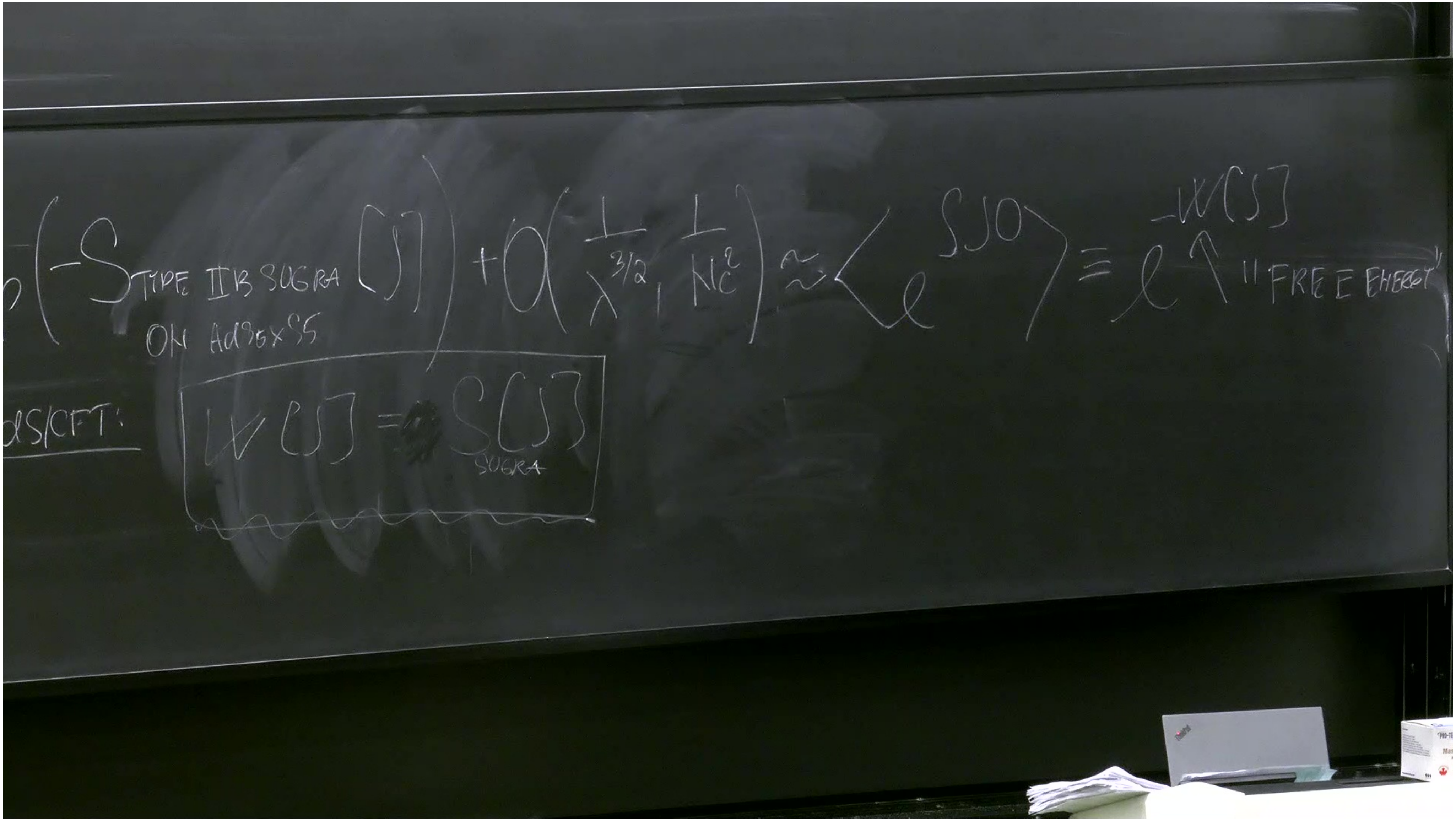
$$\int D \ldots e^{-S}$$

WEAKER VERSION:

$$Z^{\lambda\to\infty,\ N_c\to\infty}_{\mathcal{N}=4\ SO(N_c)\ SYM\ \text{IN}\ d=3+1}[J] \overset{WKB}{\simeq} \exp\left(-S_{\text{TYPE IIB SUGRA ON } AdS_5\times S^5}[J]\right) + O\left(\frac{1}{\lambda^{3/2}}\right)$$

"ANOTHER EXPANSION"

AdS/CFT: $\boxed{W[J] = S_{SUGRA}[J]}$



TYPE IIB SUGRA
ON AdS5×S5
"FREE ENERGY"
dS/CFT:
SUGRA

BUILDING THE DICTIONARY: STATE-OPERATOR CORRESPONDENCE

ON CFT: OPERATORS ... $\mathcal{O}_\Delta$ ... SPIN ℓ SCALING DIM.

$x \to \lambda x \to \boxed{\mathcal{O}_\Delta \to \lambda^{-\Delta}\mathcal{O}_\Delta}$

ON GRAVITY: FIELDS

GUIDING PRINCIPLE: "MATCH SYMMETRIES"

SCALAR $\longleftrightarrow$ SCALAR

BUILDING THE DICTIONARY: STATE-OPERATOR CORRESPONDENCE

USEFUL TABLE:

ON CFT: OPERATORS ... O_Δ ... SPIN ℓ SCALING DIM.

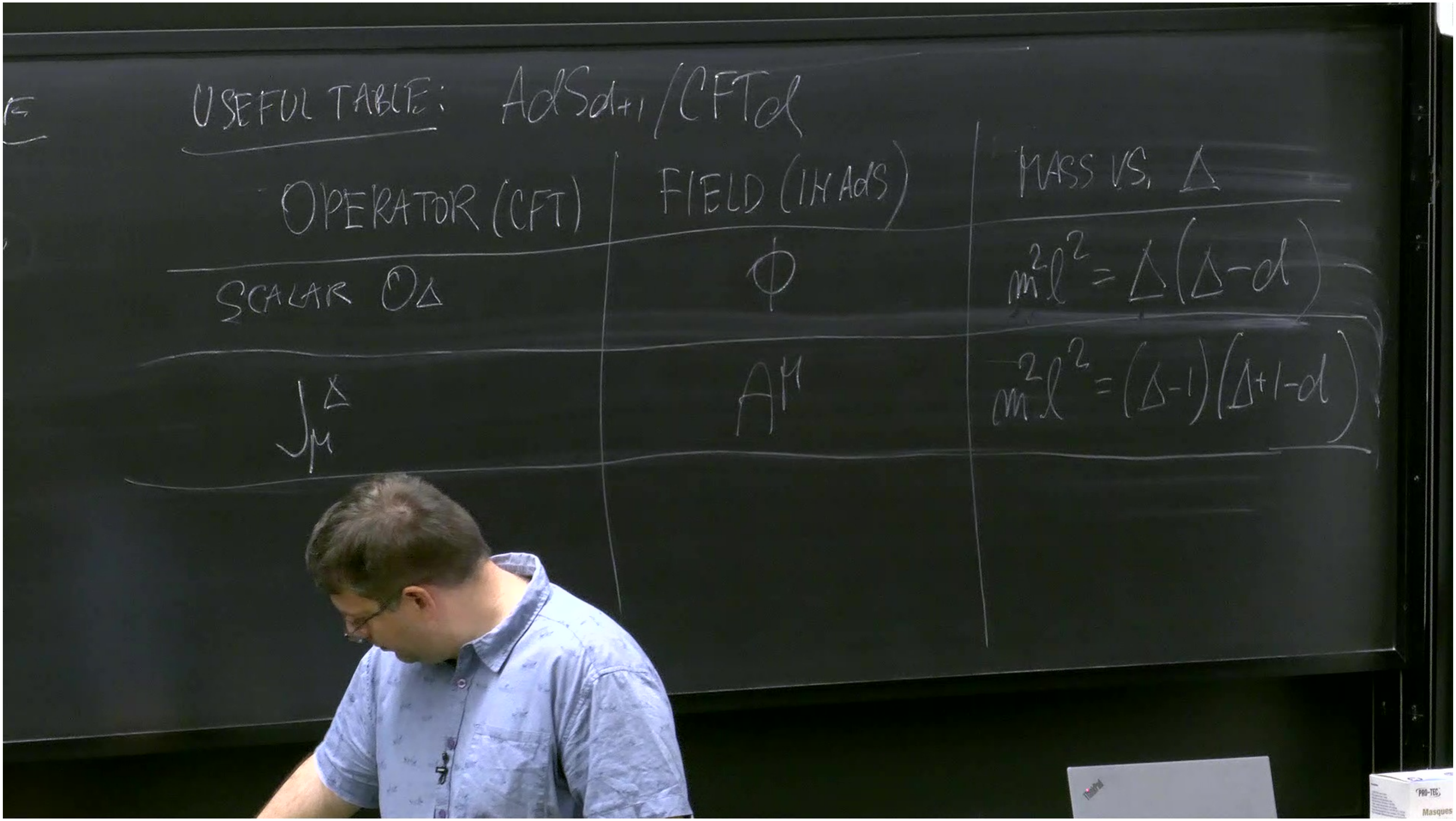
$$x \to \lambda x \to \boxed{O_\Delta \to \lambda^{-\Delta} O_\Delta}$$

ON GRAVITY: FIELDS

GUIDING PRINCIPLE: "MATCH SYMMETRIES"

SCALAR $\longleftrightarrow$ SCALAR

Δ $\longleftrightarrow$ MASS OF THE FIELD


USEFUL TABLE: AdS_{d+1}/CFT_d
OPERATOR (CFT)
FIELD (IN AdS)
MASS VS. Δ
SCALAR $\mathcal{O}_\Delta$
ϕ
$m^2 l^2 = \Delta(\Delta - d)$
J_μ^Δ
A^μ
$m^2 l^2 = (\Delta - 1)(\Delta + 1 - d)$

OPERATOR (CFT)	FIELD (IN AdS)	MASS VS. Δ
SCALAR O_Δ	ϕ	$m^2\ell^2 = \Delta(\Delta-d)$
J_μ^Δ	A^μ	$m^2\ell^2 = (\Delta-1)(\Delta+1-d)$
$T_{\mu\nu}$	$h^{\mu\nu}$	$\Delta = d$ $m^2\ell^2 = 0$

USEFUL TABLE: AdS_{d+1}/CFT_d

OPERATOR (CFT)	FIELD (IN AdS)	MASS VS. Δ
SCALAR $\mathcal{O}_\Delta$	ϕ	$m^2\ell^2 = \Delta(\Delta - d)$
J_μ^Δ	A^μ	$m^2\ell^2 = (\Delta-1)(\Delta+1-d)$
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