Title: Lecture - AdS/CFT, PHYS 777

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Subject: Quantum Fields and Strings, Quantum Gravity

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Euclidean trick (Gibbons & Hawking 1977)

• **Gravitational action**:

$$S_E = \int_\Omega \frac{d^4x\sqrt{g}R}{16\pi G} + \int_{\partial\Omega} \frac{d^3x\epsilon\sqrt{h}\mathcal{K}}{8\pi G} + \text{counter terms}$$

Einstein-Hilbert action
(gives Einstein equations)

York-Gibbons-Hawking term
(yields well posed variational principle with Dirichlet BCs)

Counter terms: "renormalize" the value of the action
(In AdS given covariantly by *holographic renormalization*. In flat space no covariant prescription exists!)

- The prescription confirms **Bekenstein's area law!**

$$S = -\frac{\partial F}{\partial T} = \frac{A}{4}$$

Euclidean trick (Gibbons & Hawking 1977)

- **Original manifold** non-singular:

$$\varphi \sim \varphi + 2\pi \quad \Leftrightarrow \quad \tau \sim \tau + \underbrace{2\pi/\kappa}_{\beta} \quad \Leftrightarrow \quad \boxed{T = \frac{\kappa}{2\pi}\,,}$$

… which is the **Hawking's temperature**.

Gravitational partition function

$$\boxed{Z = \int Dg e^{-S_E[g]} \approx e^{-S_E(g_c)}}$$

(using WKB approximation)

- **Free energy**:

$$F = -\frac{1}{\beta}\log Z \approx \frac{S_E}{\beta} \quad \Rightarrow \quad S = -\frac{\partial F}{\partial T}$$

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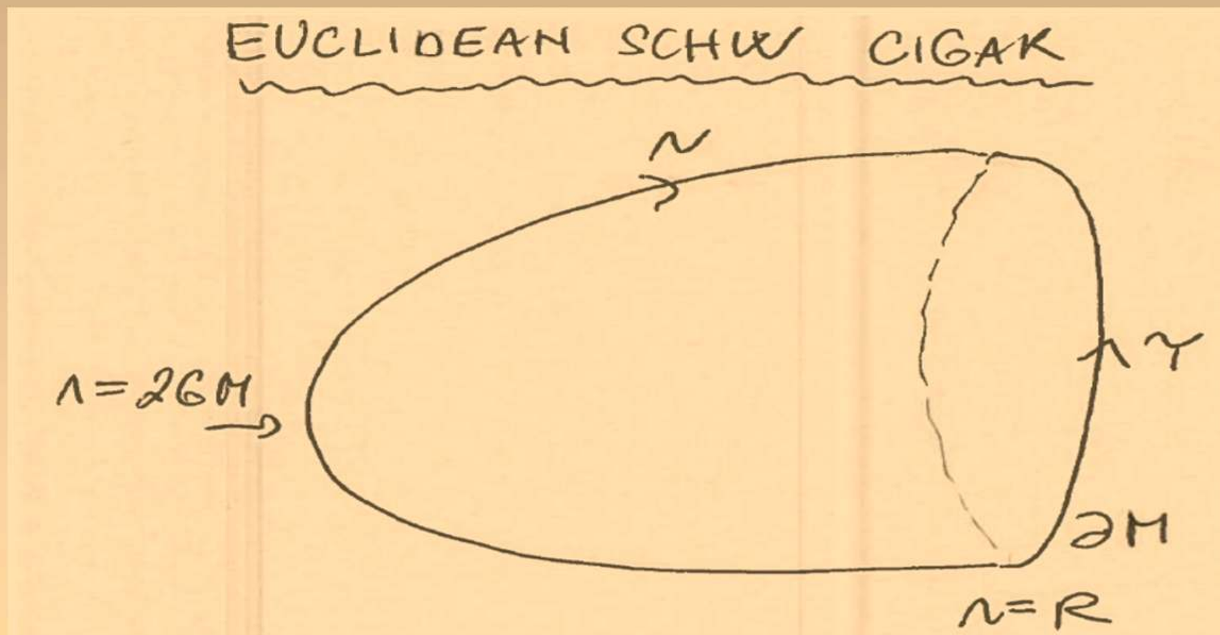
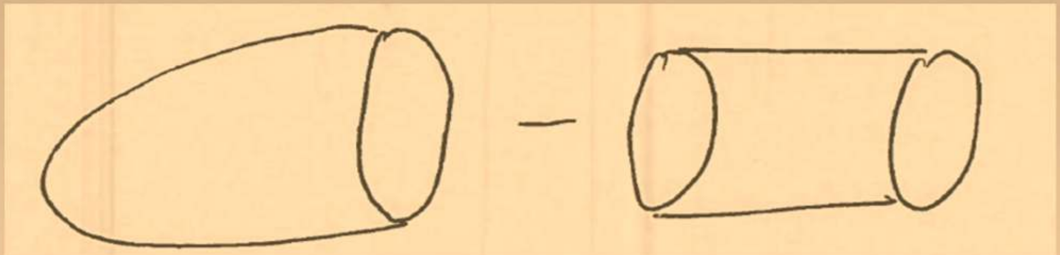
Example 1: Euclideanized Schwarzschild

$$ds^2 = f d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \quad f = 1 - \frac{2M}{r}$$



Background subtraction

$$S_E = -\frac{1}{16\pi G}\int_M d^4x\sqrt{g}R - \frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{h}(K - K_0)$$



$$F = \frac{S_E}{\beta} = \frac{M}{2} = \frac{\beta}{16\pi} = M - TS$$

$$S = -\frac{\partial F}{\partial T} = \frac{\beta^2}{16\pi} = 4\pi M^2 = \pi r_+^2 = \frac{A}{4}$$

… confirmed **Bekenstein's area law!**

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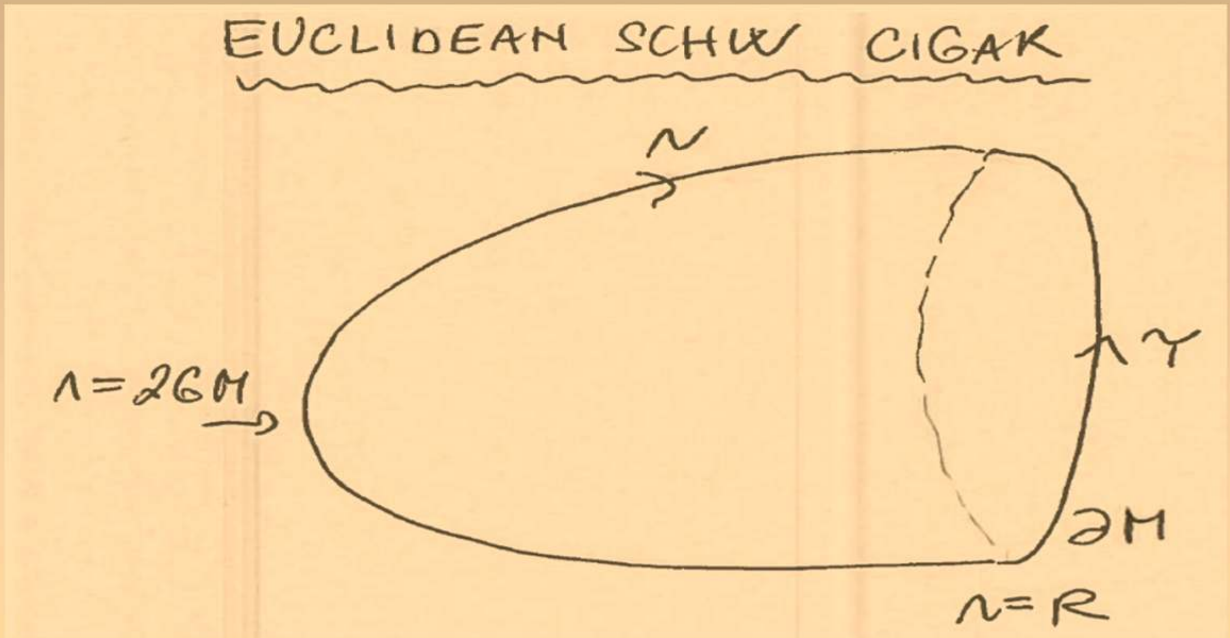


- **Absence** of conical singularity implies:

$$T = \frac{1}{\beta} = \frac{f'(r_+)}{4\pi} = \frac{1}{8\pi M}$$

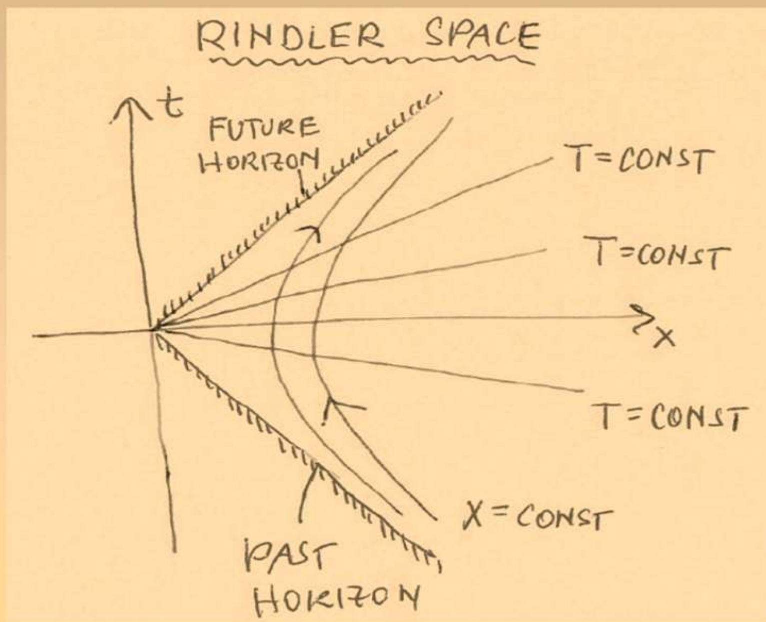
Example 2: Rindler spacetime

- Consider uniformly accelerated observer:

$$a = \sqrt{a_\mu a^\mu} = \text{const}$$

$$t = \frac{1}{a}\sinh(a\tau)\,, \quad x = \frac{1}{a}\cosh(a\tau)$$

- **Rindler frame**:



$$t = \left(\frac{1}{a} + X\right)\sinh(aT)\,,$$

$$x = \left(\frac{1}{a} + X\right)\cosh(aT)$$

$$ds^2 = -dt^2 + dx^2 = -(1 + aX)^2 dT^2 + dX^2$$

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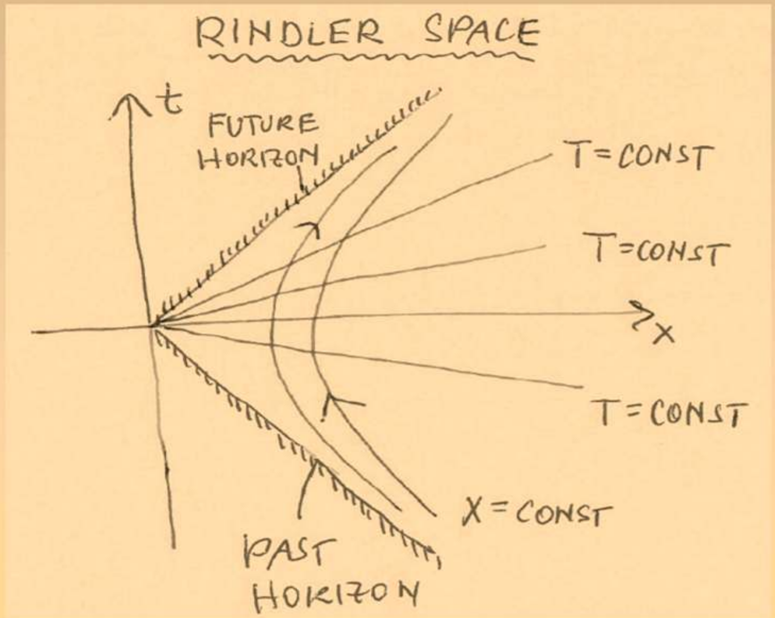

$$t = \left(\frac{1}{a} + X\right)\sinh(aT)\,,$$

$$x = \left(\frac{1}{a} + X\right)\cosh(aT)$$

Rindler horizon at:

$$X_R = -\frac{1}{a}$$

$$ds^2 = -dt^2 + dx^2 = -(1 + aX)^2 dT^2 + dX^2$$

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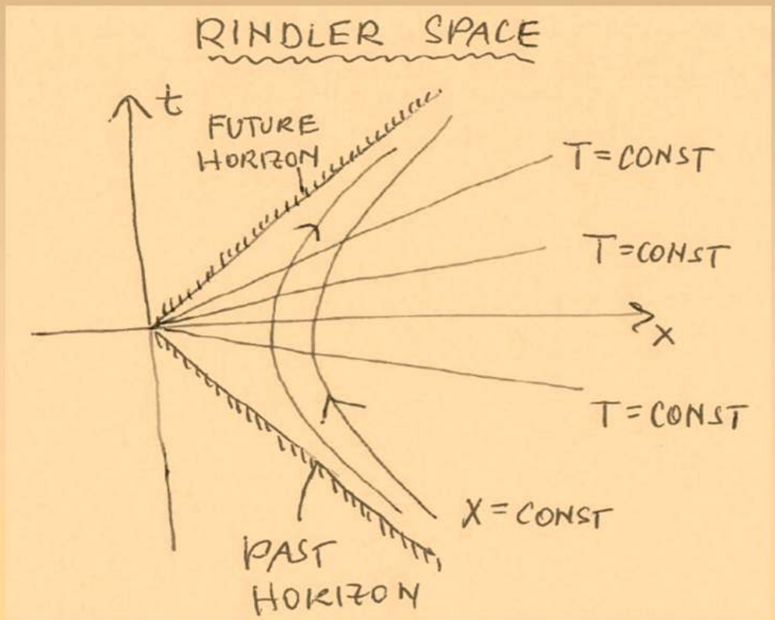


Regularity on the horizon yields

$$T = \frac{a}{2\pi}$$

… which is **Unruh's temperature**.

$$ds^2 = -dt^2 + dx^2 = -(1 + aX)^2 dT^2 + dX^2$$

Example 3: de Sitter horizon

- **de Sitter (dS) space** = maximally symmetric solution of EE with positive Lambda:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad \Lambda = \frac{3}{\ell^2}$$

- metric:

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \quad f = 1 - \frac{r^2}{\ell^2}$$

- Cosmological horizon:

$$f(r_c) = 0 \quad \Rightarrow \quad r_c = \ell$$

- repeating the Euclidean trick we get:

$$T = \frac{|f'(r_c)|}{4\pi} = \frac{1}{2\pi\ell}\,.$$

… which is **Gibbons-Hawking temperature** (note the absolute value!)

Summary

1) Black holes are **thermodynamic objects.** They can be assigned **Hawking's temperature** and **Bekenstein's entropy**. Obey the standard laws of TDs.

2) Euclidean magic predicts thermodynamics of BH, Rindler, or dS horizons. Namely:

   i. **Regularity** of the **Euclideanized manifold** fixes periodicity of the Euclidean time and yields **Hawking temperature** of the black hole.

   ii. G**ravitational partition function** yields free energy, which in WKB approximation recovers the **Bekenstein's area law** (& other conjugate quantities).

3) AF case: use **background subtraction** to calculate the action**,** ensemble not well defined!

Lecture 1: asymptotically flat BHs

- First law of black hole thermodynamics:

$$\delta M = T\delta S + \sum_i \Omega_i \delta J_i + \Phi \delta Q$$

- Smarr-Gibbs-Duhem relation:

$$\frac{d-3}{d-2}M = TS + \sum_i \Omega_i J_i + \frac{d-3}{d-2}\Phi Q$$

Basic properties:

- Thermodynamic ensemble not well defined (C<0)!
- TD behaviour interesting, yet not exactly analogous to everyday thermodynamics!
- Where is the standard PdV term? (BH chemistry)

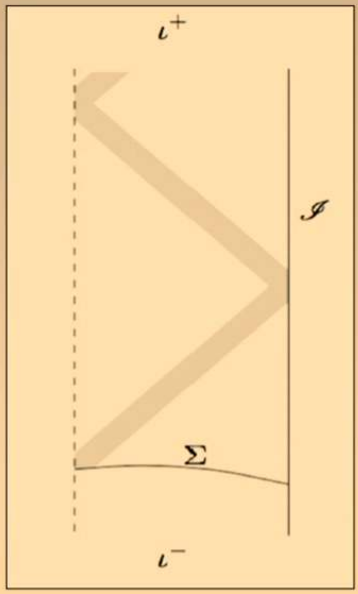
Global AdS4: a few basic facts

- **Anti de Sitter (AdS) space** = maximally symmetric solution of EE with negative Lambda:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \qquad \Lambda = -\frac{3}{\ell^2}$$

- metric:

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \qquad f = 1 + \frac{r^2}{\ell^2}$$



- Pull of the negative cosmological constant implies that AdS acts like a **confining box**

- There is a timelike conformal boundary (due to reflective BCs, **nonlinearities do not decay**)

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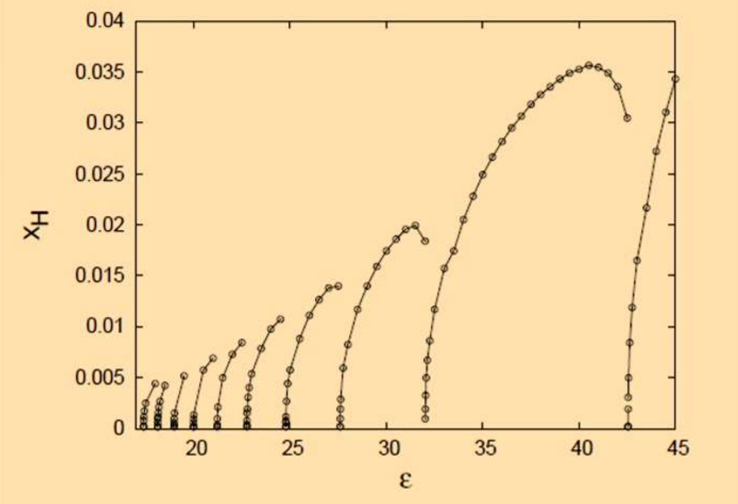
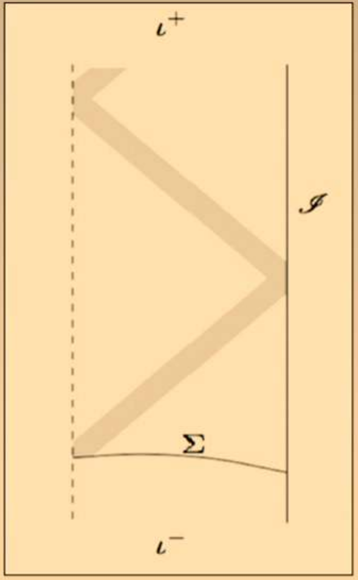


FIG. 1: Horizon radius vs amplitude for initial data (9). The number of reflections off the AdS boundary before collapse varies from zero to nine (from right to left).



Bizon and Rostworowski, *On weakly turbulent instability of anti-de Sitter space*, Phys. Rev. Lett. 107, 031102 (2011).

Schwarzschild-AdS black hole

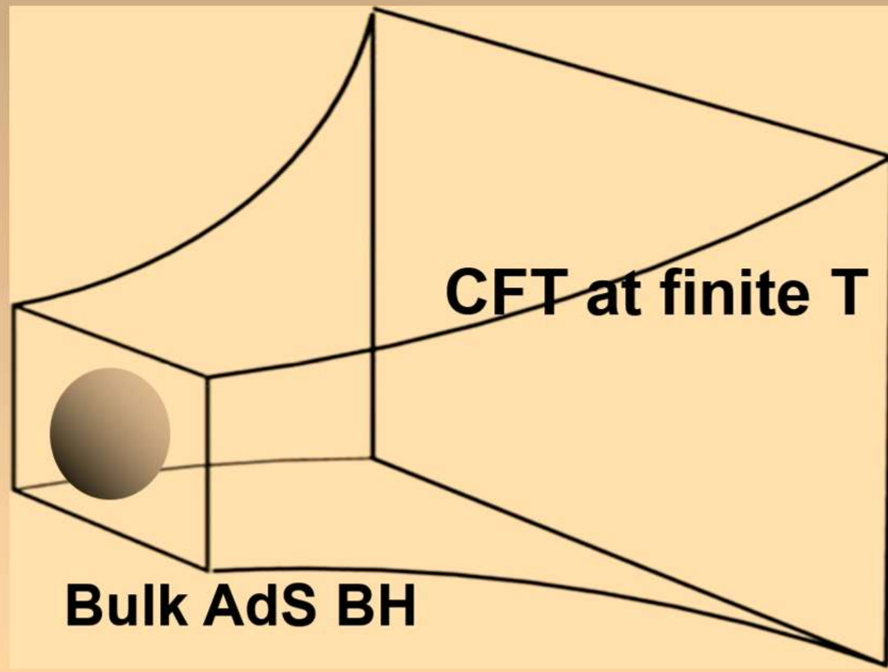
$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_k^2 \qquad f = k - \frac{2m}{r} + \frac{r^2}{\ell^2}$$

- It is an **Einstein space** (vacuum with Lambda solution)

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad \Lambda = -\frac{3}{\ell^2}$$

- Choices of $k = 0, \pm 1$ correspond to various **horizon topologies** (with $d\Omega_k^2$ the corresponding metric).
- One can add, charge, or rotation – having charge-AdS, or Kerr-AdS black holes in 4 and higher dimensions.
- These have rather interesting properties.

AdS/CFT duality (at finite temperature)



Bulk:
Hawking temp T
BH entropy S
BH mass M

⟺

Boundary:
CFT at finite T
CFT entropy S
CFT energy E

Gravitational action in AdS

$$S_E = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}\left[R + \frac{6}{\ell^2}\right] + \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{-h}\mathcal{K}$$
$$- \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{-h}\left[\frac{2}{\ell} + \frac{\ell}{2}\mathcal{R}(h)\right] ,$$

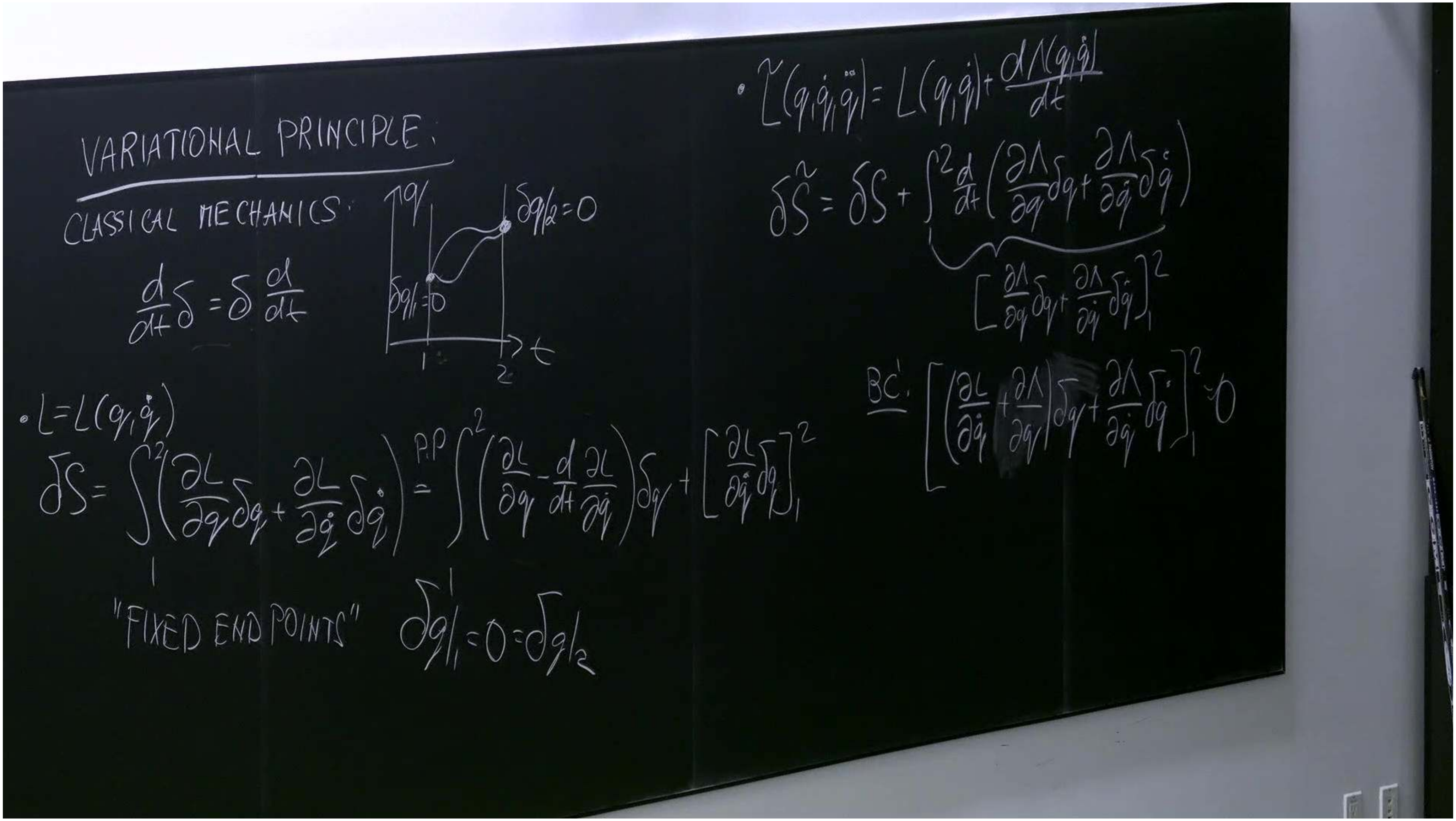
- The 2nd line are the covariant **AdS counterterms** (c.f. `vague' background subtraction in AF case)
- Variation yields:

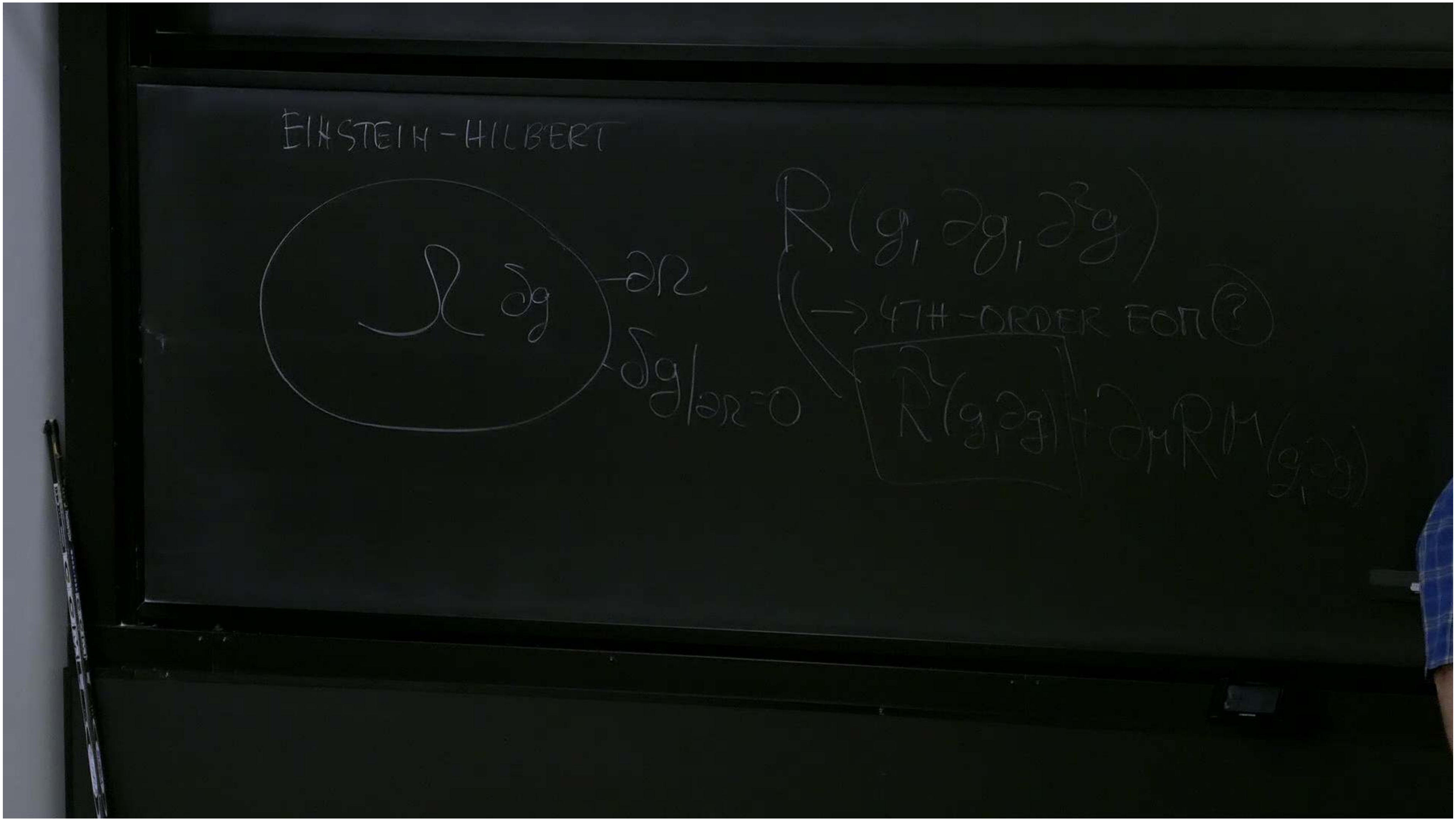
$$\delta S_E = -\frac{1}{2}\int_{\partial\Omega} d^3x\sqrt{-h}\tau_{ab}\delta h^{ab}$$ + bulk EOM

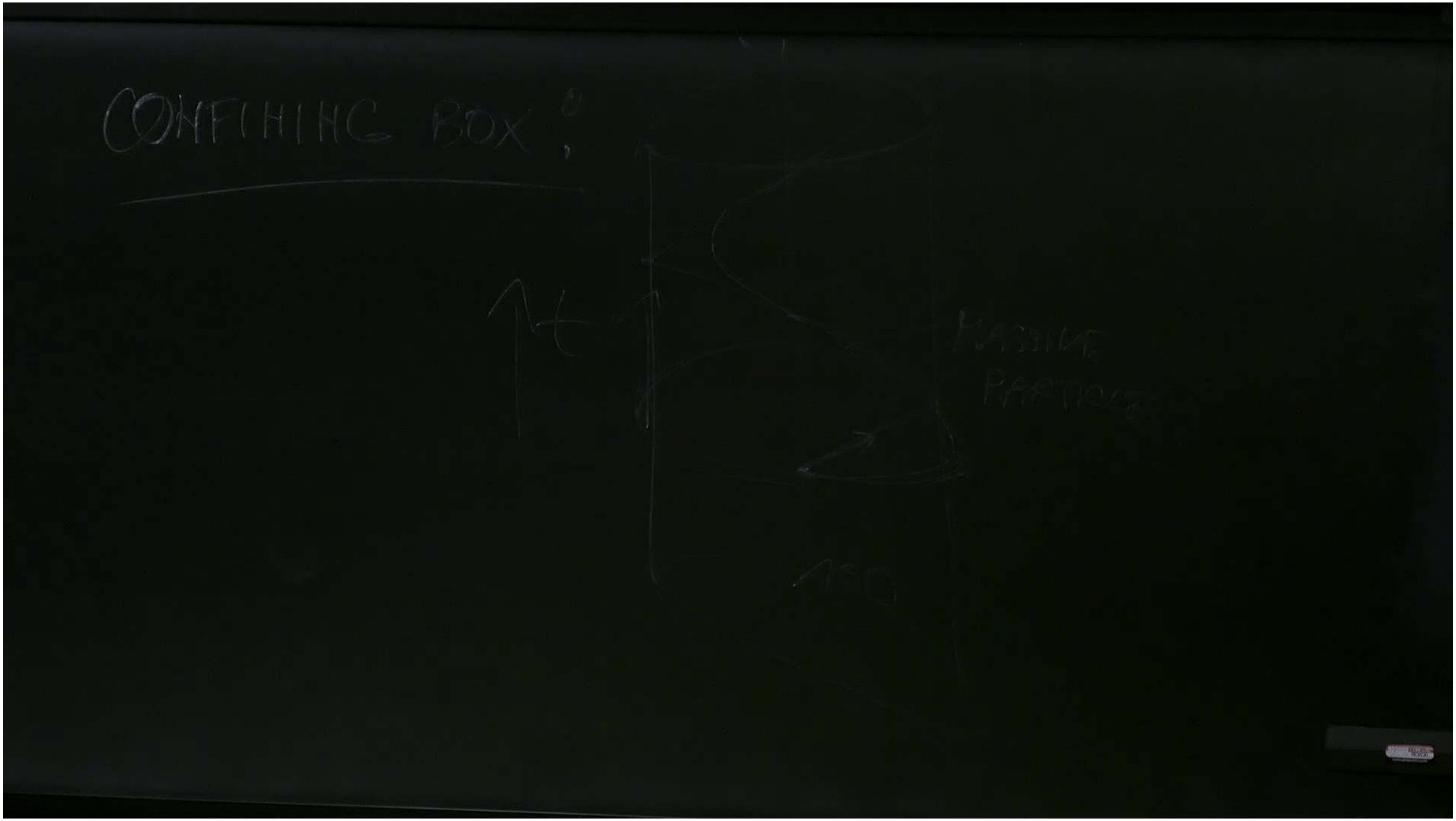
here

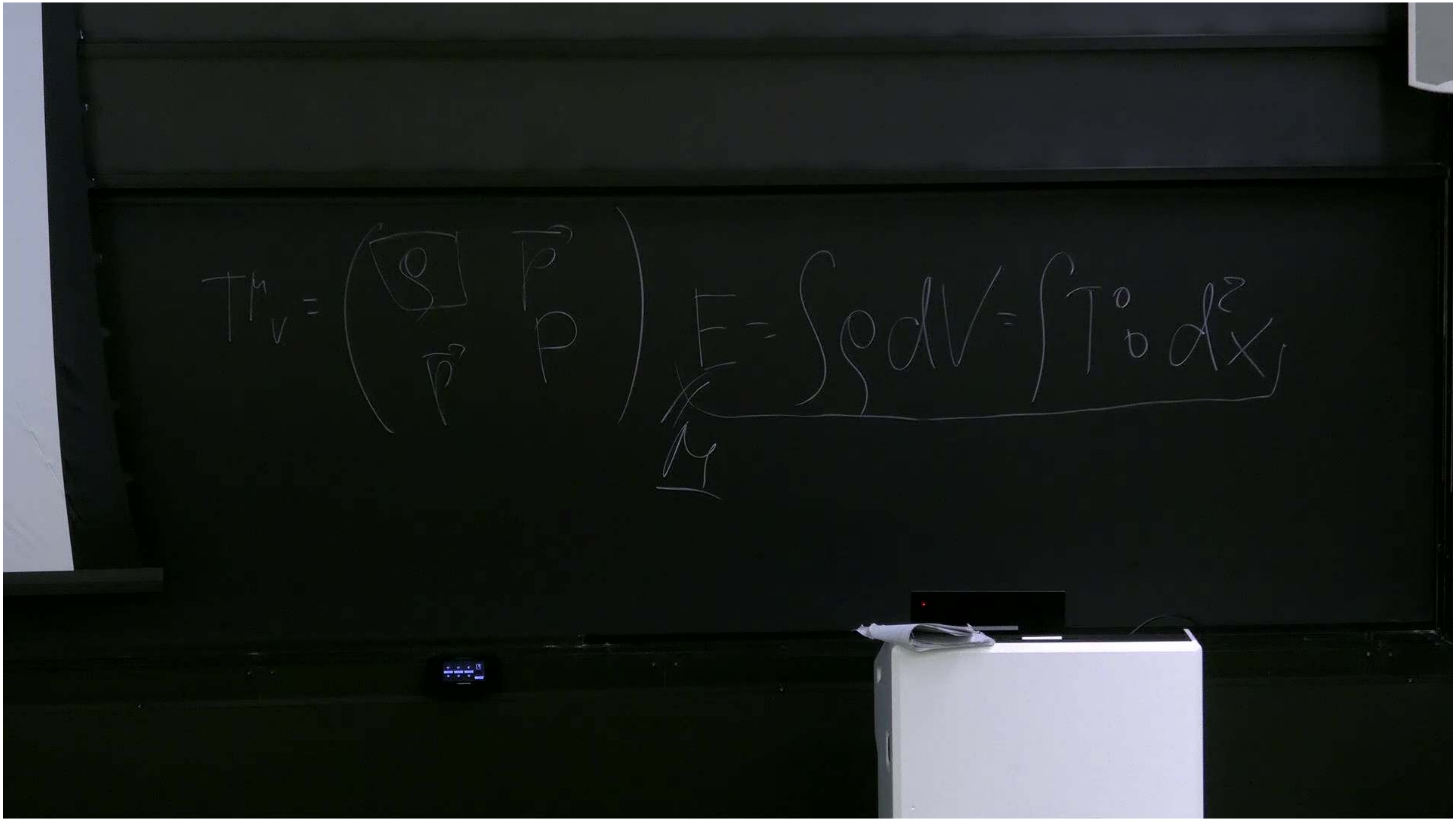
$$8\pi\tau_{ab} = \mathcal{K}h_{ab} - \mathcal{K}_{ab} + \ell G_{ab}(h) - \frac{2}{\ell}h_{ab}$$

is up to trivial (infinite) scaling the **holographic stress tensor**


VARIATIONAL PRINCIPLE:
CLASSICAL MECHANICS:
$\frac{d}{dt}\delta = \delta\frac{d}{dt}$
q
$\delta q|_2 = 0$
$\delta q|_1 = 0$
t
1
2
• $L = L(q, \dot{q})$
$\delta S = \int_1^2 \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right) \overset{P.P}{=} \int_1^2 \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q + \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_1^2$
"FIXED END POINTS" $\delta q|_1 = 0 = \delta q|_2$
• $\tilde{L}(q, \dot{q}, \ddot{q}) = L(q, \dot{q}) + \frac{d\Lambda(q, \dot{q})}{dt}$
$\delta\tilde{S} = \delta S + \int_1^2 \frac{d}{dt}\left(\frac{\partial\Lambda}{\partial q}\delta q + \frac{\partial\Lambda}{\partial\dot{q}}\delta\dot{q}\right)$
$\left[\frac{\partial\Lambda}{\partial q}\delta q + \frac{\partial\Lambda}{\partial\dot{q}}\delta\dot{q}\right]_1^2$
BC: $\left[\left(\frac{\partial L}{\partial\dot{q}} + \frac{\partial\Lambda}{\partial q}\right)\delta q + \frac{\partial\Lambda}{\partial\dot{q}}\delta\dot{q}\right]_1^2 = 0$


EINSTEIN - HILBERT
R(g, ∂g, ∂²g)
→ 4TH-ORDER EOM ?


CONFINING BOX:
MASSIVE
PARTICLE


$T^{\mu}{}_{\nu} = \begin{pmatrix} \rho & \vec{p} \\ \vec{p} & P \end{pmatrix}$
$E = \int \rho \, dV = \int T^0{}_0 \, d^3x$
M