Title: Lecture - Quantum Matter, PHYS 777

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Subject: Condensed Matter

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$\mathbb{Z}_2$ gauge theory, Majorana & Ising

qubit. $\tau_i^{x,y,z}$

c_i

Impose gauge symmetry

$$c_i \to \underline{s_i}\, c_i, \quad \tau_i^x \to s_i s_{i+1} \tau_i^x$$

$s_i = \pm 1$

gauge connection

$$\sim e^{i\int_i^{i+1} \vec{A}\cdot d\vec{x}}$$

Hamiltonian $\qquad (\nabla \times A)^2$

$$c_i = \gamma_{2i-1} + i\gamma_{2i}$$

$$H = -J\sum_i i\gamma_{2i}\gamma_{2i+1}\tau_i^x - g\sum_i i\gamma_{2i-1}\gamma_{2i}$$

We can pick a "smart" gauge

$$c_1 \to c_1, \quad c_{i>1} \to \Big(\prod_{j<i}\tau^x\Big) c_{i>1}$$

Hamiltonian. $(\nabla \times A)^2$

$\prod_{\square} \tau^x_{\cdot} \sim$

$c_i = \gamma_{2i-1} + i\gamma_{2i}$

$$H = -J\sum_i i\gamma_{2i}\gamma_{2i+1}\tau^x_i - g\sum i\gamma_{2i-1}\gamma_{2i}$$

We can pick a "smart" gauge

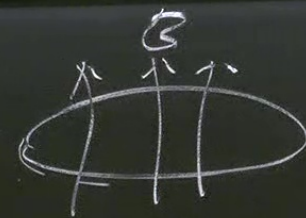
$$c_1 \to c_1, \quad c_{i>1} \to \Big(\prod_{j<i} \tau^x\Big) c_{i>1}$$

$$\Rightarrow H = -J\sum_{1\le i\le L-1} i\gamma_{2i}\gamma_{2i+1} - g\sum_i i\gamma_{2i-1}\gamma_{2i} - J\, i\gamma_{2L}\gamma_1\tau^x_L.$$

made of τ^x_i

In 1d $\mathbb{Z}_2$ gauge field, only gauge-inv. d.o.f. is the Wilson loop

$$\prod_{\square} \tau^x_i \sim e^{i\oint \vec{A}\cdot d\vec{l}}$$



gauge flux

$\ldots_{2i} - \gamma_{2i}$

γ_{2i} $\quad -J\, i\gamma_{2L}\gamma_1 \tau^x_L$.

Constraint from gauge invariance:

Total gauge charge fixed in $\mathcal{H}$.

Two choices:

① $\prod_{i=1}^{L} i\gamma_{2i-1}\gamma_{2i} = 1$

② $\underbrace{\left(\prod_{i=1}^{L} i\gamma_{2i-1}\gamma_{2i}\right)}_{(-1)^F}\left(\prod_{i=1}^{L} \tau_x\right) = 1$

JW transformed: Ising model.

$(-1)^F = \tau^x_L = \prod_i X_i$

made of τ^x_L

gauge-inv. d.o.f. is the Wilson loop

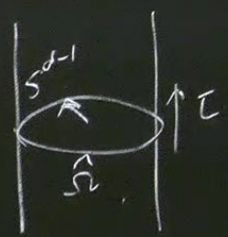
"Ising $\longleftrightarrow$ Majorana $/\mathbb{Z}_2$"

Now: $Z_i \longleftrightarrow \underbrace{\gamma_{2i-1}\,\tau^z_{i-1}}$: flux-changing operator.

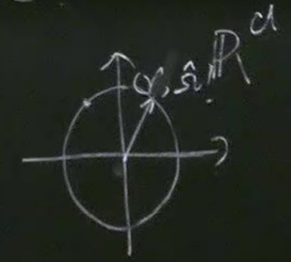
a.k.a. "instanton".

How to compute Δ_Z

...ory : State-operator correspondence.



$$r = e^{\tau}$$



Cylinder: $S^{d-1} \times \mathbb{R}$.

$\tau = -\infty \Longleftrightarrow r = 0$.

Eigenstate on $S^{d-1} \longleftrightarrow$ Operator $\mathcal{O}(r=0)$

$|\mathcal{O}\rangle$

G.S. $|\Omega\rangle \longleftrightarrow 1$.

Time-translation $\tau \to \tau + \alpha \longleftrightarrow r \to r \cdot e^{\alpha}$: scaling

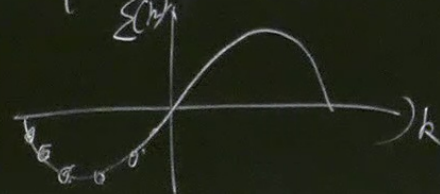
$$\boxed{\text{Energy } E \longleftrightarrow \Delta_{\mathcal{O}}}$$

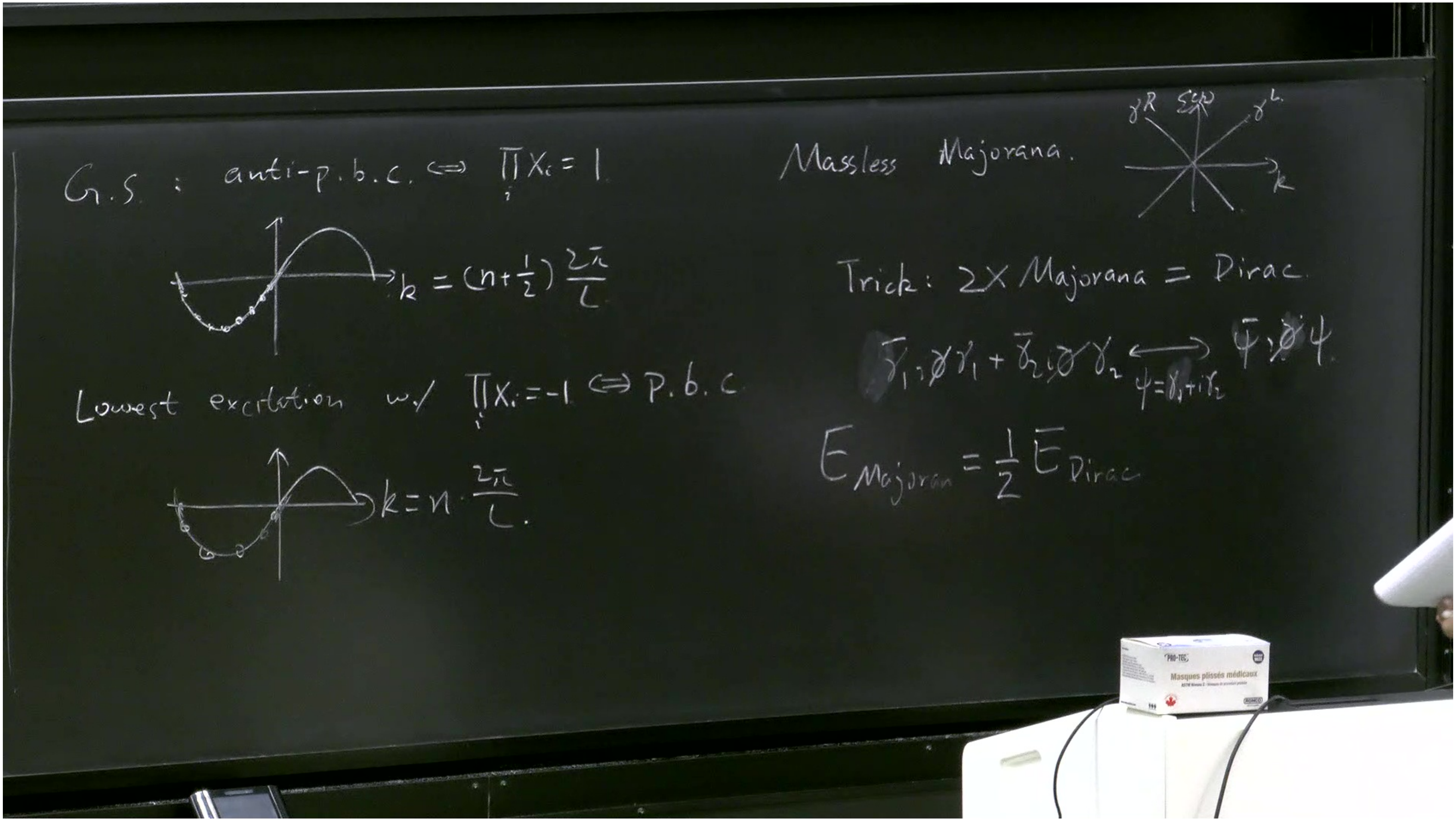
"Ising $\longleftrightarrow$ Majorana $/\mathbb{Z}_2$"

Now: $\mathbb{Z}_2 \longleftrightarrow \underbrace{\gamma_{2i-1}\,\tau^z_{i-1}}$: flux-changing operator a.k.a. "instanton"

How to compute Δ_Z?

Just compute E of some states.




G.S : anti-p.b.c. ⟺ ∏ X_i = 1
k = (n+1/2) 2π/L
Lowest excitation w/ ∏ X_i = -1 ⟺ p.b.c
k = n · 2π/L.
Massless Majorana.
γR ε(k) γL
k
Trick: 2X Majorana = Dirac
γ̄1 i∂̸ γ1 + γ̄2 i∂̸ γ2 ⟷ ψ̄ i∂̸ ψ
ψ = γ1 + iγ2
E_Majoran = 1/2 E_Dirac
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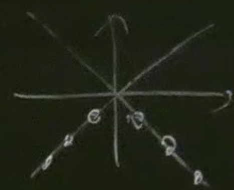
$$E_{\text{Majorana}} = \frac{1}{2} E_{\text{Dirac}}$$



$$= \frac{1}{2} \sum_{k:\varepsilon(k)<0} \varepsilon(k) \quad , \quad \varepsilon(k) = -|k|$$

$$= -\sum_{k>0} |k| \qquad k = \begin{matrix} n+\frac{1}{2} \\ n \end{matrix} \cdot \cancel{\frac{2\pi}{L}}$$

$$\text{Anti-pbc} = -\frac{1}{2}(1+3+5+7\ldots)$$

$$\text{pbc} = -(1+2+3+4\ldots)$$

Deep facts:

$$1+2+3+4\ldots = -\frac{1}{12}$$

$$1+3+5+7\ldots = \frac{1}{12}$$

$$\Rightarrow \Delta_{\mathbb{Z}} = E_{pbc} - E_{Apbc} = \frac{1}{12} + \frac{1}{24} = \frac{1}{8}.$$

Deep facts:

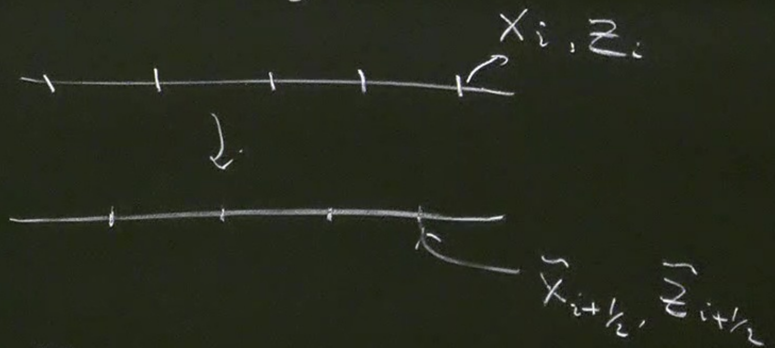
$$1+2+3+4\ldots = -\frac{1}{12}$$

$$1+3+5+7\ldots = \frac{1}{12}$$

$$\Rightarrow \Delta_Z = E_{pbc} - E_{Apbc} = \frac{1}{12} + \frac{1}{24} = \frac{1}{8}.$$

$$\langle Z_i Z_j \rangle \sim \frac{1}{|i-j|^{1/4}}.$$

Another duality for Ising : Kramers-Wannier (KW)



$\tilde{Z}$: domain-wall creation.

Z : charge creation.

$$\tilde{X}_{i+1/2} = Z_i \, Z_{i+1}$$

$$\tilde{Z}_{i+1/2} = \prod_{1\le j\le i} X_j$$

Consider a ferromagnet

$$|\cdots\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots\rangle$$

$$\tilde{Z}_{i+\frac{1}{2}}|\ldots\uparrow\uparrow\ldots\uparrow\ldots\rangle = |\downarrow\downarrow\ldots\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle,$$

$\mathbb{Z}_2$ gauge field, only gauge-inv. d.o.f. is the Wilson loop

$(-1)^F$

w)

ation.

Side note:

JW transform: $\gamma_i = \check{Z}_{i-1/2} Z_i$. Domain wall + charge $\Rightarrow$ fermion.

("Flux-attachment")

$$-J \sum Z_i Z_{i+1} - g \sum X_i \Longleftrightarrow -g \sum \check{Z}_{i-1/2} \check{Z}_{i+1/2} - J \sum \check{X}_{i+1/2}$$

KW: $g \leftrightarrow J$.

Para $\leftrightarrow$ Ferro

Ferro $\leftrightarrow$ Para.

Symmetric phase = Defect (domain wall) condensate.

Kramers-Wannier (KW)

$\hat{Z}$: domain-wall creation.

Z : charge creation.

$\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow$

ferromagnet $\pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$

$\uparrow\uparrow\uparrow \cdots \rangle$

$\uparrow\uparrow \cdots \uparrow \cdots \rangle = |\downarrow\downarrow \cdots \downarrow \uparrow\uparrow\uparrow\uparrow \rangle$

Side note:

JW transform: $\gamma_i = \hat{Z}_{i-1/2} Z_i$.

Domain wall

$\Rightarrow$

("Flux-attac

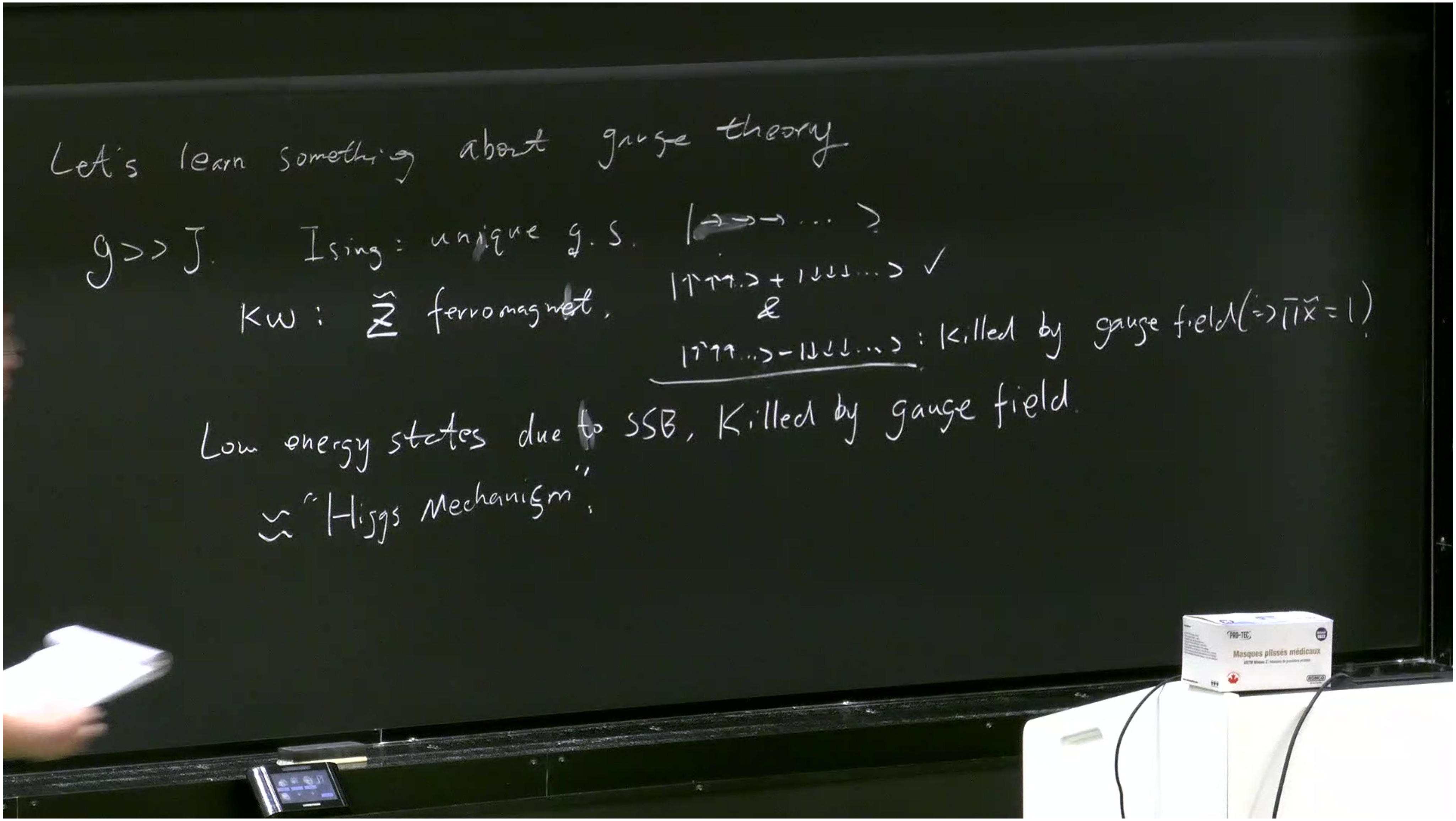
$$-J\sum Z_i Z_{i+1} - g\sum X_i \Longleftrightarrow -g\sum \hat{Z}_{i-1/2}\hat{Z}_{i+1/2} -$$

KW: $g \leftrightarrow J$

Para $\leftrightarrow$ Ferro

Ferro $\leftrightarrow$ Para.

Symmetric phase = Def


Let's learn something about gauge theory
$g >> J$. Ising: unique g.s. $|\rightarrow\rightarrow\rightarrow\ldots\rangle$
KW: $\hat{\tilde{Z}}$ ferromagnet. $|\uparrow\uparrow\uparrow\ldots\rangle + |\downarrow\downarrow\downarrow\ldots\rangle$ ✓
&
$|\uparrow\uparrow\uparrow\ldots\rangle - |\downarrow\downarrow\downarrow\ldots\rangle$: Killed by gauge field ($\Rightarrow \prod \hat{\tilde{X}} = 1$)
Low energy states due to SSB, Killed by gauge field.
$\simeq$ "Higgs Mechanism".
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$g \ll J$ Ising: G.S: $|\uparrow\uparrow\uparrow\ldots\rangle \pm |\downarrow\downarrow\downarrow\ldots\rangle$

KW: $\check{Z}$ paramagnet, at low energy $\Rightarrow$ pure $\mathbb{Z}_2$ gauge theory

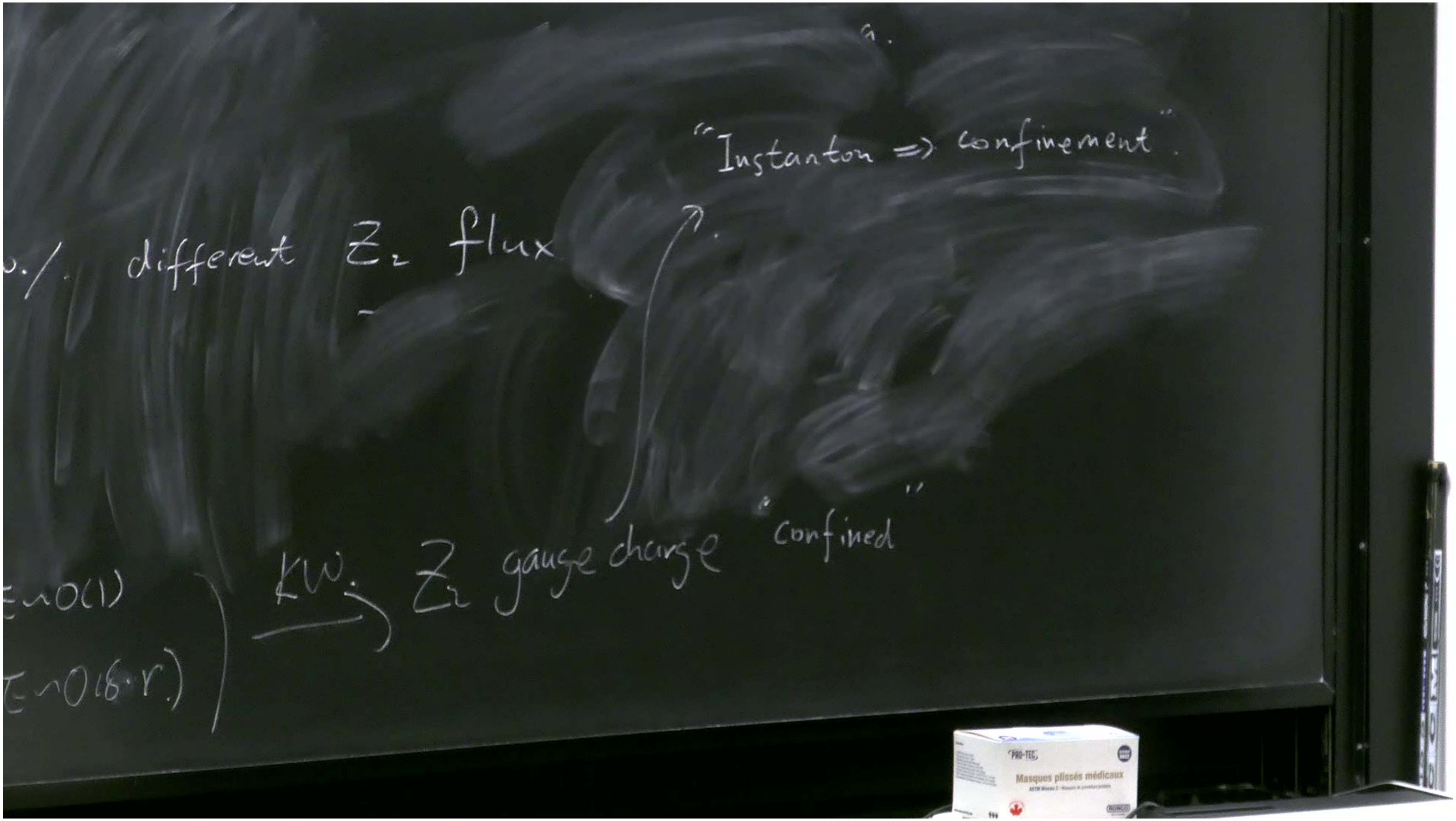
$\Rightarrow$ Pure $\mathbb{Z}_2$ gauge theory has 2-fold degeneracy w./ dif

As long as Z perturbation is not allowed.

But if I turn on $H_0 - \delta \sum_i Z_i$

$\Rightarrow |\uparrow\uparrow\uparrow\uparrow\rangle$ unique g.s $(\Delta E_{\downarrow\downarrow\downarrow} \propto \delta \cdot L)$

Domain wall $|\uparrow\uparrow\uparrow \underbrace{\downarrow\downarrow\downarrow\downarrow\downarrow}_{r} \uparrow\uparrow\uparrow\rangle$ Energy cost $\begin{cases} \delta = 0: \Delta E \sim O(1) \\ \delta \neq 0: \Delta E \sim O(\delta \cdot r) \end{cases}$


"Instanton ⇒ confinement"
different $\mathbb{Z}_2$ flux
KW: $\mathbb{Z}_2$ gauge charge "confined"
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