Title: Lecture - Quantum Matter, PHYS 777

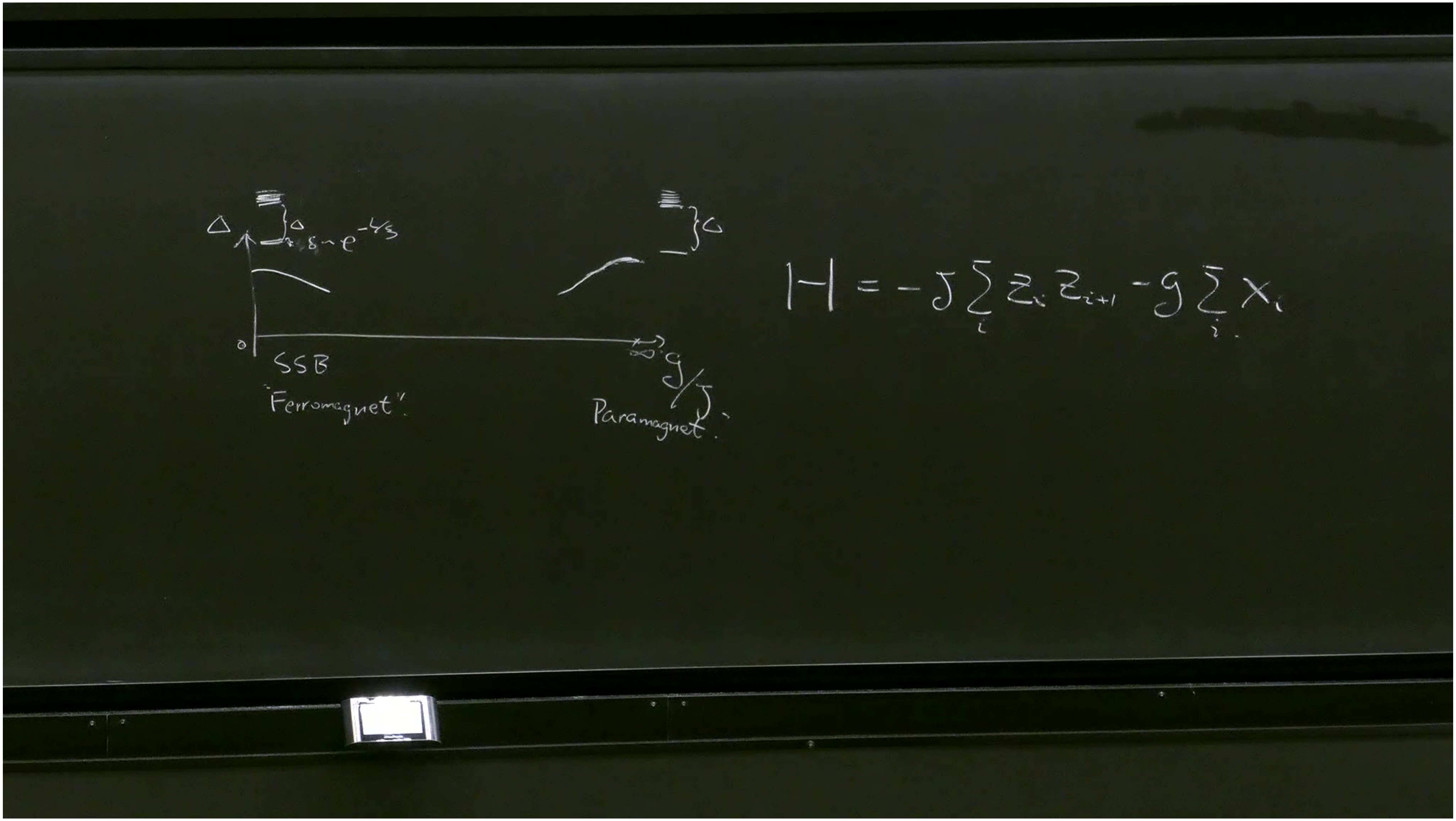
Speakers: Chong Wang

Collection/Series: Quantum Matter (Elective), PHYS 777, March 31 - May 2, 2025

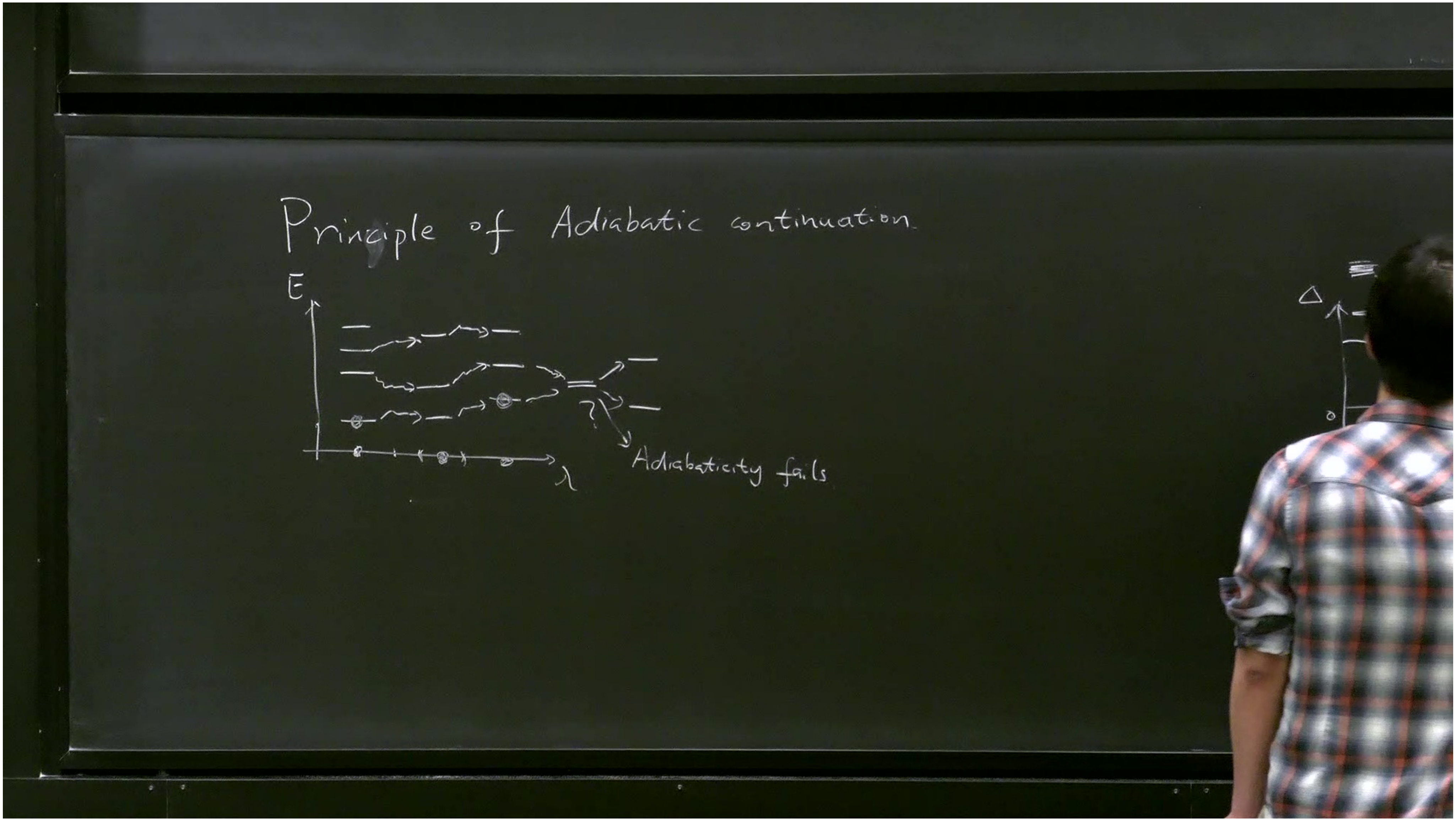
Subject: Condensed Matter

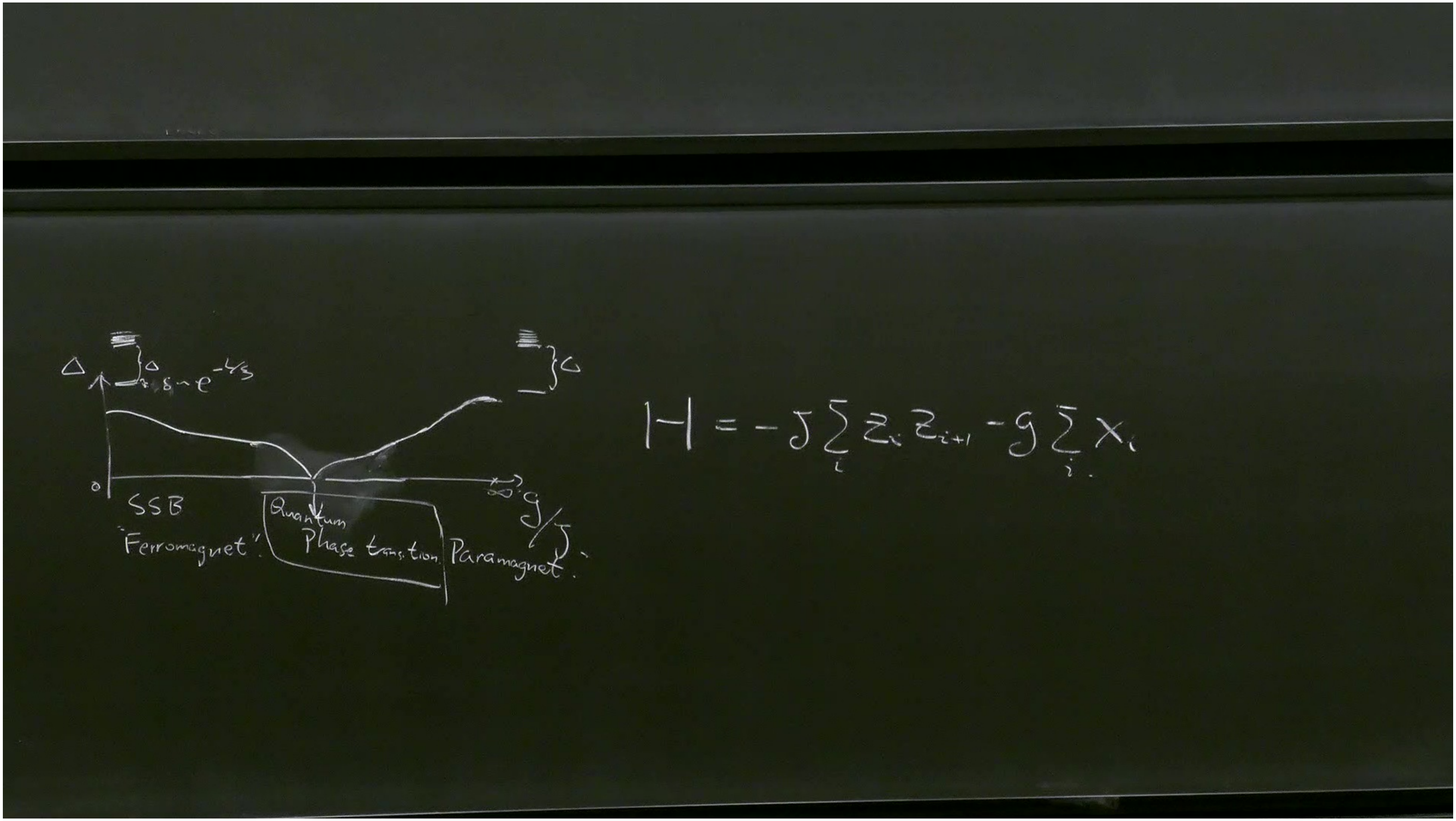
Date: April 03, 2025 - 10:15 AM

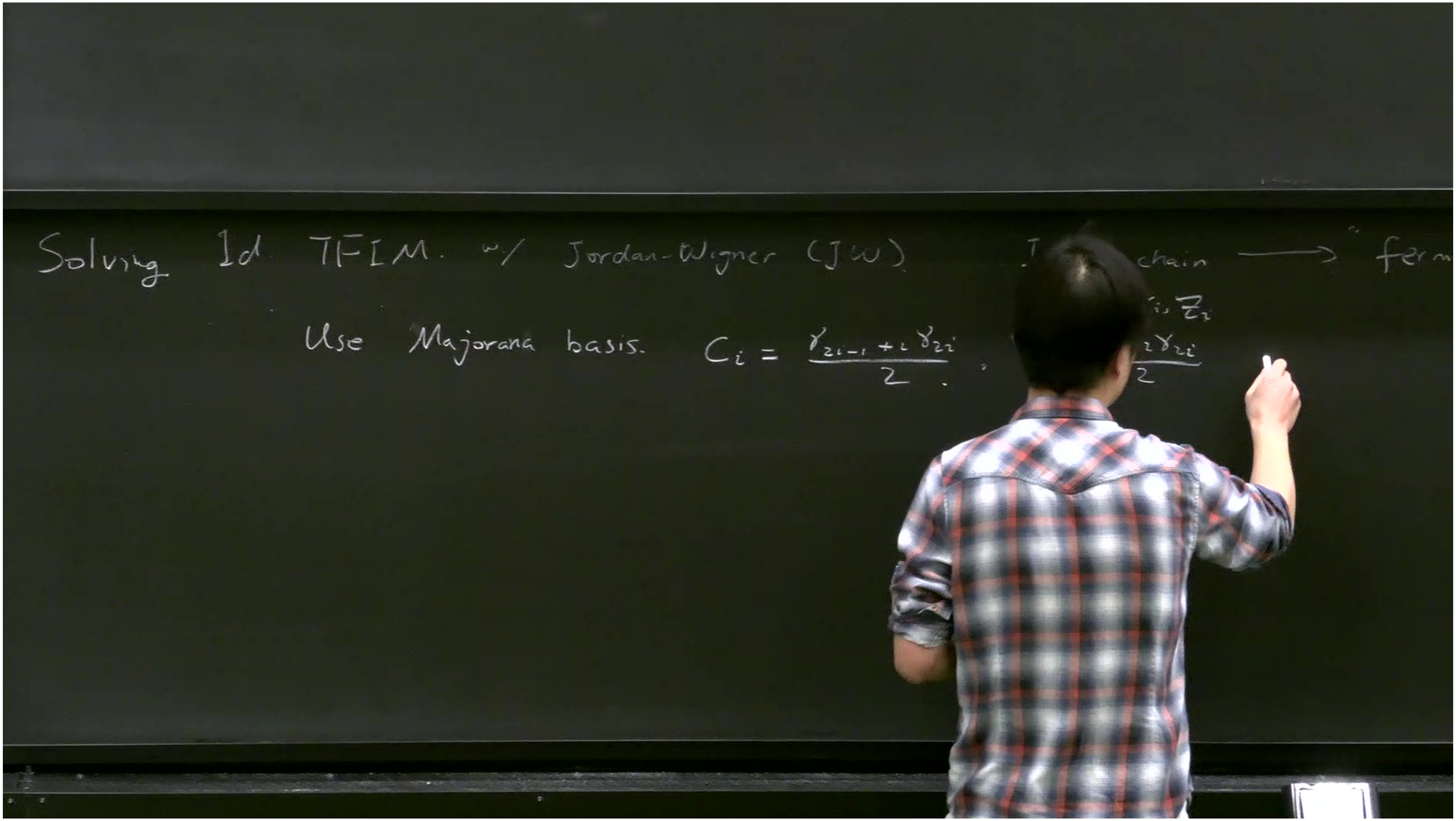
URL: https://pirsa.org/25040002

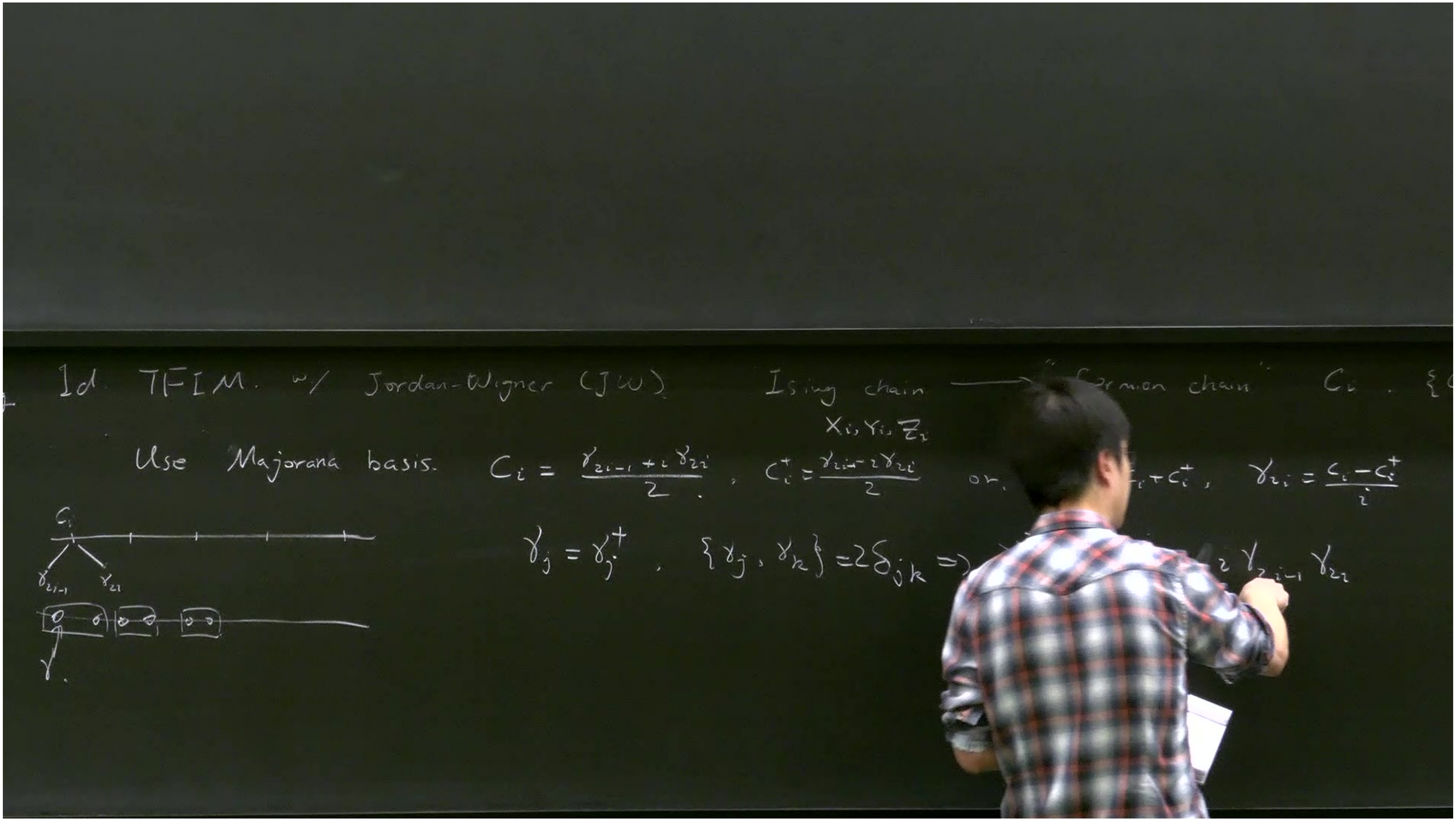


SSB
"Ferromagnet"
Paramagnet
g/J
H = -J Σ Z_i Z_{i+1} - g Σ X_i


Principle of Adiabatic continuation
E
λ
?
Adiabaticity fails
Δ
0


$H = -J \sum_i Z_i Z_{i+1} - g \sum_i X_i$
$\delta \sim e^{-L/\xi}$
SSB
"Ferromagnet"
Quantum
Phase transition
Paramagnet
g/J


Solving 1d TFIM. w/ Jordan-Wigner (JW)
chain ⟶
Use Majorana basis. $C_i = \frac{\gamma_{2i-1} + i\gamma_{2i}}{2}$


1d TFIM. w/ Jordan-Wigner (JW)
Ising chain
"Fermion chain"
X_i, Y_i, Z_i
Use Majorana basis.
c_i = (γ_{2i-1} + i γ_{2i})/2 , c_i^† = (γ_{2i-1} − i γ_{2i})/2
or
+ c_i^† , γ_{2i} = (c_i − c_i^†)/i
γ_j = γ_j^† . {γ_j , γ_k} = 2δ_{jk} =>
i γ_{2i-1} γ_{2i}
c_i
γ_{2i-1}
γ_{2i}
γ

$\longrightarrow$ "fermion chain" c_i , $\{c_i, c_j^\dagger\} = \delta_{ij}$

...or. $\gamma_{2i-1} = c_i + c_i^\dagger$, $\gamma_{2i} = \dfrac{c_i - c_i^\dagger}{i}$

$\gamma_j^2 = 1$

$\downarrow$

Majorana fermion

$i\gamma_{2i-1}\gamma_{2i} = 2c_i^\dagger c_i - 1 = (-1)^{n_i}$, $\quad c_i^\dagger c_i = n_i$

JW transform.

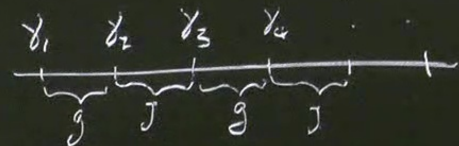
$$X_i = i\gamma_{2i-1}\gamma_{2i}$$

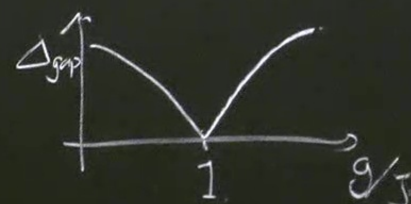
$$Z_i = \Big[\prod_{1\le j<i} (i\gamma_{2j-1}\gamma_{2j})\Big]\cdot \gamma_{2i-1}$$

Check: $X_i^2 = Z_i^2 = 1$

$$X_i Z_j = (-1)^{\delta_{ij}} Z_j X_i$$

$$H \text{ "=" } -J\sum_i i\gamma_{2i}\gamma_{2i-1} - g\sum_i i\gamma_{2i-1}\gamma_{2i}$$





free fermion model!

Take $g=J$. $H = -iJ\sum_{j=1}^{2L}\gamma_j\gamma$

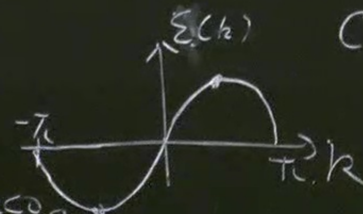
Try $\gamma_k := \frac{1}{\sqrt{4L}}\sum_j e^{-ikj}\gamma_j$, k

Can check: if $k=0,\pi$, $\sqrt{2}\gamma_k$ is

for other k, $\gamma_k^\dagger = \gamma_{-k}$,

$\Rightarrow d_k = \gamma_k$.

$$\Rightarrow H = 2J\sum_k \sin k\,\gamma_{-k}\gamma_k$$



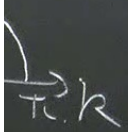
S. $n_k = d_k^\dagger d_k = \gamma_{-k}\gamma_k = \begin{cases} 1, & \pi<k<0 \\ 0, & 0<k<\pi \end{cases}$ Same condition.

free fermion model!

Take $g = J$. $H = -iJ\sum_{j=1}^{2L} \gamma_j \gamma_{j+1}$

Try $\gamma_k := \frac{1}{\sqrt{4L}} \sum_j e^{-ikj} \gamma_j$, $k = \frac{2\pi}{2L} \cdot n$, $n = 0, 1 \ldots 2L-1$.

Can check: if $k = 0, \pi$, $\sqrt{2}\gamma_k$ is a Majorana.



for other k. $\gamma_k^\dagger = \gamma_{-k}$, $\{\gamma_k, \gamma_{-k'}\} = \delta_{k,k'}$.

$\Rightarrow d_k = \gamma_k$, $d_k^\dagger = \gamma_{-k}$

ition.

Low energy ($E << J$) : Expand near $k \simeq 0, \pi$ $\Rightarrow H \simeq 2J \int_0^L dx$

Take a "cutoff" momentum Λ, $\frac{2\pi}{L} << \Lambda << \pi$

For $\delta k << \Lambda$, define

$$\gamma^L_{\delta k} := \gamma_{\delta k}, \quad \gamma^R_{\delta k} = \gamma_{\delta k + \pi}$$

$$\Rightarrow H \simeq \sum_{|k|<\Lambda} 2J \left[k\, \gamma^L_{-k} \gamma_k - k\, \gamma^R_{-k} \gamma^R_k \right]$$

Define "continuum" fermion $\gamma^{L,R}(x) := \sqrt{\frac{\pi}{L\Lambda}} \sum_k e^{ikx} \gamma^{L,R}_k$

$\downarrow$

Hermitian.

$$\Rightarrow H \simeq 2J \int_0^L dx \; \gamma^L(-i\partial_x)\gamma^L - \gamma^R(-i\partial_x)\gamma^R .$$

$\begin{pmatrix}\gamma^L \\ \gamma^R\end{pmatrix}$: Relativistic, Massless Majorana fermion in (1+1)d.

Lorentz inv. emerges at low energy.

In Lagrangian form:

$$S = \int dt\, dx \left[\gamma^L (i\partial_t - i\partial_x)\gamma^L + \gamma^R (i\partial_t + i\partial_x)\gamma^R \right].$$

For $\delta k \ll \Lambda$, define

$$\gamma^L_{\delta k} := \gamma_{\delta k}, \quad \gamma^R_{\delta k} = \gamma_{\delta k+\pi}$$

$$\Rightarrow H \simeq \sum_{|k|<\Lambda} 2J \left[k\, \gamma^L_{-k} \gamma_k - k\, \gamma^R_{-k} \gamma^R_k \right]$$

Define "continuum" fermion $\underset{\downarrow \text{ Hermitian.}}{\gamma^{L,R}}(x) := \sqrt{\frac{\pi}{L\Lambda}} \sum_k e^{ikx} \gamma^{L,R}_k$

(γ^R): $\underbrace{\text{Relativistic}}$, Massless Majorana fermion in (1+1)d.

Lorentz inv. emerges at low energy.

In Lagrangian form:

$$S = \int dt\, dx \left[\gamma^L \underbrace{(i\partial_t - i\partial_x)}\gamma^L + \gamma^R \underbrace{(i\partial_t + i\partial_x)}\gamma^R \right] \quad (*)$$

Relevant op.

only one: $i\gamma_L \gamma_R$: mass. $\Delta_m = 1 < 2 \Leftrightarrow$

Scaling invariant: $x \to bx$, $t \to bt$, $\gamma^{L,R} \to b^{-1/2} \gamma^{L,R}$ $\Rightarrow$ Renormalization group (RG) fixed point.

Recall in RG: scaling opertor $\mathcal{O} \to b^{-\Delta_\mathcal{O}} \mathcal{O}$. $\Delta_\mathcal{O}$: scaling dim.

As a perturbation $S + \lambda \int dt\, dx\, \mathcal{O}$, $\lambda \to b^{2-\Delta_\mathcal{O}} \lambda$.

$\Delta_\mathcal{O} < 2$: relevant

$\Delta_\mathcal{O} > 2$: irrelevant

$\Delta_\mathcal{O} = 2$: marginal.

$\Delta_\mathcal{O}$ also controlls many other things,

e.g. $\langle \mathcal{O}^\dagger(x)\, \mathcal{O}(y) \rangle \sim \frac{1}{|x-y|^{2\Delta_\mathcal{O}}}$

op.

only one $i\gamma_L\gamma_R$: mass. $\Delta_m = 1. < 2$ $\Leftrightarrow$ $Z_i Z_{i+1} - X_i$

$\Rightarrow$ Renormalization group (RG) fixed point.

...ing dim.

$b^{2-\Delta_O}\lambda$.

$\Delta_O < 2$: relevant

$\Delta_O > 2$: irrelevant

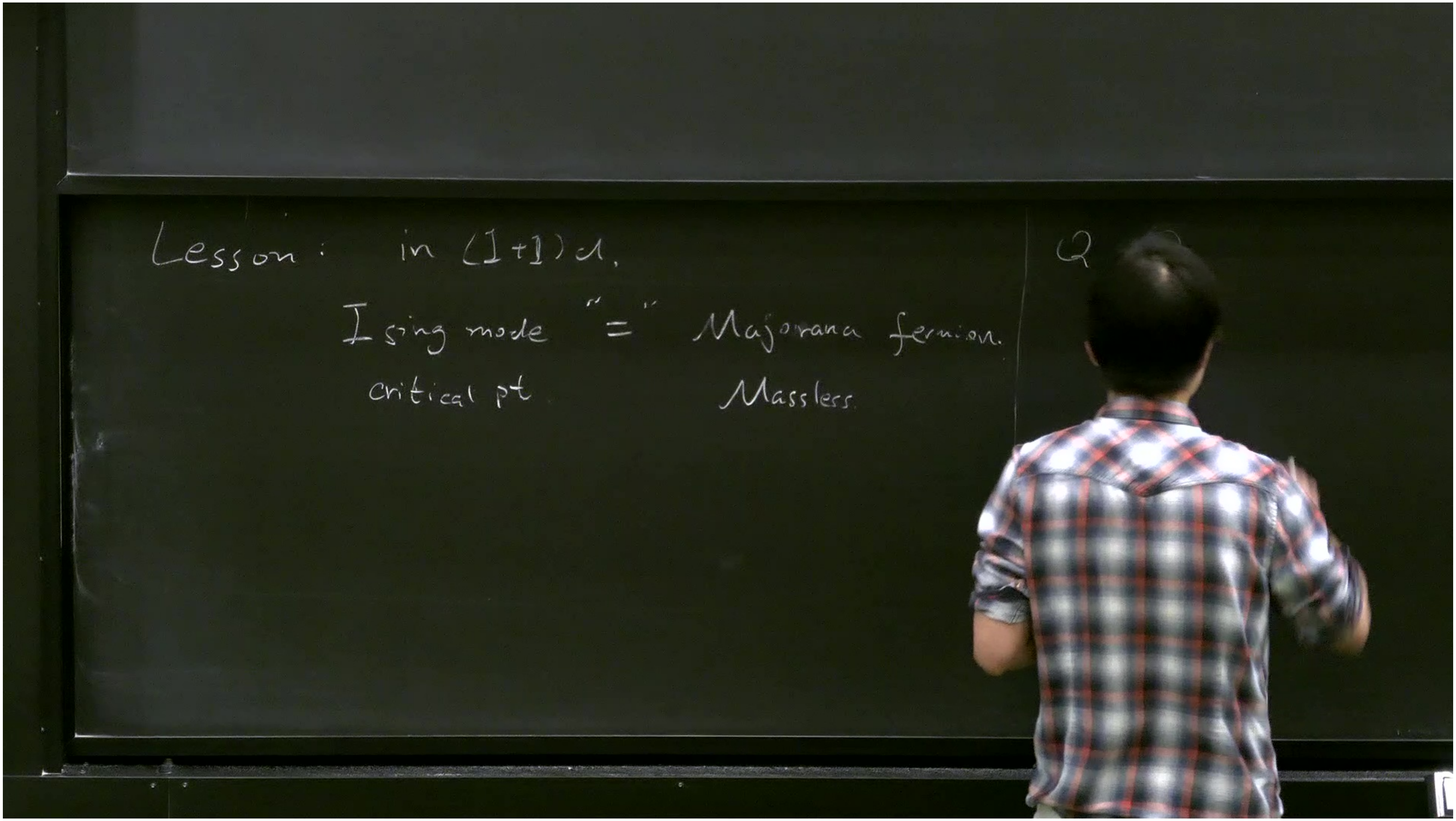
$\Delta_O = 2$: marginal.

When $m > 0$, energy gap $\Delta(m) > 0$.

$$\left.\begin{array}{l}\text{Under scaling: } m \to b^{2-\Delta_m} m = b\, m \\ \Delta \to b\Delta\end{array}\right\}$$

$t \to b t$, $\varepsilon \to b^{-1} t$

$\Rightarrow \Delta \propto m^{\#=1}$


Lesson: in (1+1)d.
Ising model "=" Majorana fermion.
critical pt
Massless.

"Ferromagnet" | Quantum Phase transition | Paramagnet. g/J

Q: ① How could a spin model ("bosonic" $[O_i, O_j]=0$) be equivalent to a fermion model?

② What is Z_i operator in fermion mode? How to compute Δ_Z?

Back to JW transform

$$\gamma_{2i-1} = \Big(\prod_{1\le j<i} X_j\Big) Z_i$$

$$\gamma_{2i} = \Big(\prod_{1\le j<i} X_j\Big) Y_i$$

$\boxed{\text{Fermions are non-local}}$

p.b.c if $X=-1$.

Anti-pbc if $X=+1$.

P.b.c. $-J Z_{2L} Z_1 = -J\Big(\underbrace{\prod_{i=1}^{L} X_i}_{X}\Big)(-i\gamma_{2L}\gamma_1) = -J(-X)\,i\gamma_{2L}\gamma_1$, can absorb $(-X)$ by defining $\gamma_{2L+1} = \gamma_1(-X)$

$\boxed{\text{B.C. of fermion is a dynamical d.o.f.}}$ $\to$ Signs of a gauge theory (Gauge group $= \mathbb{Z}_2$)