Title: Lecture - Quantum Matter, PHYS 777

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Collection/Series: Quantum Matter (Elective), PHYS 777, March 31 - May 2, 2025

Subject: Condensed Matter

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Two. Simp

Quantum matter.

Week 1. Ising model in (1+1) d.

qubit.

Pauli matrices: X, Y, Z

$\sigma^x, \sigma^y, \sigma^z$

$$H = -\sum_{i=1}^{L} Z_i Z_{i+1} - g \sum_{i=1}^{L} X_i$$

Transverse field Ising model

Periodic b.c. $i = 1, 2, \ldots, L$

$i + L \simeq i$

Two Simple limits.

① $g \to \infty$ $\quad H_0 = -\sum_i X_i$

Ground state (g.s). $|\Omega\rangle = \bigotimes_i |X_i = +1\rangle = |\rightarrow\rightarrow\rightarrow \dots\rangle$

Lowest excitation: $|\rightarrow\rightarrow \underset{x+h}{\leftarrow} \rightarrow \rightarrow \rightarrow\rangle := |x\rangle$. L degenerate states.

Excite energy. $\Delta = 2$: remains finite (nonzero) as $L \to \infty$

Now consider $g >> 1$. Perturbation in $O(1/g)$

$$H = -\sum X_i - \frac{1}{g}\sum Z_i Z_{i+1} = H_0 + \delta H$$

$$E_\Omega = E_0 + \langle\Omega|\delta H|\Omega\rangle = E_0$$

X_i

del

L

$\ldots\rangle$

enerate states

$L \to \infty$

What about $\{|x\rangle\}$?

Degenerate perturbation:

$$\langle x|\delta H|y\rangle = -\frac{1}{g}\left(\delta_{x+1,y} + \delta_{x-1,y}\right)$$

$$= -\frac{1}{g}\begin{pmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ 1 & & & 1 & 0 \end{pmatrix}$$

Diagonalize δH in $\mathrm{Span}\{|x\rangle\}$.

Claim: eigenstate takes the form

$$|k\rangle = \frac{1}{\sqrt{L}} \sum_{x=1}^{L} e^{ikx}|x\rangle.$$

k: "lattice momentum".

$$\delta H\,|k\rangle = -\frac{1}{g}\frac{1}{\sqrt{L}}\sum_x e^{ikx}\left(|x-1\rangle + |x+1\rangle\right)$$

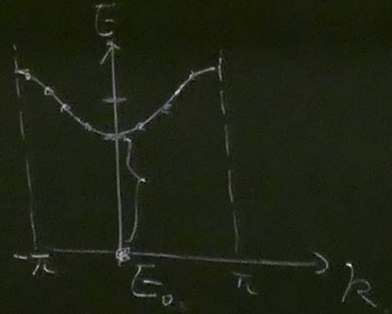
$$= -\frac{1}{g}\frac{1}{\sqrt{L}}\sum_x \left(e^{ik}+e^{-ik}\right)e^{ikx}\,|x\rangle$$

$$= \underbrace{-\frac{2}{g}\cos k}\,|k\rangle$$

$E_\Omega = E_0 + \langle \Omega | \delta H | \Omega \rangle = E_0$

k : lattice momentum

P. b. c. : $L + x \simeq x \Rightarrow kL = 2\pi n \Rightarrow k = \frac{2\pi n}{L}$, $n = 0, 1, 2, \ldots, L-1$.



Gap : $\Delta = 2 - \frac{2}{g}$. for small $1/g$, Δ remains finite.

Two comments:

(1). The fact that Δ remains finite under small perturbation: example of "universal behavior".

Finite gap doesn't requir fine-tuning.

Generally, it is believed that if

$$H = \sum_i H_i \quad \text{has finite } \Delta \text{ as } L \to \infty, \text{ then}$$

$$H + \lambda \sum_i \delta H_i \quad \text{has finite } \Delta \text{ for sufficiently small } \lambda.$$

Rigorous proof available for special classes of H.

②. When diagonalizing δH in $\text{span}\{|x\rangle\}$, $|k\rangle$ is useful, because.

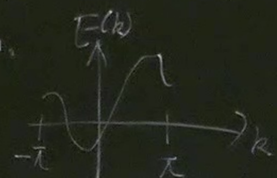
H has discrete translation symmetry. $T_x : |x\rangle \longrightarrow |x+1\rangle$, $[H, T_x] = 0$.

$|k\rangle$ is an eigenstate of T_x, w/ eigenvalue e^{-ik}.

Lesson: when solving single-particle H w/ lattice translation symm.

always try. $\frac{1}{\sqrt{L}} \sum_x e^{ikx} |u^k\rangle_x$ ← Bloch wavefunction.

Bloch Thm.



(B). $g \to 0$. $H_0 = -\sum Z_i Z_{i+1}$ G.S. $|\Omega_\uparrow\rangle = \bigotimes_i |Z_i = +1\rangle = |\uparrow\uparrow\uparrow \ldots\rangle$

$|\Omega_\downarrow\rangle = \bigotimes_i |Z_i = -1\rangle = |\downarrow\downarrow\downarrow \ldots\rangle$

lowest excitation $|\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\uparrow\uparrow\rangle$ $\Delta = 4$.

domain wall

Symmetr

Now $0 < g \ll 1$. As before $\Delta = 4 - O(g)$

$X|\uparrow$

Need degenerate perturbation in $\mathrm{Span}\{|\Omega_\uparrow\rangle, |\Omega_\downarrow\rangle\}$

Eigenst

$= |\uparrow\uparrow\uparrow\ldots\rangle$

$|\downarrow\downarrow\downarrow\ldots\rangle$ or any superposition $\alpha|\Omega_+\rangle + \beta|\Omega_-\rangle$.

in n'th order perturbation

$$\Delta = 4. \qquad \Big]_{\delta \sim e^{-L/\xi}} \rightarrow \delta \sim \langle\Omega_+|SH^n|\Omega_+\rangle - \langle\Omega_-|SH^n|\Omega_-\rangle = \begin{cases} 0 & \text{for } n \ll L. \\ g^L \sim e^{-L/\xi} \end{cases}$$

Symmetry of Ising model: Z_2 sym generated by $X = \bigotimes_i X_i$

$$X|\uparrow\downarrow\uparrow\uparrow\ldots\rangle = |\downarrow\uparrow\downarrow\downarrow\ldots\rangle, \quad X^\dagger Z_i X = -Z_i \quad X^\dagger X_i X = X_i$$

Eigenstates: $|\Omega_+\rangle = |\uparrow\uparrow\uparrow\ldots\rangle + |\downarrow\downarrow\downarrow\ldots\rangle$ If $\mathrm{tr}_{L\text{-spin}}$: $\rho = \frac{1}{2}|\uparrow\uparrow\uparrow\ldots\rangle\langle\uparrow\uparrow\uparrow\ldots| + \frac{1}{2}|\downarrow\downarrow\downarrow\ldots\rangle\langle\downarrow\downarrow\downarrow\ldots|$

$|\Omega_-\rangle = |\uparrow\uparrow\uparrow\ldots\rangle - |\downarrow\downarrow\downarrow\ldots\rangle$

C.D.

What's special about $\alpha|\Omega_+\rangle + \beta|\Omega_-\rangle$?

"Off-diagonal long-range order".

$$\langle Z_i Z_j \rangle \xrightarrow{|i-j| \text{ large}} 1 - O(g) = \text{const} \neq 0.$$

For symmetric states $|\Omega_+\rangle$ or $|\Omega_-\rangle$

$$\langle Z_i \rangle = \langle Z_j \rangle = 0$$

$$\Rightarrow \langle Z_i Z_j \rangle_c = \langle Z_i Z_j \rangle - \langle Z_i \rangle \cdot \langle Z_j \rangle = \text{Const} \neq 0.$$

Violating "Cluster decomposition" (CD) condition.

D. Satisfied for $|\uparrow\uparrow\uparrow\ldots\rangle$, or $|\downarrow\downarrow\downarrow\ldots\rangle$.

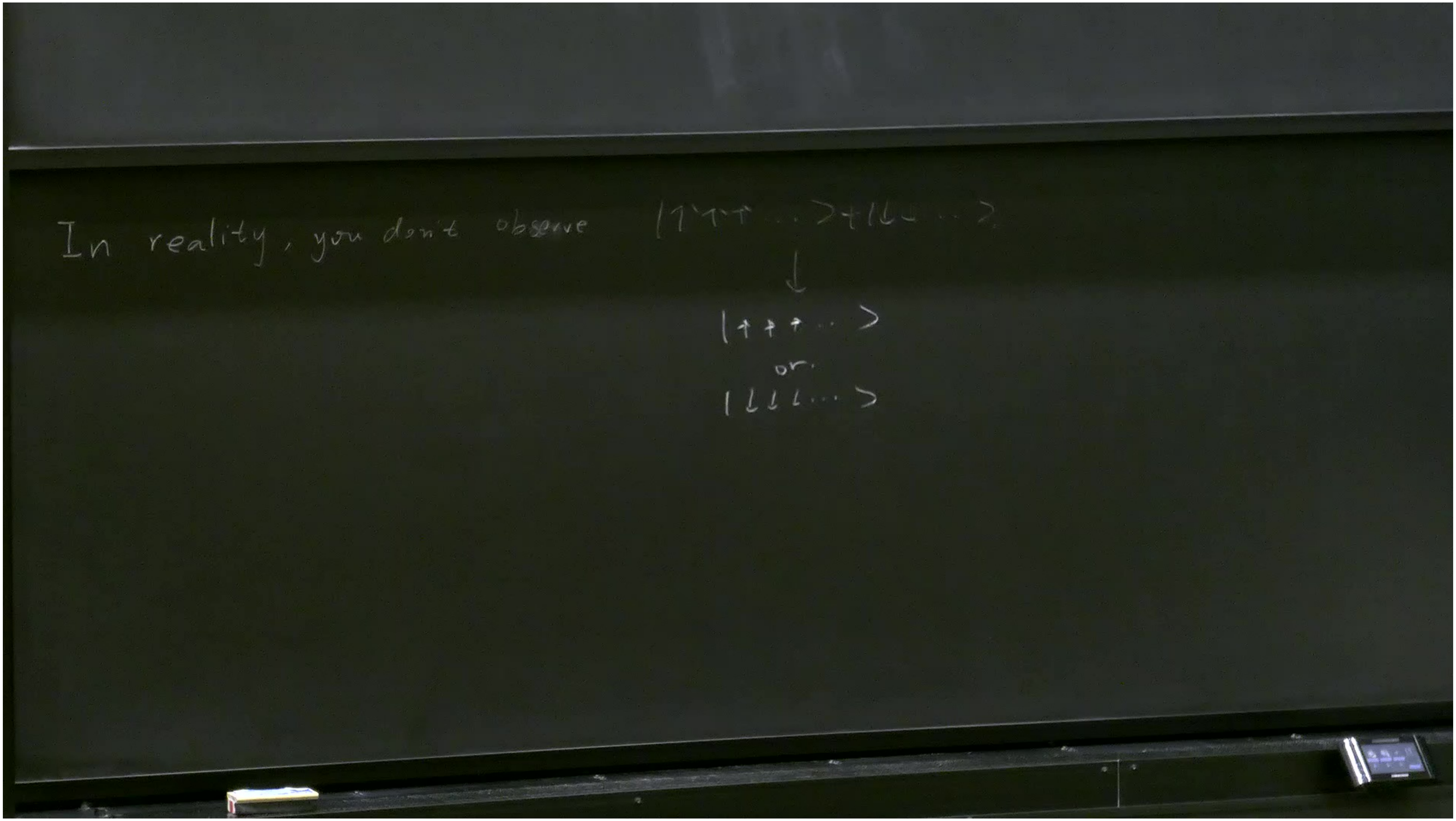
$$\langle Z_i \rangle = \langle Z_j \rangle = \pm 1.$$

$$\langle Z_i Z_j \rangle_c \to 0.$$

But breaks $\mathbb{Z}_2$ symmetry.

This conflict between symmetry of states & CD. is called "Spontaneous symmetry breaking".

SSB.


In reality, you don't observe $|\uparrow\uparrow\uparrow\ldots\rangle + |\downarrow\downarrow\downarrow\ldots\rangle$
$\downarrow$
$|\uparrow\uparrow\uparrow\ldots\rangle$
or
$|\downarrow\downarrow\downarrow\ldots\rangle$