Title: Ideas in Multiplicative Non-abelian Hodge theory

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Abstract:

Non-abelian Hodge theory is a profound three-way equivalence between topological, smooth and holomorphic objects, i.e. representations of the fundamental group, flat connections and Higgs bundles. It is natural to explore a group-theoretic or multiplicative version — an enterprise that has been untaken by Soibelman, Kontsevich, Mochizuki and others. In this talk, we will review the current landscape of multiplicative non-abelian Hodge theory and discuss some outstanding questions.

Ideas in Multiplicative Non-abelian Hodge Theory

§1. Linear NAHT : X compact R.S.

$$\left[\begin{array}{lll} M_B(X,r) \xrightarrow{\text{analytic} \cong} & M_{dR}(X,r) \xrightarrow[\text{stable locus}]{\simeq,\ \text{diffeo on}} & \\ = \text{Reps } \pi_1(X) \to GL(r,\mathbb{C}) & \text{Flat bundles} & \end{array}\right.$$

on-abelian Hodge Theory

mpact R.S.

$(\mathcal{E}, D^\lambda \text{ s.t. } D^\lambda(fs) = fD^\lambda s + \lambda s \otimes df)$

$(X,r) =$ s.s. λ-flat bundles w/ 0 Chern classes

$M_{dR}(X,r) \xrightarrow[\text{stable locus}]{\simeq,\ \text{diffeo on}} M_{Dol}(X,r)$

s.s. Higgs bundles, w/ 0 Chern classes

bundles

Deligne: $\pi: M_{Hod}(X,r) \to \mathbb{A}^1$ can be extended as $TW(X,r) \longrightarrow \mathbb{P}$

= obtained by gluing

§2 <u>Multiplicative Version</u>: G complex red group $\supset T$

$\supset$ Dolbeault

$Z \subset X$ finite set of pts.

<u>Defn</u>: A mult. Higgs bundle = $(E \in Bun_G(X),$

$M_{Hod}(X,r) \longrightarrow \mathbb{A}^1$ can be extended as $TW(X,r) \longrightarrow \mathbb{P}^1$

= obtained by gluing $M_{Hod}(X,r)$ to M

plicative version: G complex red group $\supset T$ maximal torus

t $Z \subset X$ finite set of pts.

mult. Higgs bundle = $\left(E \in Bun_G(X),\ \text{merom. } \varphi \in H^0(X, E \times_G G) \right)$

$\varphi : E|_{X \setminus Z} \cong E|_{X \setminus Z}$

$$\mathrm{Higgs}(X,G) = \underline{\mathrm{Maps}}(X, [\mathfrak{g}/G])$$

$$\mathrm{mHiggs}(X,G) = \underline{\mathrm{Maps}}(X|Z, G/G) = \underline{\mathrm{Maps}}(X|Z \times S^1, BG)$$

$$G/G = \mathcal{L}(BG) = \mathrm{Maps}(S^1, BG) \qquad (X \times S^1)|\tilde{Z}$$

$\Rightarrow$ Harmonic

Defn: A monopole on $Y = (S^1 \times X) \setminus \{p_1, \ldots, p_n\}$

$/G])$

$G/G) = \underline{\mathrm{Maps}}(X|Z \times S^1, BG)$

$\shortparallel \qquad (X \times S^1)|\tilde{Z}$

$\mathrm{aps}(S^1, BG)$

$\mathcal{E} \in \mathrm{Bun}_G(Y)$ satisfies

$= (S^1 \times X) \setminus \{p_1, \dots, p_n\}$ is $(\mathcal{E}, \nabla, \phi)$ where ∇ G-Conn

$\phi \in H^0(Y, \mathrm{ad}(E))$

$$\mathrm{Higgs}(X,G) = \underline{\mathrm{Maps}}(X, [\mathfrak{g}/G])$$

$$\mathrm{mHiggs}(X,G) = \underline{\mathrm{Maps}}(X|Z, \mathfrak{g}/G) = \underline{\mathrm{Maps}}(X|Z \times S^1, BG)$$

$$(X \times S^1)|\tilde{Z}$$

$$\mathcal{L}(BG) = \mathrm{Maps}(S^1, BG)$$

$\Rightarrow$ Harmonic

Defn: A monopole on $Y = (S^1 \times X) \setminus \{p_1, \dots, p_n\}$ is $(\mathcal{E}, \nabla, \phi)$ where

HEB eqn: $F(\nabla) - iC\omega_X = * d_\nabla \phi,$

Thrm: (Charbonneau-Hurtubise, Smith)

Irred mono poles on $Y = (X \times S^1) \setminus \{p_1, \ldots, p_n\}$ $\longleftrightarrow$ Mult Higgs bundles on X

+ sings $p_i = (z_i, t_i)$ of μ_i-Dirac type

+ sing. $z_i \in Z$ are controlled by μ

$(\mathcal{E}, \nabla, \Phi) \longmapsto \left(\mathcal{E}|_{X \times \{0\}},\ \rho_{0,T} : E|_{\{0\} \times X \setminus Z} \xrightarrow{\ \cong\ } \right.$

se, Smith)

, p_n } $\longleftrightarrow$ Mult Higgs bundles on X

+ sing. $z_i \in Z$ are controlled by $\mu_i \in X_*(T)$

length of the circle

c type

$\longrightarrow \left(\mathcal{E}|_{X \times \{0\}},\ \rho_{0,T} : E|_{\{0\} \times X \setminus Z} \xrightarrow{\cong} E|_{\{T\} \times X \setminus Z} \right)$

scattering maps = parallel transport induced by

$\nabla_t - i\phi$

Thrm: (Charbonneau-Hurtubise, Smith)

Irred mono poles on $Y = (X \times S^1) \setminus \{p_1, \dots, p_n\}$ $\longleftrightarrow$ Mult Higgs bundles on X

+ sings $p_i = (z_i, t_i)$ of μ_i-Dirac type

+ sing. $z_i \in Z$ are controlled

$$(\mathcal{E}, \nabla, \Phi) \longmapsto \left(\mathcal{E}|_{X \times \{0\}},\ \rho_{0,T} : E|_{\{0\} \times}\right.$$

scattering m

Q: What are dR and Betti?

§3. NAHT of periodic monopoles

Let $\Gamma \subseteq \mathbb{R}^3$ discrete

$$\mathcal{M} = \mathbb{R}^3/\Gamma \supseteq Z \text{ finite subset}$$

Defn: A monopole on $\mathcal{M} \setminus Z$ is (E, h, ∇, Φ) s.) Herm es

subset

s (E, h, ∇, ϕ) s.T (E, h) Herm. v.b satisfies standard Bog-eqn

∇ unitary conn

ϕ Skew-Herm end.

Let $\Gamma \subseteq \mathbb{R}^3$ discrete

$\mathcal{M} = \mathbb{R}^3/\Gamma \supseteq Z$ finite subset

Defn: A monopole on $\mathcal{M} \setminus Z$ is (E, h, ∇, Φ) s.t. (E, h) Herm

∇ unitary c

Φ Skew-H

$$F(\nabla) = * d_{\nabla} \Phi .$$

Remark: On $S^1 \times X$, it is compact & global smooth slns are sim

It is periodic monopole if $\Gamma = \mathbb{Z}$, $M = \mathbb{R}^3/\mathbb{Z} = S^1 \times \mathbb{R}^2 = S^1 \times \mathbb{C}$

doubly periodic if $\Gamma = \mathbb{Z}^2$, $M = \mathbb{R}^3/\mathbb{Z}^2 = \mathbb{R} \times (S^1)^2$

triply periodic if $\Gamma = \mathbb{Z}^3$, $M = (S^1)^3$ $\quad$ $\mathrm{rk}\,\Gamma$

$\rhd$ de Rham : difference modules

R comm. algebra, $\Phi \in \mathrm{Aut}(R)$

__Defn:__ A diff module over (R, Φ) ... Φ is ...

eriodic monopole if $\Gamma = \mathbb{Z}$, $M = \mathbb{R}^3/\mathbb{Z} = S^1 \times \mathbb{R}^2 = S^1 \times \mathbb{C}$

ubly periodic if $\Gamma = \mathbb{Z}^2$, $M = \mathbb{R}^3/\mathbb{Z}^2 = \mathbb{R} \times (S^1)^2$

ply periodic if $\Gamma = \mathbb{Z}^3$, $M = (S^1)^3$

$\mathrm{rk}\,\Gamma = \#$ periodicities

m : difference modules

omm. algebra, $\Phi \in \mathrm{Aut}(R)$

s.t. $\Phi_V(fs) = \Phi(f)\Phi_V(s)$, $f \in R$, $s \in V$

A diff module over (R, Φ) is $\left(V \in \mathrm{Mod}_R,\ \Phi_V \text{ is a } \mathbb{C}\text{-linear isom}\right)$

Examples:

- $R = \mathbb{C}[x]$, $(\Phi(f))(x) = f(x+\lambda)$, $\lambda \in \mathbb{C}$

  Additive λ-diff mod $= (V \in \mathrm{Mod}_{\mathbb{C}[x]}, \Phi_V)$

- $R = \mathbb{C}[x, x^{-1}]$, $(\Phi(f))(x) = f(qx)$, $q \in \mathbb{C}^*$

  Mult. q-diff mod $= (V \in \mathrm{Mod}_{\mathbb{C}[x,x^{-1}]}, \Phi_V)$

- $R = K(E)$, E elliptic curve, $(\Phi(f))(x) = f(x+\alpha)$, $\alpha \in E$

  Elliptic α-diff mods

s :

], $(\Phi(f))(x) = f(x+\lambda)$, $\lambda \in \mathbb{C}$

λ-diff mod $= (V \in \mathrm{Mod}_{\mathbb{C}[x]}, \Phi_V) \Longleftrightarrow A_\lambda$-mod

$$A_\lambda = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}[x]\,\Phi_V^n$$

$$A_q = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}[x,x^{-1}]\,\Phi_V^n$$

x^{-1}], $(\Phi(f))(x) = f(qx)$, $q \in \mathbb{C}^*$ $\Longleftrightarrow A_q$-mod

ff mod $= (V \in \mathrm{Mod}_{\mathbb{C}[x,x^{-1}]}, \Phi_V)$

), E elliptic curve, $(\Phi(f))(x) = f(x+\alpha)$, $\alpha \in E$

α-diff mods

...ple if $\Gamma = \mathbb{Z}$, $M = \mathbb{R}^3/\mathbb{Z} = S^1 \times \mathbb{R}^2 = S^1 \times \mathbb{C}$

$n\cdot(t, w) = (t+n, q^n w)$

if $\Gamma = \mathbb{Z}^2$, $M = \mathbb{R}^3/\mathbb{Z}^2 = \mathbb{R} \times (S^1)^2 = \mathbb{R} \times \mathbb{C}^*/q^{\mathbb{Z}}$

if $\Gamma = \mathbb{Z}^3$, $M = (S^1)^3$

$\mathrm{rk}\,\Gamma = \#$ periodicities

$q \in \mathrm{Aut}(X)$ q-diff conn $= \underline{\mathrm{Maps}}(X \times_q S^1, BG)$

modules

$\Phi \in \mathrm{Aut}(R)$

s.t. $\Phi_V(fs) = \Phi(f)\,\Phi_V(s)$, $f \in R$, $s \in V$

...e over (R, Φ) is $\left(V \in \mathrm{Mod}_R,\ \Phi_V \text{ is a } \mathbb{C}\text{-linear isom}\right)$

$$\begin{tikzcd} X \sqcup X \arrow[r, "\mathrm{id} \sqcup \mathrm{id}"] \arrow[d, "q \sqcup \mathrm{id}"'] & X \arrow[d] \\ X \arrow[r] & X \times_q S^1 \end{tikzcd}$$

$\Gamma = \mathbb{Z}, \quad M = \mathbb{R}^3/\mathbb{Z} = S^1 \times \mathbb{R}^2 = S^1 \times \mathbb{C}$

$\Gamma = \mathbb{Z}^2, \quad M = \mathbb{R}^3/\mathbb{Z}^2 = \mathbb{R} \times (S^1)^2 = \mathbb{R} \times \mathbb{C}^\times/\mathbb{Z}$

$\Gamma = \mathbb{Z}^3, \quad M = (S^1)^3$

$\mathrm{rk}\,\Gamma = \#$ periodicities

$q \in \mathrm{Aut}(X) \quad q\text{diff Conn} = \underline{\mathrm{Maps}}(X \times_q S^1, BG)$

$\in R \qquad \in V$

$\mathrm{Rep}(R)$ s.t. $\Phi_V(fs) = \Phi(f)\Phi_V(s)$

(R, Φ) is $\left(V \in \mathrm{Mod}_R,\ \Phi_V \text{ is a } \mathbb{C}\text{-linear isom}\right)$

$$\begin{tikzcd}
X \sqcup X \arrow[r, "\mathrm{id} \sqcup \mathrm{id}"] \arrow[d, "q \sqcup \mathrm{id}"'] & X \arrow[d] \\
X \arrow[r] & X \times_q S^1
\end{tikzcd}$$

$X^{\mathrm{an}} \to \mathbb{A}^1$

Thm (Mochizuki, 2019, 21)

① Periodic monopoles on $S^1 \times \mathbb{C} \longleftrightarrow$ Additive $(2i\lambda T)$-diff mod parabolic, deg 0

$S^1 = \mathbb{R}/T\mathbb{Z}$

- Dirac type sing
- . . . .

② Doubly periodic monopoles on $\mathbb{R} \times (S^1)^2 \longleftrightarrow$ Mult q diff mod parabolic, deg 0

- Dirac type sing

$2i)$

$\times \mathbb{C} \longleftrightarrow$ Additive $(2i\lambda T)$-diff mod

parabolic, deg 0

Betti

deRham $\qquad$ KS, $|q| \neq 1$ $\qquad$ Locally free $\mathcal{O}_{\mathbb{C}^*/q^{\mathbb{Z}}}(*D)$-mod

on $\mathbb{R} \times (S^1)^2 \longleftrightarrow$ Mult q diff mod $\xleftrightarrow[\text{Riemann-H}]{\text{KS, } |q| \neq 1}$ E w/ 2 anti-Harder Narasimhan filtrations

parabolic, deg 0

Qdiff conn: $q \in \mathrm{Aut}(X)$

$$\left(E \in \mathrm{Bun}_G(X),\ \varphi : E|_{X\setminus Z} \cong q^* E|_{X\setminus Z}\right)$$

$$q\mathrm{Diff\,Conn}(X, G) = \mathrm{Maps}\left(X \times_q S^1, BG\right)$$

$$\Big\downarrow {\scriptstyle q=1}$$

$$m\mathrm{Higgs}(X, G) = \mathrm{Maps}\left(X \times S^1, BG\right)$$