Title: Fermions and Gaussianity ; Resources and Simulability

Speakers: Andrew Projansky

Collection/Series: Quantum Information

Subject: Quantum Information

Date: March 26, 2025 - 11:00 AM

URL: https://pirsa.org/25030179

Abstract:

Matchgates are a well studied class of quantum circuits tied to the time dynamics of Free Fermion Hamiltonians. It is important to note however that Matchgates specifically come from representing Free Fermions with the Jordan-Wigner encoding. When we represent our fermionic systems with other encodings besides Jordan-Wigner, we still are considering the time dynamics of Free Fermion solvable Hamiltonians, but we can introduce complexity in how we encode our fermionic information. This gives us a test ground for clarifying what physical properties make time dynamics hard to simulate, even when Hamiltonians can be exactly diagonalized. In this talk I will discuss the theory behind matchgates, fermionic encodings, and recent results in the simulability of Clifford/matchgate hybrid circuits (arxiv:2312.08447, arxiv:2410.10068). These results clarify resources for Free Fermions represented beyond the Jordan-Wigner encoding, as well as an overall perspective of what it means for a state to be Gaussian.

Fermions and Gaussianity; Resources and Simulability

Andrew Projansky
Ph.D. Student in the group of James Whitfield
Dartmouth College

Perimeter Institute
3/26/25

Based on work done with Jason Necaise, Joshuah Heath, and James Whitfield

[Projansky 25']

[Projansky 24']

Quantum Information Science at Dartmouth

U.S. DEPARTMENT OF ENERGY

Outline

1. What is a Fermion?
2. Review of Simulable Circuits
3. Combining/Unifying Cliffords and Matchgates

2

Fermions: Fundamentals

Indistinguishable:

Antisymmetric:

$$\Psi_F(..., \mathbf{r_i}, ..., \mathbf{r_j}, ...) = -\Psi_F(..., \mathbf{r_j}, ..., \mathbf{r_i}, ...)$$

$$\Psi_F(\mathbf{r_1}, \mathbf{r_2}, ..., \mathbf{r_N}) = \begin{vmatrix} \chi_1(\mathbf{r_1}) & \chi_2(\mathbf{r_1}) & \cdots & \chi_N(\mathbf{r_1}) \\ \chi_1(\mathbf{r_2}) & \chi_2(\mathbf{r_2}) & \cdots & \chi_N(\mathbf{r_2}) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_1(\mathbf{r_N}) & \chi_2(\mathbf{r_N}) & \cdots & \chi_N(\mathbf{r_N}) \end{vmatrix}$$

Second quantization to the rescue! How many fermions in a state?

$$|\Psi_F\rangle = |n_1, n_2, n_3, ...\rangle$$

Fermions: Two Choices

Anticommutator $\{\cdot,\cdot\}$: $\{A,B\} = AB + BA$

Dirac Fermions

$$a^\dagger|0\rangle = |1\rangle$$
$$a|1\rangle = |0\rangle$$

$$\{a_j, a_k^\dagger\} = \delta_{jk}$$
$$\{a_j, a_k\} = 0$$
$$\{a_j^\dagger, a_k^\dagger\} = 0$$

Anticommuting!

Majorana operators

$$c_{2k-1} = a_k + ia_k^\dagger$$
$$c_{2k} = a_k - ia_k^\dagger$$

$$\{c_j, c_k\} = 2\delta_{jk}$$

Mutually anticommuting

Operators defined on Fock Space: space of mode occupancies for indistinguishable particles

Fermionic Hamiltonians

Molecular Hamiltonian: interacting

$$H_M = \sum_{j,k} h_{j,k} a_j^\dagger a_k + \sum_{j,k,l,m} V_{j,k,l,m} a_j^\dagger a_l^\dagger a_m a_k$$

Free Fermion: No quartic term, efficiently diagonalizible!

$$H_F = \sum_{j,k} h_{j,k} a_j^\dagger a_k = \sum_x \epsilon_x (b_x^\dagger b_x - b_x b_x^\dagger)$$

Spins and Spin Hamiltonians

Set of Spins: Pauli operators $P_i = p_1 \otimes p_2 \otimes ... \otimes p_n$
for $p_i \in I, X, Y, Z$.

General spin Hamiltonian: $H_S = \sum_{i=1}^{4^n} h_i P_i$

Example: 1D Transverse Field Ising:

$$H = -J \left(\sum_{i=1}^{n} Z_i Z_{i+1} + g \sum_{j} X_j \right)$$

→ Solvable via mapping to Fermions!

6

Fermion-to-Qubit Transforms

Goal: Map from Fock space to qubit space with operators which share anticommutation structure

Most common example: Jordan-Wigner transformation

$$a_j = \prod_{k<j} Z_k \otimes |0\rangle\langle 1| = \frac{1}{2}\prod_{k<j} Z_k (X_j + iY_j)$$

$$a_j^\dagger = \prod_{k<j} Z_k \otimes |1\rangle\langle 0| = \frac{1}{2}\prod_{k<j} Z_k (X_j - iY_j)$$

Can also define Jordan-Wigner Majoranas

$$c_{2i-1} \rightarrow \left(\prod_{j=1}^{i-1} Z_j\right) X_i, \qquad c_{2i} \rightarrow \left(\prod_{j=1}^{i-1} Z_j\right) Y_i.$$

Beyond Jordan-Wigner

Other sets of Mutually anticommuting Paulis exist...

$$\left\{\begin{matrix} X \\ Y \\ ZX \\ ZY \\ ZZX \\ ZZY \\ \vdots \end{matrix}\right\} \qquad \left\{\begin{matrix} Z \\ Y \\ XZ \\ XY \\ XXZ \\ XXY \\ \vdots \end{matrix}\right\}$$

JW $\qquad$ E'

$$\sum_{i=0}^{n-1} J_i(X_i X_{i+1} + Y_i Y_{i+1}) \qquad \sum_{i=0}^{n-1} J_i(Z_i Z_{i+1} + Y_i Y_{i+1})$$

JW $\qquad$ E'

Free Fermion Solvable

Spins induce Fermions, Fermions induce spins

Beyond Jordan-Wigner: Locality

Jordan-Wigner: Pauli weight $O(n)$, but occupancy stored locally

$$a_j = \prod_{k<j} Z_k \otimes |0\rangle\langle 1| = \frac{1}{2}\prod_{k<j} Z_k (X_j + iY_j)$$

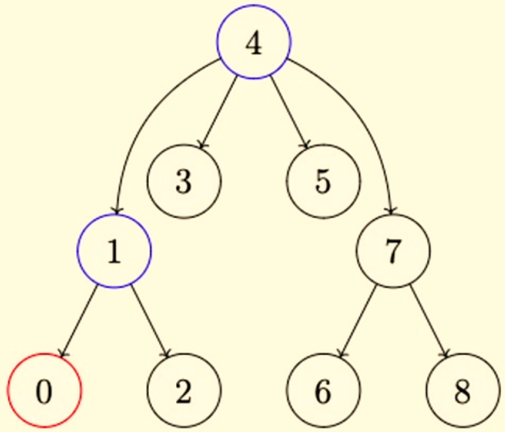
$$a_j^\dagger = \prod_{k<j} Z_k \otimes |1\rangle\langle 0| = \frac{1}{2}\prod_{k<j} Z_k (X_j - iY_j)$$

$O(n)$ Z strings

Occupancy updates are local

Beyond Jordan-Wigner: Locality

Ternary Tree/Sierpinski: optimal Pauli weight $O(log_3(n))$,
but occupancy stored non-locally [1] [2]



Occupancy of mode 0 stored in:
0, 1, 4

[1] *Optimal fermion-to-qubit mapping via ternary trees with applications to reduced quantum states learning*, Jiang, Kalev, Mruczkiewicz, Neven

[2] *A Sierpinski Triangle Fermion-to-Qubit Transform*, Harrison, Chiew, Necaise, **Projansky**, Strelchuk, Whitfield

Free Fermions and Dynamics

$$H_{S_1} = \sum_{i=0}^{n-1} J_i(X_i X_{i+1} + Y_i Y_{i+1})$$

$$\downarrow$$

$$U = e^{iH_{S_1}}$$

Dynamics of Free Fermion Solvable Hamiltonian

Simulable?

$$H_{S_2} = \sum_{i=0}^{n-1} J_i(Z_i Z_{i+1} + Y_i Y_{i+1})$$

$$\downarrow$$

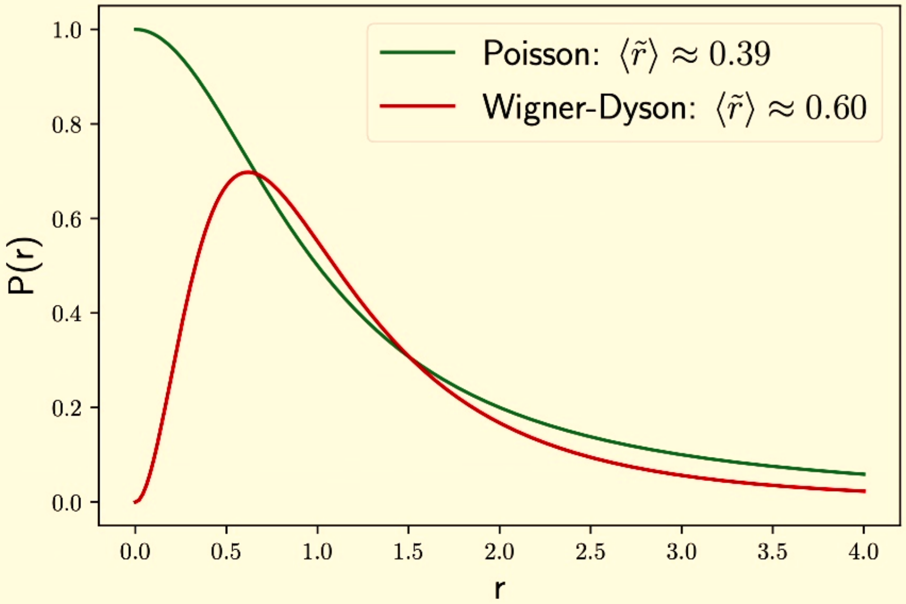
$$U = e^{iH_{S_2}}$$

Dynamics of Free Fermion Solvable Hamiltonian

Simulable?

Entanglement Spectra

Entanglement spectra: Take eigenvalues of bipartite reduced density matrix, study distribution



Distributions of eigenvalues can be characterized by single number related to ratios of level spacings

$$\tilde{r}_k = \frac{\min(\delta_k, \delta_{k+1})}{\max(\delta_k, \delta_{k+1})}.$$

for $\delta_k = p_{k-1} - p_k$ and $p_k \geq p_{k+1}$

Entanglement Spectra and Simulability

In [3], argued that entanglement spectral statistics are a signature of complexity of quantum state.

Poisson distributed spectra $\rightarrow$ Simulable task (ex, Clifford circuits)

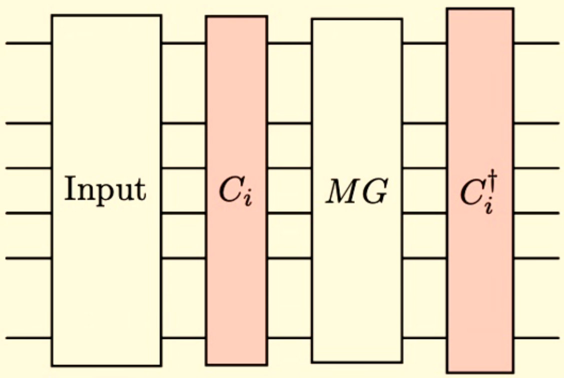
- Integrable entanglement: $\langle \tilde{r} \rangle \approx 0.39$

Wigner-Dyson distributed spectra $\rightarrow$ Non-simulable tasks (ex, Clifford + T)

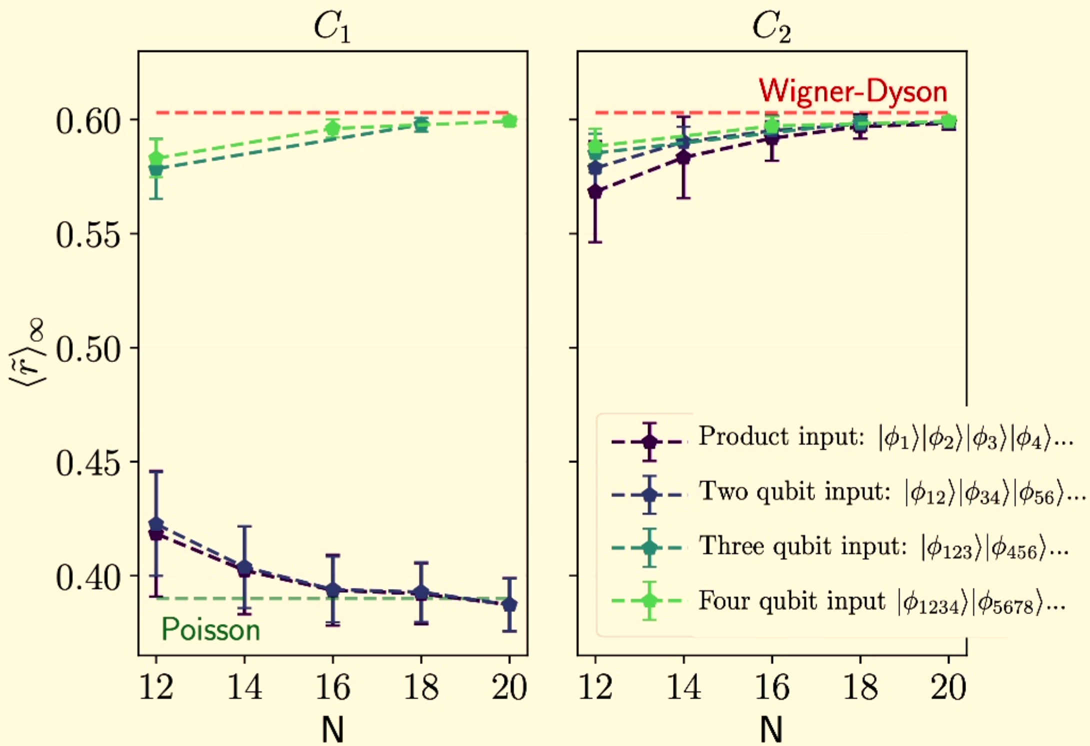
- Non-integrable entanglement: $\langle \tilde{r} \rangle \approx 0.60$

[3] *Irreversibility and entanglement spectrum statistics in quantum circuits*, Shaffer, Chamon, Hamma, Mucciolo

Clifford Conjugated Matchgate Circuits

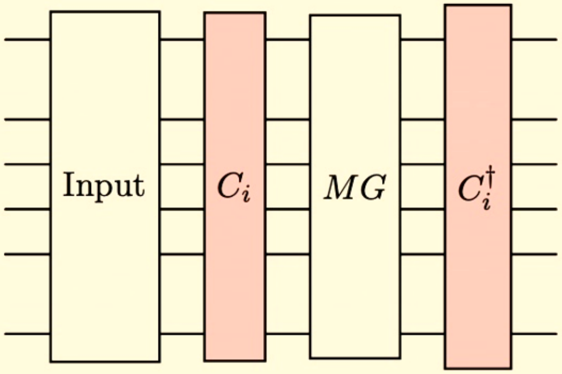


Non Jordan-Wigner solvable
free Fermion Hamiltonian



[**Projansky 24'**]: *Entanglement spectrum of matchgate circuits with universal and non-universal resources*, **Projansky**, Heath, Whitfield

What's behind the curtain?



- Comment: Gap between chaotic signatures and complexity: (can prove for any Free Fermion time dynamics, *some* Pauli expectation values can be simulated classically).
- **Question: What's the resource of product states/other inputs to these free fermion solvable circuits?**

Let's find out!

Outline

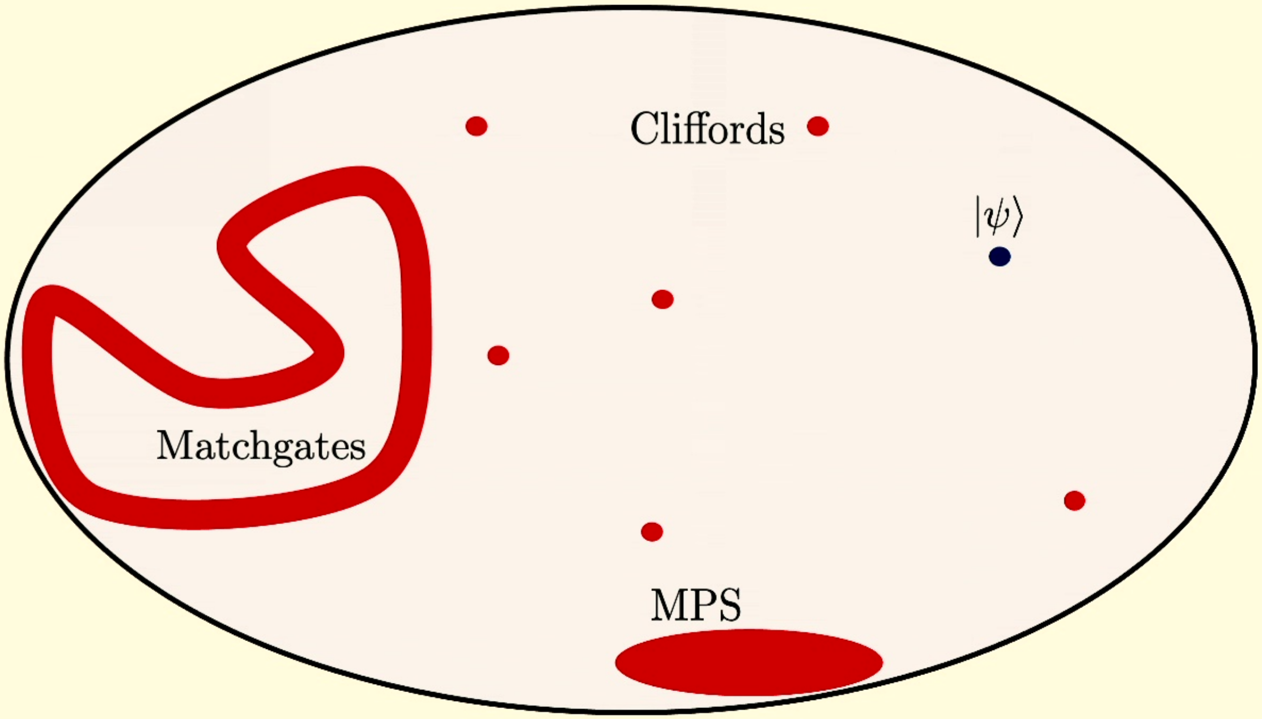
1. What is a Fermion?
2. Review of Simulable Circuits
3. Combining/Unifying Cliffords and Matchgates

Fermions, Cliffords, and matchgates, oh my!

Quantum/Classical Divide

Hilbert space over N qubits

Set of classically simulable quantum states



Cliffords and Stabilizer States

Clifford circuits: discrete group generated by unitary operators

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

the Hadamard, Phase, and CNOT gates.

Stabilizer State: $C|00...0\rangle$ for Clifford C

Why are Cliffords Simulable?

$$p_i = g_i \cdot p_{i_1} \otimes p_{i_2} \otimes ... \otimes p_{i_n}$$

for $p_{i_j} \in \{X, Y, Z, I\}$, $g_i \in \{\pm 1, \pm i\}$. Let $p_i \in \mathcal{P}_n$ define the Pauli group.

For any Clifford C and Pauli P_i, $C^\dagger P_i C \in \mathcal{P}_n$.

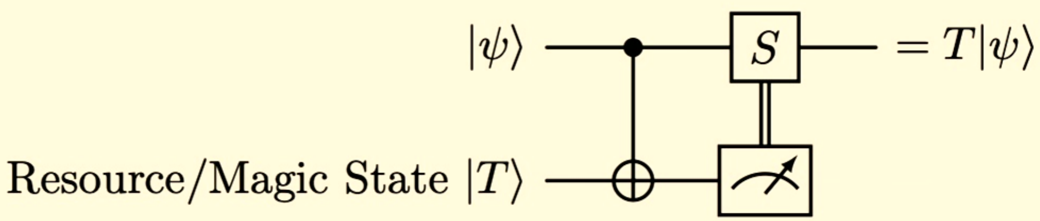
Gottesmann-Knill Theorem [4]: For any stabilizer state, bitstring and Pauli expectation value outputs are classically simulable.

[4]: *The Heisenberg Representation of Quantum Computers*, Gottesmann, 1998

Cliffords and Stabilizer States

Clifford circuits + T gate (RZ($\frac{\pi}{4}$)): universal quantum circuits

T gate: Magic resource, resource which enables universal quantum computation. Often performed with magic state gadget.



Free Fermions: Time Evolution

Not only diagonalizible, also easy time evolution! Let

$$H_F = \sum_{j=1,k=1}^{2n} h_{j,k} c_j c_k$$

For $U_{MG} = e^{-iH_F}$

$$U_{MG}^{\dagger} c_v U_{MG} = \sum_{v=1}^{2n} R_{uv} c_v$$

$$R_{uv} = e^{4H_F} \in O(2n)$$

→ Conceptually: Free fermion Lie algebra exponentially smaller than Lie algebra for arbitrary unitary gates

Matchgates

$$H_F = i \sum_{k>j=1}^{4} h_{j,k} c_j c_k$$

$$H_S$$

Jordan-Wigner Mapping

$$e^{iH_S} = G(A, B) = \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}$$

$$A \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B \equiv \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\det(A) = \det(B)$$

$$\text{fswap} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

SWAP $\neq$ Matchgate

Analog to T gate

Matchgates: Marginals/Bitstring Outputs

Matchgates can be simulated efficiently! [5]

$$p(y^*|x) = \langle 00...0|U_{MG}^\dagger|y^*\rangle\langle y^*|U_{MG}|00...0\rangle$$

$$p(y^*|x) = \sum_{a,b,...,k} R_{i_1,a}R_{i_2,b}...R_{i_k,k}\langle 00..0|a_a a_b...a_k|00...0\rangle$$

$$p(y^*|x) = \mathrm{Pf}(M) \ = \sqrt{\det(M)}$$

Reduces problem down to Pfaffian of antisymmetric matrix

[5]. *Classical simulation of noninteracting-fermion quantum circuits*, Terhal, Divincenzo, 2002

Matchgates: Expectation Value Outputs

Covariance Matrix: $\Gamma_{j,k}(\rho) = -\frac{i}{2}\mathrm{Tr}([c_j, c_k]\rho)$

Can be written for arbitrary state...
But for Matchgates,

$$\Gamma\left(U_{\mathrm{MG}}\rho_o U_{\mathrm{MG}}^\dagger\right) = R\Gamma(\rho_o)R^\dagger = \Gamma(|\psi\rangle\langle\psi|) = R\bigoplus_{j=1}^{n}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} R^\dagger$$

For any Pauli P $c_{\mu_1}c_{\mu_2}...c_{\mu_{2k}}$

$$(-i)^k\mathrm{Tr}(c_{\mu_1}c_{\mu_2}...c_{\mu_{2k}}\rho) = \mathrm{Pf}(\Gamma(|\psi\rangle\langle\psi|)_{\hat{\mu}\hat{\mu}})$$

Free Fermions: Clifford Cojnugated Matchgates

Other sets of Mutually anticommuting Paulis exist...

$$\left\{\begin{matrix} X \\ Y \\ ZX \\ ZY \\ ZZX \\ ZZY \\ \vdots \end{matrix}\right\} \qquad\qquad \left\{\begin{matrix} Z \\ Y \\ XZ \\ XY \\ XXZ \\ XXY \\ \vdots \end{matrix}\right\} = C^\dagger \left\{\begin{matrix} X \\ Y \\ ZX \\ ZY \\ ZZX \\ ZZY \\ \vdots \end{matrix}\right\} C$$

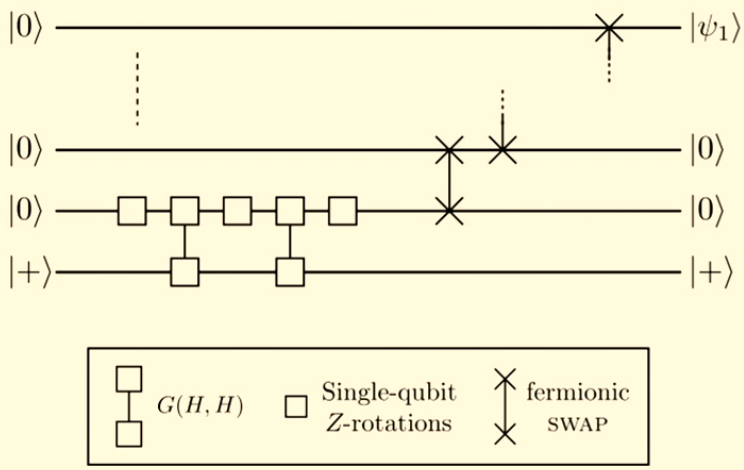
JW $\qquad\qquad$ E'

Dynamics over free Fermion solvable
Hamiltonians $\rightarrow$ Clifford conjugated matchgates

Outline

1. What is a Fermion?
2. Review of Simulable Circuits
3. Combining/Unifying Cliffords and Matchgates

Product States and Matchgates



[6]

Product states: simulable inputs for matchgates!

- Utilize single $|+\rangle$ state ancilla
- Map $H_F = i\sum_{j,k=1}^{2n} h_{j,k} c_j c_k + \sum_{j=1}^{2n} h_j^l c_j$

$$\downarrow$$

$$H_F' = i\sum_{j,k=1}^{2n+1} h'_{j,k} d_j d_k$$

What fermionic encodings have the generators requierd to make arbitrary product states?

[6]: *Efficient classical simulation of matchgate circuits with generalized inputs and measurements*, Brod, 2016

Product States and Fermionic Encodings

Result: The *only* encodings such that product states are Gaussian are the Jordan-Wigner encoding, and SWAP conjugated Jordan-Wigner encoding **[Projansky 25']**

$$\mathcal{L}^*_{1,2} = \{c_j, c_j c_k | j,k \in [1,2,3...,2n]\}$$

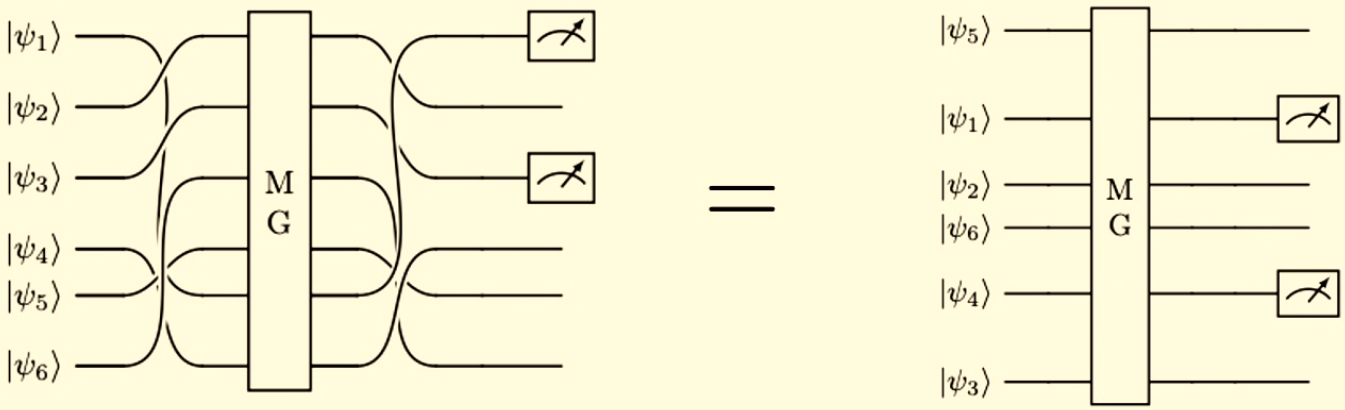
Proposition 1. The generators of arbitrary single qubit rotations cannot be in $\mathcal{L}^*_{1,2}$ for more than two qubits.
Proposition 2. Access to some swap-like gate implies some XX and YY generators are in $\mathcal{L}^*_{1,2}$.
Proposition 3. Given the results of Proposition 1 and Proposition 2, there are limits to what XX and YY can be simultaneously present in $\mathcal{L}^*_{1,2}$.
Proposition 4. Given the result of Proposition 3, there are only certain forms a set of Majoranas can take such that the necessary XX and YY generators are in $\mathcal{L}^*_{1,2}$.

[Projansky 25']: *Gaussianity and Simulability of Cliffords and Matchgate Circuits*, **Projansky**, Necaise, Whitfield

Product States and Fermionic Encodings

Result: The *only* encodings such that product states are Gaussian are the Jordan-Wigner encoding, and SWAP conjugated Jordan-Wigner encoding **[Projansky 25']**

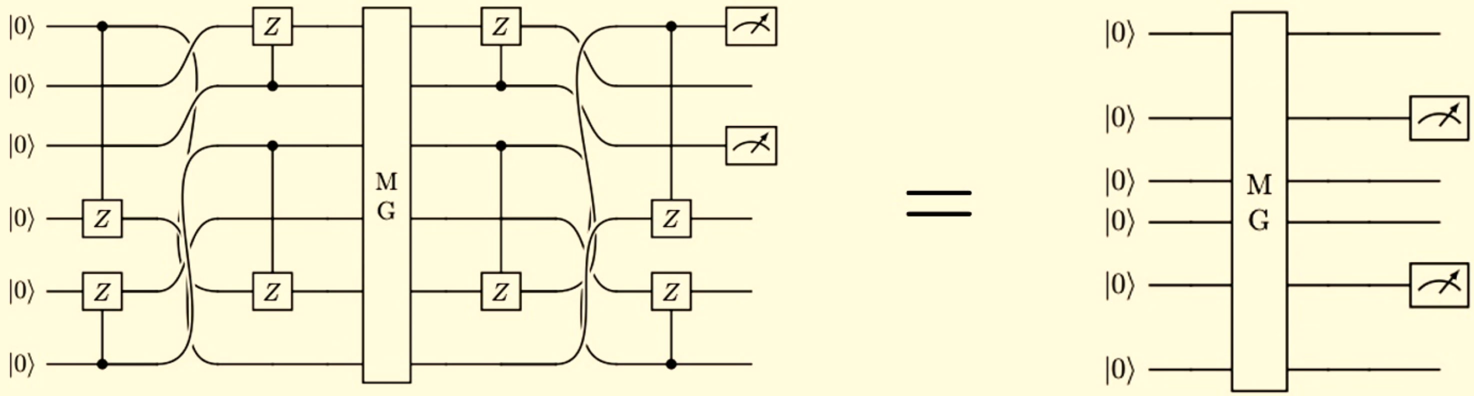
Corollary: For SWAP conjugated matchgate circuits, bitstring outputs are simulable on product state inputs



[Projansky 25']: *Gaussianity and Simulability of Cliffords and Matchgate Circuits*, **Projansky**, Necaise, Whitfield

CZ-Conjugated Matchgates

Corollary: For SWAP+CZ conjguated matchgate circuits, bitstring outputs are simulable on computational basis state inputs



CZ-Conjugated Matchgates, Concept

Corollary: For SWAP+CZ conjguated matchgate circuits, bitstring outputs are simulable on computational basis state inputs

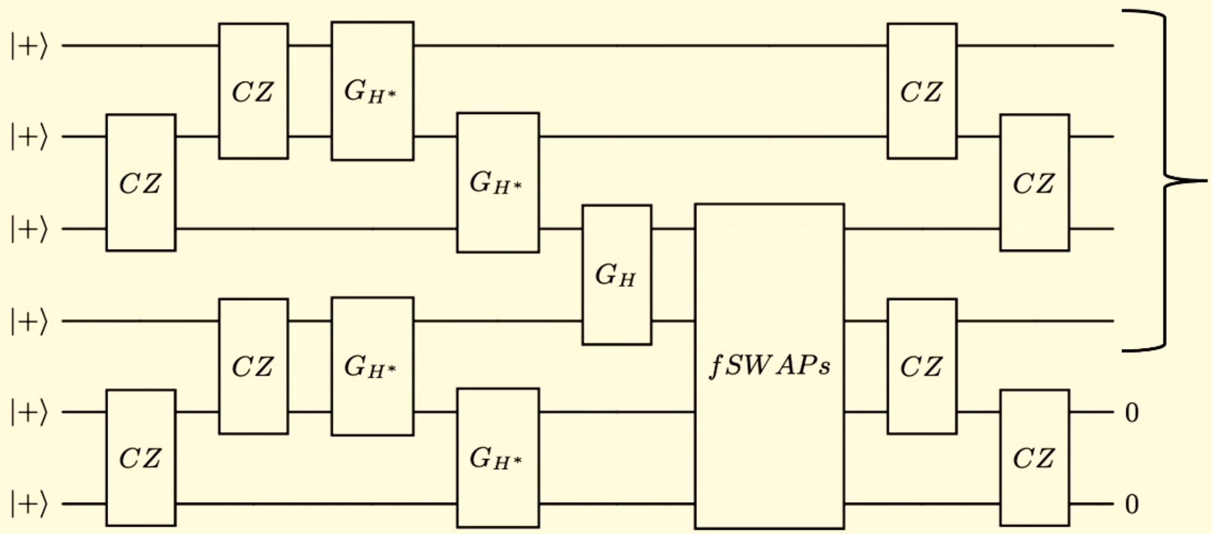
$$\begin{pmatrix} c_1^{X,Y} \\ c_2^{X,Y} \\ c_3^{X,Y} \\ c_4^{X,Y} \\ \dots \end{pmatrix} = \begin{pmatrix} \{X,Y\} & p_2^1 & p_3^1 & p_4^1 & \dots \\ p_1^2 & \{X,Y\} & p_3^2 & p_4^2 & \dots \\ p_1^3 & p_2^3 & \{X,Y\} & p_4^3 & \dots \\ p_1^4 & p_2^4 & p_3^4 & \{X,Y\} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $p_i^j = Z$ or I, and if $p_i^j = Z$, $p_j^i = I$ and vice versa.

All encodings such that fermionic occupancy stored locally [**Projansky 25'**]

CZ-Conjugated Matchgates

Result: For SWAP+CZ conjguated matchgate circuits, bitstring outputs are *not* simulable on product state inputs unless quantum computing is classically simulable [**Projansky 25'**].



$|GHZ_4\rangle$
Magic state for matchgates

Cliffords Following Matchgates

$$\langle \psi_{prod} | U_{MG}^{\dagger} C^{\dagger} p C U_{MG} | \psi_{prod} \rangle$$

$$\downarrow$$

$$\langle \psi_{prod} | U_{MG}^{\dagger} p' U_{MG} | \psi_{prod} \rangle$$

Maps to matchgate expectation value trivially

$$\begin{aligned} [C(\rho)]_{jk} &= -\frac{i}{2} \mathrm{Tr}([c_j', c_k'] C^{\dagger} U_{\mathrm{MG}} |\mathbf{0}\rangle\langle\mathbf{0}| U_{\mathrm{MG}}^{\dagger} C) \\ &= -\frac{i}{2} \mathrm{Tr}(C^{\dagger} [c_j, c_k] C C^{\dagger} U_{\mathrm{MG}} |\mathbf{0}\rangle\langle\mathbf{0}| U_{\mathrm{MG}}^{\dagger} C) \\ &= -\frac{i}{2} \mathrm{Tr}([c_j, c_k] U_{\mathrm{MG}} |\mathbf{0}\rangle\langle\mathbf{0}| U_{\mathrm{MG}}^{\dagger}) \end{aligned}$$

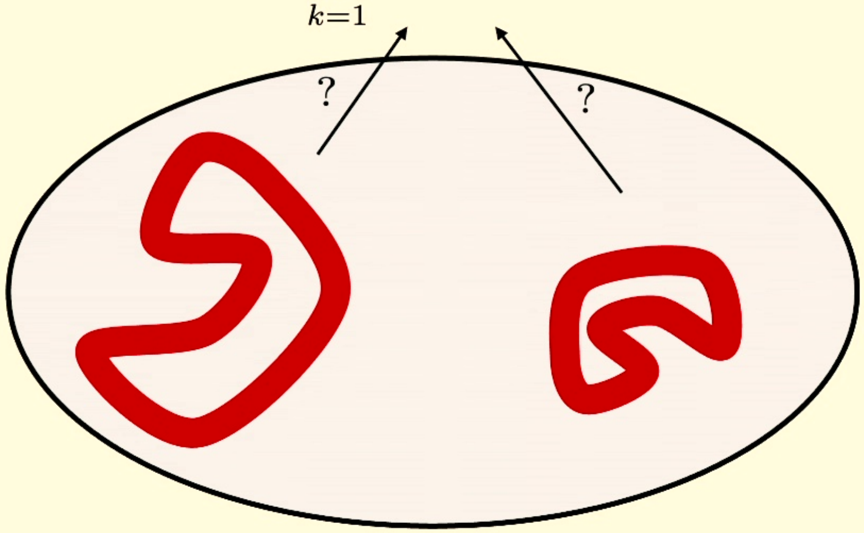
Can instead re-write covariance matrix with respect to conjugated majoranas

Perspective Shift?

Gaussianity: State $|\psi\rangle$ is Gaussian if and only if [7]

$$\sum_{k=1}^{2n} c_k \otimes c_k |\psi\rangle|\psi\rangle = 0$$

Hilbert space over N qubits



What states are Gaussian?

[7]: *Lagrangian representation for fermionic linear optics*, Bravyi, 2004

Perspective Shift?

Observation: *Any* state is Gaussian if we allow for our Majoranas c_i to be complicated enough. For $|\psi\rangle = U|00..0\rangle$,

$$\sum_{k=1}^{2n} c_k^* \otimes c_k^* |\psi\rangle|\psi\rangle = (U \otimes U) \sum_{k=1}^{2n} c_{k_{JW}} \otimes c_{k_{JW}} (U^\dagger \otimes U^\dagger)(U \otimes U)|00...0\rangle|00...0\rangle$$

$$= (U \otimes U) \left[\sum_{i=1}^{2n} c_{k_{JW}} \otimes c_{k_{JW}} |00...0\rangle|00...0\rangle \right]$$

$$= 0$$

All states are Gaussian if we allow for arbitrarily complex Majoranas

Perspective Shift?

Place restriction: All majoranas need to be single Paulis. Then shifted perspective gives two corollaries

Corollary 1: $CU_{MG}|\psi_{prod}\rangle$ are the set of all states in Hilbert space such that they are Gaussian with respect to some choice of anticommuting Paulis [**Projansky 25'**].

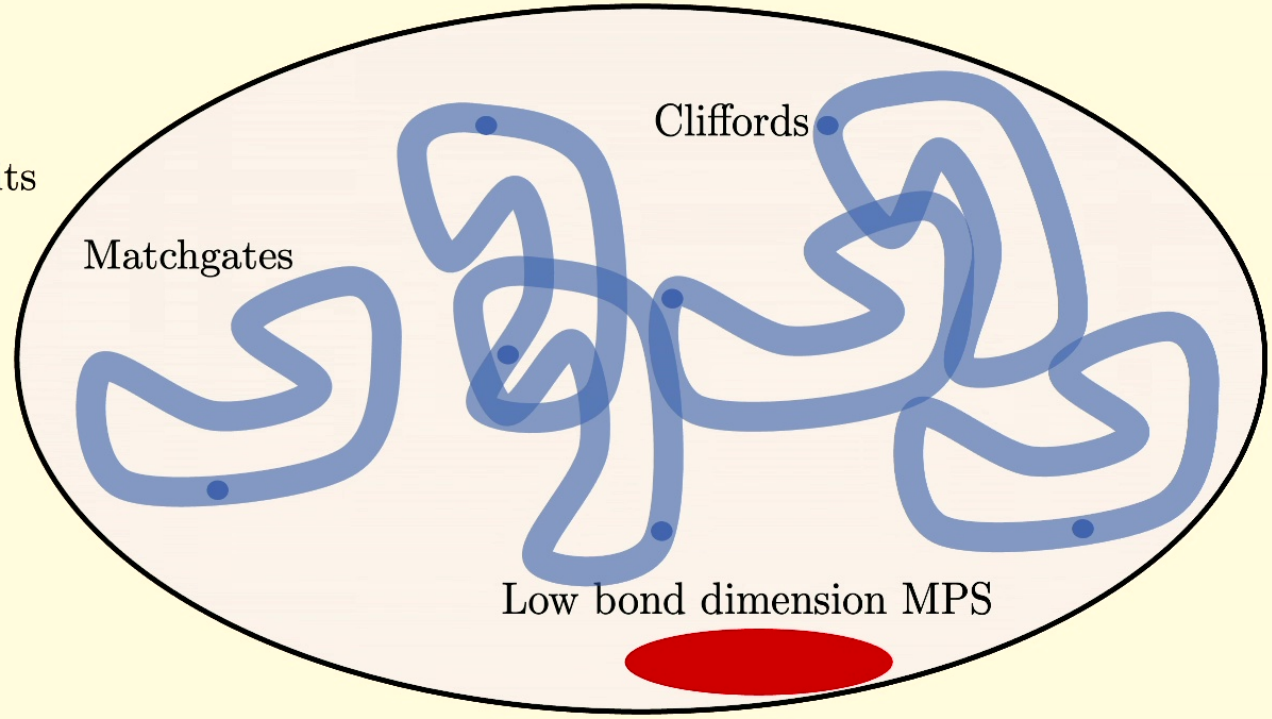
Corollary 2: For all stabilizer states $|\psi_{stab}\rangle$, there exist choice of $2n$ mutually anticommuting Paulis such that [**Projansky 25'**]

$$\sum_{i=1}^{2n} c_k \otimes c_k |\psi_{stab}\rangle|\psi_{stab}\rangle = 0$$

All Stabilizers are Gaussian!

Gaussianity and Simulability

Hilbert space over N qubits



We have access to more of Hilbert space than we did with just matchagtes/Cliffords alone... we have *all* Gaussian states!

Thank you!

- The Jordan-Wigner encoding is special
- Product states for most Clifford conjugated matchgate circuits are resourceful
- All stabilizer states are Gaussians states

This work was supported by the Office of Science, Office of Advanced Scientific Computing Research under programs Fundamental Algorithmic Research for Quantum Computing and Optimization, Verification, and Engineered Reliability of Quantum Computers project.

Quantum Information Science at Dartmouth

U.S. DEPARTMENT OF ENERGY

Questions? Email andrew.m.projansky.gr@dartmouth.edu