Title: The sewing-factorization theorem for C_2-cofinite VOAs

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Abstract:

In this talk, I will present a sewing-factorization theorem for conformal blocks in arbitrary genus associated to a (possibly nonrational) C_2-cofinite VOA V. This result gives a higher genus analog of Huang-Lepowsky-Zhang's tensor product theory. Moreover, I will explain the relation between our result and pseudotraces, and confirm some of the conjectures by Gainuditnov-Runkel. The relationship between our result and coends will also be discussed. The talk is based on an ongoing project (arXiv: 2305.10180, 2411.07707) joint with Bin Gui.

The sewing-factorization theorem for C_2-cofinite VOAs

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Joint work with Bin Gui
arXiv:2305.10180, 2411.07707

The category $\mathrm{Rep}(\mathbb{V})$

- We start with a C_2-cofinite and self-dual vertex operator algebra $\mathbb{V}$, which is not necessarily rational.
- $\mathrm{Rep}(\mathbb{V})$, the tensor category of (grading-restricted generalized) $\mathbb{V}$-modules defined by Huang-Lepowsky-Zhang, is a Grothendieck-Verdier category (Allen, Lentner, Schweigert, Wood 2021). $\mathrm{Rep}(\mathbb{V})$ is not necessarily semisimple, but is conjectured to be rigid.
- The tensor product of $\mathrm{Rep}(\mathbb{V})$ is denoted by $\boxtimes$, called fusion product. $\otimes$ means the usual tensor product over $\mathbb{C}$.
- The Deligne product $\mathrm{Rep}(\mathbb{V}) \otimes^{\mathsf{Del}} \mathrm{Rep}(\mathbb{V})$ is equivalent to $\mathrm{Rep}(\mathbb{V} \otimes \mathbb{V})$ (McRae 2023) with a bi-functor $\otimes : \mathrm{Rep}(\mathbb{V}) \times \mathrm{Rep}(\mathbb{V}) \to \mathrm{Rep}(\mathbb{V} \otimes \mathbb{V})$.

Conformal blocks

- Choose an N-pointed compact Riemann surface with local coordinates $\mathfrak{X} = (C; x_1, \cdots, x_N; \eta_1, \cdots, \eta_N)$, or equivalently, a Riemann surface with N boundary circles. Associate a $\mathbb{V}^{\otimes N}$-modules $\mathbb{W}$ to $x_1, \cdots, x_N$. A **conformal block** is a linear map $\psi : \mathbb{W} \to \mathbb{C}$ invariant under certain action of $\mathbb{V}$ and $\mathfrak{X}$ on $\mathbb{W}$. The spaces of conformal blocks is denoted by $CB(\mathfrak{X}, \mathbb{W})$, or

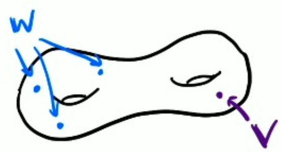
$CB($  $)$

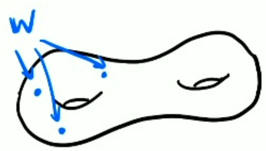
- In particular, you may choose $\mathbb{W} = \mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N$. In general, $\mathbb{W}$ cannot be expressed as a tensor product of $\mathbb{V}$-modules.

Propagation of conformal blocks

Theorem (Propagation of CB (Zhu 94))

The linear map $\phi \mapsto \tilde{\phi}$ *defined by* $\tilde{\phi}(w) = \phi(w \otimes \mathbf{1})$ *gives an isomorphism*

$CB($ 

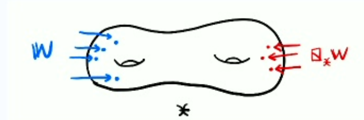
$) \xrightarrow{\simeq} CB($ 

$)$

Propagation of CB allows us to add points with vacuum inputs to conformal blocks freely.

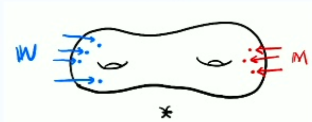
Higher genus dual fusion products

Theorem (Gui-Z. 23, arXiv:2305.10180)

Let $\mathfrak{X}$ be an $(N+L)$-pointed surface with N incoming points and L outgoing points. Let $\mathbb{W}$ be a $\mathbb{V}^{\otimes N}$-module. Then there exists a $\mathbb{V}^{\otimes L}$-module $\boxbslash_{\mathfrak{X}}\mathbb{W}$ and $\omega_{\mathfrak{X}} \in CB($ 

$)$ satisfying:

for any $\mathbb{V}^{\otimes L}$-module $\mathbb{M}$, the linear map

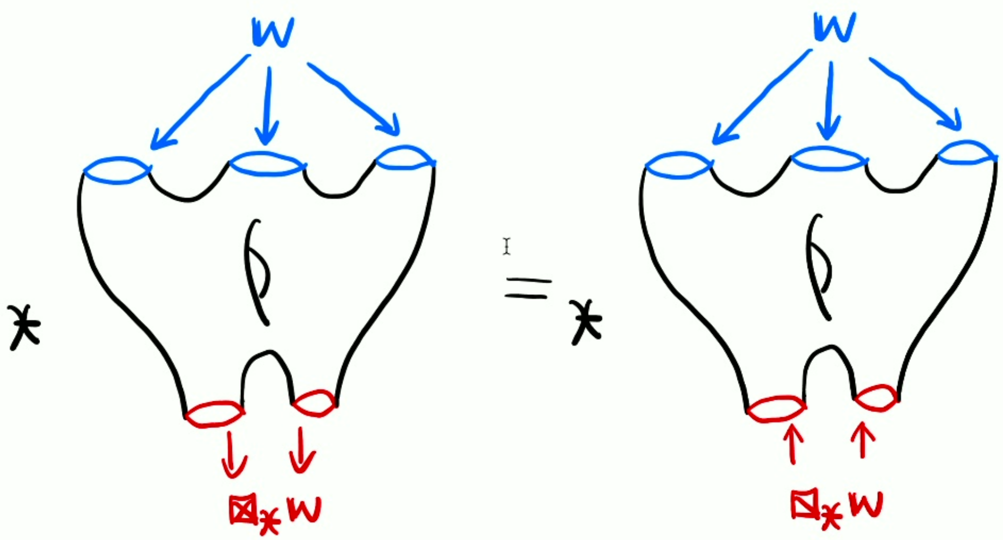
$\mathrm{Hom}_{\mathbb{V}^{\otimes L}}(\mathbb{M}, \boxbslash_{\mathfrak{X}}\mathbb{W}) \to CB($ 

$)$ *given by*

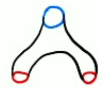
$\varphi \mapsto \omega_{\mathfrak{X}} \circ (\mathbf{1} \otimes \varphi)$ *is an isomorphism.*

$\boxbslash_{\mathfrak{X}}\mathbb{W}$ is called **dual fusion product**. $\omega_{\mathfrak{X}}$ is called **canonical conformal block**. $\boxtimes_{\mathfrak{X}}\mathbb{W} = (\boxbslash_{\mathfrak{X}}\mathbb{W})'$ is called **fusion product**.

A useful picture



Geometric realization of coends

Write $\mathfrak{P} =$  and $\mathfrak{Q} =$  .

Theorem (Gui-Z. to appear)

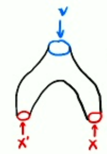
- $\boxtimes_{\mathfrak{P}} : \mathrm{Rep}(\mathbb{V} \otimes \mathbb{V}) \to \mathrm{Rep}(\mathbb{V})$ *is the lift of* $\boxtimes : \mathrm{Rep}(\mathbb{V}) \times \mathrm{Rep}(\mathbb{V}) \to \mathrm{Rep}(\mathbb{V})$ *to the Deligne product.*
- $\boxbslash_{\mathfrak{Q}}(\mathbb{V}) = \int^{\mathbb{X}} \mathbb{X}' \otimes \mathbb{X} \in \mathrm{Rep}(\mathbb{V} \otimes \mathbb{V})$.
- $\boxtimes_{\mathfrak{P}}(\boxbslash_{\mathfrak{Q}}(\mathbb{V})) = \int^{\mathbb{X}} \mathbb{X}' \boxtimes \mathbb{X} := L \in \mathrm{Rep}(\mathbb{V})$.

If $\mathbb{V}$ is in addition rational, then $\boxbslash_{\mathfrak{Q}}(\mathbb{V}) = \bigoplus_{\mathbb{X} \in \mathrm{Irr}} \mathbb{X}' \otimes \mathbb{X}$ and $L = \bigoplus_{\mathbb{X} \in \mathrm{Irr}} \mathbb{X}' \boxtimes \mathbb{X}$.

Dinatural transformation of coends

- By the universal property of dual fusion products and propagation, we have an isomorphism

$$\mathrm{End}_{\mathbb{V}}(\mathbb{X}) \simeq CB\Big($$



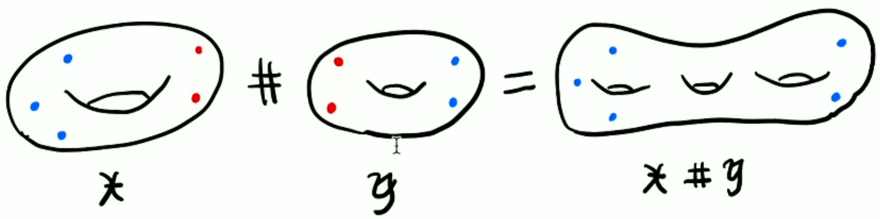
$$\Big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathbb{X}' \otimes \mathbb{X}, \boxbslash_{\mathfrak{Q}}\mathbb{V})$$

for each $\mathbb{X} \in \mathrm{Rep}(\mathbb{V})$.

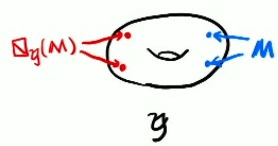
- The identity map of $\mathbb{X}$ corresponds to a morphism $\iota_{\mathbb{X}} : \mathbb{X}' \otimes \mathbb{X} \to \boxbslash_{\mathfrak{Q}}\mathbb{V}$ in $\mathrm{Rep}(\mathbb{V} \otimes \mathbb{V})$.
- Applying to functor $\boxtimes_{\mathfrak{P}} : \mathrm{Rep}(\mathbb{V} \otimes \mathbb{V}) \to \mathrm{Rep}(\mathbb{V})$, we get a morphism $\iota_{\mathbb{X}} : \mathbb{X}' \boxtimes \mathbb{X} \to L$ in $\mathrm{Rep}(\mathbb{V})$.

Towards sewing-factorization theorem

- Let $\mathfrak{X}$ be an $(N+L)$-pointed surface and $\mathfrak{Y}$ be an $(L+K)$-pointed surface. We can sew $\mathfrak{X}$ and $\mathfrak{Y}$ to get $\mathfrak{X}\#\mathfrak{Y}$, which is an $(N+M)$-pointed surface.



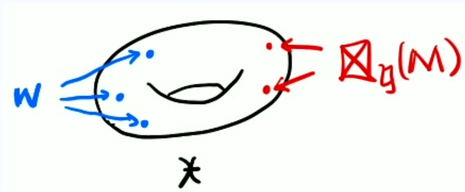
- Choose a $\mathbb{V}^{\otimes K}$-module $\mathbb{M}$ and canonical conformal block

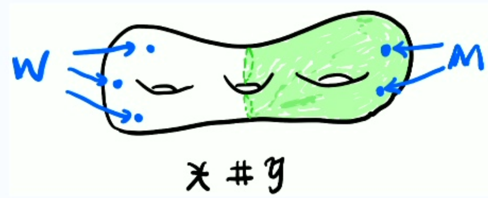
$\omega_{\mathfrak{Y}} \in CB($  $)$, $\quad \omega_{\mathfrak{Y}} : \boxbslash_{\mathfrak{Y}}(\mathbb{M}) \otimes \mathbb{M} \to \mathbb{C}$

Sewing-factorization (SF) theorem

Theorem (SF theorem, Gui-Z. to appear)

'Sewing conformal blocks' $\psi \mapsto \psi\#\omega_{\mathfrak{Y}}$ *gives an isomorphism*

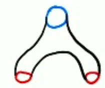
$CB($ 

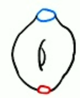
$) \xrightarrow{\simeq} CB($ 

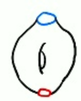
$)$

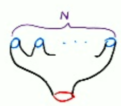
It's highly nontrivial that sewing conformal blocks is convergent and hence well-defined. This is proved in my joint paper (arXiv:2411.07707) with Gui. The convergence of pseudo q-traces (Miyamoto 04', Fiordalisi 16') corresponds to the convergence of sewing conformal blocks in this theorem.

SF theorem for coends

Recall that $\mathfrak{P} =$  and $\mathfrak{Q} =$  .

- SF theorem implies that $\boxtimes$  $(\mathbb{V}) \simeq \boxtimes_{\mathfrak{P}}(\boxtimes_{\mathfrak{Q}}(\mathbb{V}))$
- *Assume that* $\mathrm{Rep}(\mathbb{V})$ *is rigid.* We can prove that $\boxtimes_{\mathfrak{Q}}(\mathbb{V})$ is self-dual and

$\boxtimes$  $(\mathbb{V}) \simeq L$

- Write $\mathfrak{P}_N =$  so that $\mathfrak{P}_2 = \mathfrak{P}$. This gives a functor $\boxtimes_{\mathfrak{P}_N} : \mathrm{Rep}(\mathbb{V}^{\otimes N}) \to \mathrm{Rep}(\mathbb{V})$.

Let $\mathfrak{X}$ be an N-pointed surface with genus g and associate a $\mathbb{V}^{\otimes N}$-module $\mathbb{W}$ to the points of $\mathfrak{X}$.

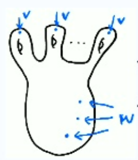
Theorem (Gui-Z. to appear, motivated by Fuchs-Schweigert)

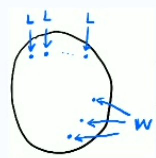
Assume that $\mathrm{Rep}(\mathbb{V})$ *is rigid. We have an isomorphism*

$$CB(\mathfrak{X}, \mathbb{W}) \simeq \mathrm{Hom}_{\mathbb{V}}\Big(L^{\boxtimes g} \boxtimes \big(\boxtimes_{\mathfrak{P}_N}(\mathbb{W})\big), \mathbb{V}\Big).$$

Proof.

By SF theorem and propagation, $CB(\mathfrak{X}, \mathbb{W})$ is isomorphic to

$CB($ 

$) \simeq CB($ 

$) \simeq \mathrm{Hom}_{\mathbb{V}}\Big(L^{\boxtimes g} \boxtimes \big(\boxtimes_{\mathfrak{P}_N}(\mathbb{W})\big), \mathbb{V}\Big).$

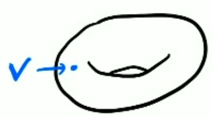
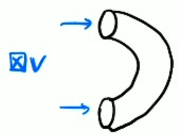
□

Genus 1 CB and symmetric linear functionals

Recall the $\mathbb{V}^{\otimes 2}$-module $\boxtimes_{\mathfrak{Q}}\mathbb{V}$ with left and right actions. They descend to a well-defined multiplication of $\boxtimes_{\mathfrak{Q}}\mathbb{V}$. This makes $\boxtimes_{\mathfrak{Q}}\mathbb{V}$ a non-unital associative algebra. From now on we omit the subscript $\mathfrak{Q}$ of $\boxtimes_{\mathfrak{Q}}\mathbb{V}$. SLF means symmetric linear functionals.

Corollary (Gui-Z. to appear)

We have a canonical SF isomorphism

$CB($  $) \xrightarrow{\simeq} CB($  $) = SLF(\boxtimes\mathbb{V}).$

Left $\boxtimes\mathbb{V}$-modules and $\mathrm{Rep}(\mathbb{V})$

- Choose a $\mathbb{V}$-module $\mathbb{M}$. Recall that we have $\iota_{\mathbb{M}} : \mathbb{M} \otimes \mathbb{M}' \to \boxbslash\mathbb{V}$ given by the dinatural transformation. Its transpose gives a linear map $\boxtimes\mathbb{V} \to \mathbb{M} \otimes \mathbb{M}' \simeq \mathrm{End}^0(\mathbb{M})$, where $\mathrm{End}^0(\mathbb{M})$ is the algebra of "finite rank" linear operators of $\mathbb{M}$. One can show that this linear map is an algebra homomorphism. Thus, $\mathbb{M}$ gives rise to a left $\boxtimes\mathbb{V}$-module $\mathfrak{F}(\mathbb{M})$.
- One can show that if $\mathbb{M}$ is a projective generator in $\mathrm{Rep}(\mathbb{V})$, then the homomorphism $\boxtimes\mathbb{V} \to \mathbb{M} \otimes \mathbb{M}' \simeq \mathrm{End}^0(\mathbb{M})$ is faithful. Therefore, we can use the algebraic structure on $\mathrm{End}^0(\mathbb{M})$ to give an explicit characterization of the algebraic structure on $\boxtimes\mathbb{V}$.

$\mathrm{Coh}_L(\boxtimes\mathbb{V}) \simeq \mathrm{Rep}(\mathbb{V})$

- Recall that we assume that $\mathbb{M}$ is grading restricted. We can see $\mathfrak{F}(\mathbb{M})$ is a coherent left $\boxtimes\mathbb{V}$-module in the sense of following.
- A left $\boxtimes\mathbb{V}$-module is called **quasicoherent** if it is a quotient module of $\bigoplus_{i\in I}(\boxtimes\mathbb{V})e_i$, where e_i are idempotents of $\boxtimes\mathbb{V}$. A quasicoherent left $\boxtimes\mathbb{V}$-module is called **coherent** if it is finitely generated. The category of quasicoherent (resp. coherent) left $\boxtimes\mathbb{V}$-modules is denoted as $\mathrm{QCoh}_L(\boxtimes\mathbb{V})$ (resp. $\mathrm{Coh}_L(\boxtimes\mathbb{V})$).

Theorem (Gui-Z. to appear)

$\mathrm{Coh}_L(\boxtimes\mathbb{V})$ *is closed under taking quotient and quasicoherent submodules. Moreover, the functor* $\mathfrak{F} : \mathrm{Rep}(\mathbb{V}) \to \mathrm{Coh}_L(\boxtimes\mathbb{V}), \mathbb{M} \mapsto \mathfrak{F}(\mathbb{M})$ *is an equivalence of abelian categories.*

Towards pseudotraces

- Choose a projective generator $\mathbb{G} \in \mathrm{Rep}(\mathbb{V})$ and set $B := \mathrm{End}_{\boxtimes\mathbb{V},-}(\mathbb{G})^{op} = \mathrm{End}_{\mathbb{V}}(\mathbb{G})^{op}$, which is a finite dimensional unital associative algebra. One can show that $\mathbb{G}$ is projective as a right B-module.
- Therefore, the pseudotrace construction gives us a linear map $SLF(B) \to SLF(\boxtimes\mathbb{V})$, and also a linear map $SLF(\boxtimes\mathbb{V}) \to SLF(B)$.

Theorem (Gui-Z. to appear)

The above linear maps $SLF(B) \to SLF(\boxtimes\mathbb{V})$ and $SLF(\boxtimes\mathbb{V}) \to SLF(B)$ defined by pseudotraces are inverse to each other.

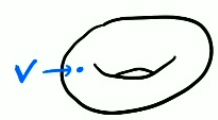
Pseudotraces and genus 1 conformal blocks

When $\boxtimes\mathbb{V}$ is replaced by a finite dimensional unital algebra, the above theorem is due to Beliakova-Blanchet-Gainutdinov 18. We can show that it is still true for $\boxtimes\mathbb{V}$.

Theorem (Gui-Z. to appear. Conjectured by Gainutdinov-Runkel 16)

The combination of the SF isomorphism and the pseudotrace construction (for associative algebras) provides a linear isomorphism of the following spaces

$$CB(\quad\quad) \simeq SLF(\mathrm{End}_{\mathbb{V}}(\mathbb{G}))$$



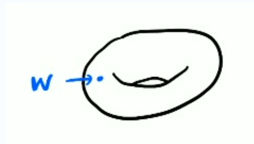
defined by pseudotraces.

Pseudotraces and genus 1 conformal blocks

The previous theorem can be generalized. Using the trick of square zero extension by Fiordalisi and Huang, we can prove:

Theorem (Gui-Z. to appear)

Suppose that $\mathbb{W}$ is a $\mathbb{V}$-module. The combination of the SF isomorphism and the pseudotrace construction (for associative algebras) provides a linear isomorphism of the following spaces

$CB($ 

$)$

$$\simeq \{\varphi : \mathrm{Hom}_{\mathbb{V}}(\mathbb{G}, \mathbb{W} \boxtimes \mathbb{G}) \to \mathbb{C} | \varphi((1 \boxtimes y)T) = \varphi(Ty),$$
$$\forall y \in \mathrm{End}_{\mathbb{V}}(\mathbb{G}), \forall T \in \mathrm{Hom}_{\mathbb{V}}(\mathbb{G}, \mathbb{W} \boxtimes \mathbb{G})\}.$$

Thank you for listening to my talk!