Title: Lecture - Quantum Field Theory III - PHYS 777 (extra Lecture)

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Subject: Quantum Fields and Strings

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Recap Last time - conf. transformations in 2d.

Equations: $\tilde{x}^\mu = x^\mu + \epsilon^\mu \Rightarrow \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \omega \eta_{\mu\nu} \quad \exists \omega$

$\Rightarrow \boxed{\frac{\partial \epsilon}{\partial \bar{z}} = 0} \quad \epsilon = \epsilon^1 + i\epsilon^0, \quad z = x^1 + i x^0$

- Most non-infinitesimal are non-invertible. Only invertible: $f(z) = \frac{az+b}{cz+d}$

  Global: $SL(2,\mathbb{C}) \cong SO(1,3)$ (cf. $SO(1,d+1)$)

- Infinitesimal generators: $l_n = -z^{n+1}\partial_z, \quad \bar{l}_n = -\bar{z}\partial_{\bar{z}}$

  $[l_n, l_m] = (n-m)\, l_{n+m}$

  $[\bar{l}_n, \bar{l}_m] = (n-m)\, \bar{l}_{n+m}$

Fields → Quasiprimary – transform covariantly under global conf. transf

$\cdot\ \tilde{\Phi}(e^{i\varphi}z, e^{-i\varphi}\bar{z}) = e^{-is\varphi}\phi(z,\bar{z})$ under $SO(2)$ – rotations

s – spin of ϕ

$\cdot\ \tilde{\Phi}(e^{\sigma}z, e^{\sigma}\bar{z}) = e^{-\Delta\sigma}\phi(z,\bar{z})$ under $\mathbb{R}_{>0}$ – dilatations

Δ – scaling dimension.

→ Primary fields $\tilde{\Phi}(w(z), \bar{w}(\bar{z})) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z})$

$(h,\bar{h}) = \left(\frac{\Delta+s}{2}, \frac{\Delta-s}{2}\right)$ – conformal weights (dimensions).

Remark e.g. $T_{\mu\nu}$ is quasi – but not primary $w = e^{i\varphi+\sigma}z$ – global transf.

Infinitesimally: $\delta_\epsilon\phi(z) = \tilde{\Phi}(z) - \phi(z) = \left[-(h\partial\epsilon + \epsilon\partial) - (\bar{h}\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial})\right]\phi(z,\bar{z})$

$w = z + \epsilon(z)$

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$w = z + \epsilon(z)$ $\qquad \partial = \frac{\partial}{\partial z} \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}$

Noether theorem 1-parametric family of symmetries $\Rightarrow$ conserved current

generator $\partial_\mu j^\mu = 0$ (on EOM)

Symmetry: $\delta_\omega \phi = \phi'(x) - \phi(x) = -i\omega\, G\phi(x)$

parameter $\in \mathbb{R}$.

$\Rightarrow S[\phi(x) - i\omega(G\phi)(x)] - S[\phi(x)] = O(\omega^2)$ (without EOM)

To derive j^μ let's do some trick. $\omega = \omega(x)$ for now.

$\delta S = S[\phi(x) - i\omega(x)\,(G\phi)(x)] - S[\phi(x)] = \int A[\phi](-i\omega G\phi) - \int j^\mu \partial_\mu \omega + O(\omega^2)$

(under assump. that $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$)

$=0$ since $\omega = $ const $\Rightarrow$ symmetry.

EOM $\Rightarrow \delta S = 0 \Rightarrow \int j^\mu \partial_\mu \omega = 0$

$\Rightarrow \boxed{\partial_\mu j^\mu = 0}$ (because $\omega = \omega(x)$)

$z + \epsilon(z)$ $\partial = \frac{\partial}{\partial z}$ $\bar\partial = \frac{\partial}{\partial \bar z}$

The general expression for the conserved current

$$j^\mu = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu\phi - \delta^\mu_\nu \mathcal{L}\right)\frac{\delta x^\nu}{\delta\omega} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta\mathcal{F}}{\delta\omega}$$

where $\tilde\phi(\tilde x) = \mathcal{F}(\phi,\omega)$, $\tilde x = \tilde x(x,\omega)$

On the quantum level — Ward identities

Lets $X = \phi_1(x_1)\dots\phi_n(x_n)$, and we have 1-parametric family of symm.

$\langle X\rangle = \frac{1}{Z}\int[d\phi]\, X \exp(-S[\phi])$, let's do change of variables in the path integral $\phi(x)\longrightarrow\tilde\phi(x) = \phi(x) - i\omega G\phi$

$X \longrightarrow X + \delta X$, $S[\phi]\longrightarrow S[\phi] - \int d^dx\, \omega(x)\partial_\mu j^\mu(x)$ $[d\tilde\phi] = [d\phi]$

$\Rightarrow \boxed{\partial_\mu j^\mu = 0}$ (because $\omega = \omega(x)$)

Expanding in ω.

$$-i \sum_{i=1}^{n} \omega(x_i)\, G_i \langle X \rangle = \int d^d x \; \omega(x)\, \partial_\mu \langle j^\mu(x)\, X \rangle$$

$$\parallel$$

$$-i \int d^d x \; \omega(x) \sum_{i=1}^{n} \delta(x - x_i)\, G_i \langle X \rangle$$

ω - any $\Rightarrow$ $$\boxed{\partial_\mu \langle j^\mu(x)\, X \rangle = -i \sum_{i=1}^{n} \delta(x - x_i)\, G_i \langle X \rangle}$$ - Ward identity.

$\Rightarrow x \neq x_i : \langle \partial_\mu j^\mu(x)\, X \rangle = 0$

$X \to X + \delta X$, $S[\Phi] \to S[\Phi] - \int d^d x\, \underline{\omega(x)}\, \partial_\mu j^\mu(x)$ $[d\Phi] = [d\Phi]$

Now let's apply this to conformal symmetry:

$\tilde{z} = z + \epsilon(z)$, $j^\mu = T^{\mu\nu} \epsilon_\nu(z)$

$\mathrm{tr}\, T = 0$ $\quad T^{\mu\nu} = T^{\nu\mu}$

<u>Remark</u> This can be rewritten as.

$$\partial_\mu(\epsilon_\nu T^{\mu\nu}) = \underbrace{\epsilon_\nu \partial_\mu T^{\mu\nu}}_{\text{local translation}} + \underbrace{\tfrac{1}{2}(\partial_\beta \epsilon^\beta)\eta_{\mu\nu} T^{\mu\nu}}_{\text{local dilatation}} + \underbrace{\tfrac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha \epsilon_\beta\, \epsilon_{\mu\nu} T^{\mu\nu}}_{\text{local rotation}}$$

<u>Remark</u> Stokes theorem in complex coordinates.

$$\int_{D\subset\mathbb{C}} d^2x\, \partial_\mu F^\mu = \int_{\partial D} dn_\mu F^\mu = \int_{\partial D} dx^\nu \varepsilon_{\nu\mu} F^\mu = \frac{i}{2}\int_{\partial D} \left(-dz\, F^{\bar{z}} + d\bar{z}\, F^{z}\right)$$

The Ward id. for conformal transformations: $(\Sigma = \Sigma)$

$$\delta_\epsilon \langle X \rangle = \int d^2x \, \partial_\mu \langle T^{\mu\nu}(x) \, \epsilon_\nu(x) \, X \rangle$$

variation under conf transform for all fields not only primary

$$= \frac{i}{2} \int_{\partial\Sigma} -dz \, \epsilon_z \langle T^{\bar z z} X \rangle - dz \, \epsilon_{\bar z} \langle T^{\bar z \bar z} X \rangle + d\bar z \, \epsilon_{\bar z} \langle T^{z \bar z} X \rangle + d\bar z \, \epsilon_z \langle T^{zz} X \rangle$$

let's use that T is symmetric, traceless and conserved on a quantum level up to "boundary terms":

$$\langle T_{z\bar z} X \rangle \sim \delta$$

$$\langle (\partial_z T_{\bar z \bar z} + \partial_{\bar z} T_{zz}) X \rangle \sim \sum \delta^{(k)}(z - z_i) \# \langle X \rangle, \quad \langle T_{\bar z z} X \rangle \sim \delta$$

$$\Rightarrow \langle \partial_z T_{\bar z \bar z} X \rangle \sim \delta \qquad \langle \partial_{\bar z} T_{zz} X \rangle \sim \delta$$

$$\Rightarrow \quad \delta_{\epsilon,\bar{\epsilon}}\langle X\rangle = -\frac{1}{2\pi i}\oint_C \epsilon(z)\langle T(z)X\rangle dz + \frac{1}{2\pi i}\oint_C \bar{\epsilon}\,\langle \bar{T}(\bar{z})X\rangle d\bar{z}$$

where $\epsilon = \epsilon_z(z)$, $\bar{\epsilon} = \epsilon_{\bar{z}}(\bar{z})$, $T = -2\pi T_{zz}(z)$

$\bar{T} = -2\pi T_{\bar{z}\bar{z}}(\bar{z})$

contour C includes all the insertion points $\{z_i\}$

— conformal Ward identity.

What happens with primary fields? $\delta_\epsilon \phi = -(\epsilon \partial \phi + h \partial \epsilon \cdot \phi) - (\bar{\epsilon} \bar{\partial} \phi + \bar{h} \bar{\partial} \bar{\epsilon} \phi)$

This is consistent with Ward id. iff:

$$\langle T(z) X \rangle = \sum_{i=1}^{n} \left\{ \frac{1}{z - z_i} \partial_{z_i} \langle X \rangle + \frac{h_i}{(z - z_i)^2} \langle X \rangle \right\} + \text{regular in } (z - z_i)$$

What happens with stres-energy tensor? $T(z) = -2\pi T_{zz}$ $\quad \Delta = 2,\ s = 2 \Rightarrow h = 2,\ \bar{h} = 0$

In fact, on a quantum level, T appears to be not a primary.

To see this we can revers- engeneer Ward identities.

this is an example of Operator Product Expansion (OPE):

$$T(z)\cdot\phi_h(w) = \frac{1}{z-w}\partial_w\phi(w) + \frac{h}{(z-w)^2}\phi(w) + \text{reg.} \rightarrow$$

OPE with T encodes transformation properties of ϕ under conformal transforms.

$$T(z)\,T(w) \sim \frac{c/2}{(z-w)^4} + \cancel{\frac{A(w)}{(z-w)^3}} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}$$

c - central charge. It is a quantum correction which makes T to be only quasiprimary