Title: The geometry and combinatorics of cosmological integrals

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Collection/Series: Quantum Fields and Strings

Subject: Quantum Fields and Strings

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Abstract:

In this talk, I will describe how ideas from geometry and combinatorics can help us understand the mathematical and physical properties of cosmological integrals. From efficiently deriving canonical differential equations to a systematic method for finding a minimal basis for the physical subspace. Time permitting, I will also comment on how geometry and combinatorics control the zeros of cosmological integrands.

BROWN

A physical basis for cosmological correlators from cuts

Andrzej Pokraka

Based 2411.09695 with Shounak De

Perimeter Institute: Strings and QFT seminar

Geometry ∩ Amplitudes ∩ cosmology

Algebraic Geometry

(Relative) Twisted Cohomology

Intersection Theory

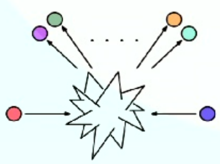
$$\int_\gamma \underbrace{T_1^{\mathbb{C}\setminus\mathbb{Z}}\cdots T_n^{\mathbb{C}\setminus\mathbb{Z}}}_{\substack{u \\ \text{multi-valued function}}} \times \underbrace{\Big(\text{holomorphic top form}\Big)}_{\text{on } \mathbb{C}^k\setminus V(T_1\cdots T_n)\setminus\text{poles}}$$

Positive Geometry

$$\gamma \in H_n \mapsto \Omega[\gamma] \in H^n$$

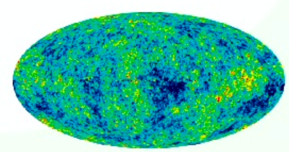
dlog-form

Cosmological Integrals



Scattering Amplitudes

Make predictions for colliders (LHC)



Wavefunction Coefficients & Cosmological Correlators

Statistical patterns in CMB/LSS

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Outline

- Introduction
  - What are FRW wavefunction coefficients
  - Introduce toy model of FRW cosmology
  - How to compute FRW wavefunction coefficients?
  - Twisted cohomology and an over counting problem
- More twisted cohomology and intersection theory
  - *Relative* twisted cohomology $\supset$ cuts
  - Cuts provide good organizing principles
- Physical, degenerate and unphysical cuts
- Diagrammatic algorithm
- Discussion and outlook

Cosmological correlators

Time-evolution/causality presents unique challenges for developing quantum theories of cosmologies

It is not well understood how causal time evolution is reflected in the boundary observables at infinity

Can causal time evolution and locality be emergent phenomena? [Nima]

[N. Arkani-Hamed, P. Benincasa, A. Postnikov (2017)] : for toy models of cosmology, wavefunction coefficients are the canonical form of a positive geometry called the cosmological polytope

Defined without reference to Feynman rules (with causality + locality backed in)

The toy model

Conformally-coupled scalar field in a power-law FRW cosmology with non-conformal polynomial interactions in $(d+1)$-dimensional spacetime

[Arkani-Hamed&Maldacena '15, AH&Benincasa '17, AH&Hillman '19, + many more]

$$S[\phi] = \int \mathrm{d}^d x\, \mathrm{d}\eta \left[\frac{1}{2}(\partial\phi)^2 - \sum_{k\geq 3} \frac{\lambda_k(\eta)}{k!}\phi^k \right]$$

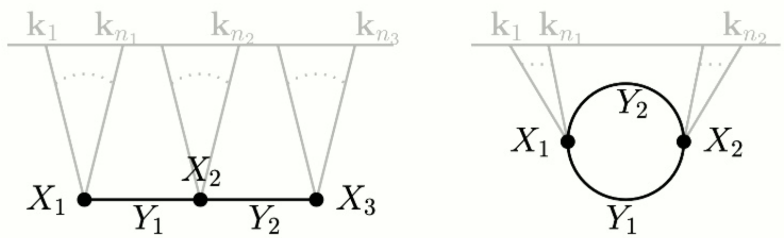
$$\mathrm{d}s^2 \equiv g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2(\eta)\left[-\mathrm{d}\eta^2 + \mathrm{d}x_i \mathrm{d}x^i\right]$$

$$a(\eta) = \frac{1}{\eta^{1+\varepsilon}} \qquad \begin{cases} \varepsilon = 0 & (\mathrm{dS}) \\ \varepsilon = -1 & (\mathrm{flat}) \\ \varepsilon = -2 & (\mathrm{RD}) \\ \varepsilon = -3 & (\mathrm{MD}) \end{cases}$$

$$\lambda_k(\eta) = \lambda_k\left[a(\eta)\right]^{(2-k)\frac{d-1}{2}+2}$$

$$\lambda_3(\eta)\Big|_{d=3} = \frac{\lambda_3}{\eta^{1+\varepsilon}}$$

Feynman representation



$$X_i = \sum_{m=n_{i-1}+1}^{n_i} |\mathbf{k}_m|, \quad Y_i = \left| \sum_{m=n_{i-1}+1}^{n_i} \mathbf{k}_m \right|$$

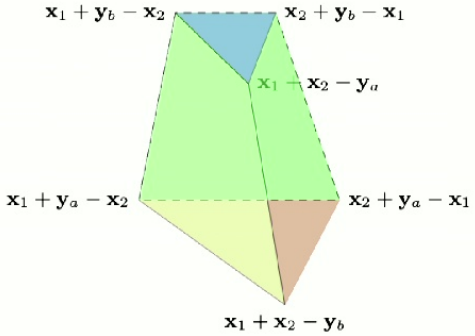
$$\psi^{(0)}_{3,\text{flat}} = \frac{X_1 + 2X_2 + X_3 + Y_1 + Y_2}{(X_1+X_2+X_3)(X_1+Y_1)(Y_1+X_2+Y_2)(Y_2+X_3)(X_1+X_2+Y_2)(Y_1+X_2+X_3)}$$

$$\psi^{(1)}_{2,\text{flat}} = \frac{2(X_1 + X_2 + Y_1 + Y_2)}{(X_1 + X_2)(X_1 + Y_1 + Y_2)(X_2 + Y_1 + Y_2)(X_1 + X_2 + 2Y_1)(X_1 + X_2 + 2Y_2)}$$

Energy is not conserved

Positive geometry and cosmological polytopes

Flat space wavefunction coefficients **are** the canonical forms of the positive geometries called cosmological polytopes [Arkani-Hamed, Benincasa, Postnikov, Baumann, Pimentel + many more]



$$\psi^{(\ell)}_{n,\mathsf{flat}} \frac{\mathrm{d}^n\mathbf{X} \wedge \mathrm{d}^{n-1+\ell}\mathbf{Y}}{\mathsf{GL}(1)} = \left(4^{n-1} Y_1 \cdots Y_{n+\ell-1}\right) \times \Omega^{(\ell)}_n$$

$\Omega^{(\ell)}_n$ defined without ever referencing “Feynman rules”

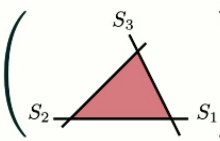
Universal integrand from a purely combinatoric-geometric origin

What is a canonical form?

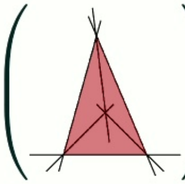
Unique differential form associated to a bounded region with logarithmic singularities on all of its boundaries

$\Omega\,\Big($ 

$\Big) = \mathrm{d}\log\dfrac{x-a}{x-b}$

$\Omega\,\Bigg($ 

$\Bigg) = \mathrm{d}\log\dfrac{S_1}{S_2}\wedge\mathrm{d}\log\dfrac{S_2}{S_3}$

$\Omega\,\Bigg($  $\Bigg) = \mathrm{d}\log\dfrac{S_1}{S_2}\wedge\mathrm{d}\log\dfrac{S_2}{S_3}\wedge\mathrm{d}\log\dfrac{S_3}{S_4}$

$$\vdots \qquad\qquad \vdots$$

$$\mathrm{Res}_{S_i}\Big[\Omega[\gamma]\Big] = \Omega\Big[\partial_{S_i}\gamma\Big]$$

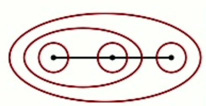
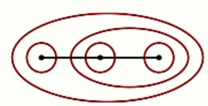
Cosmological canonical form from combinatorics of tubes

Energy factors (propagators)

$$S_i = S_{\tau_i} = \sum_{v \in \mathcal{V}_{\tau_i}} X_v + \sum_{e \in \mathcal{E}_{\tau_i}} Y_e$$

$$\Omega_n^{(\ell)} = \left(\sum_{\text{complete tubings } T} \frac{1}{\prod_{\tau \in T} S_\tau} \right) \frac{\mathrm{d}^n \mathbf{X} \wedge \mathrm{d}^{n-1+\ell} \mathbf{Y}}{\mathsf{GL}(1)}$$

3-site example:

$\Omega_3^{(0)} \propto$  $\Big|_{\frac{1}{\times}} +$  $\Big|_{\frac{1}{\times}} = \frac{1}{S_1 \cdots S_4} \left(\frac{1}{S_5} + \frac{1}{S_6} \right)$

FRW correlators ($\varepsilon \neq -1$) from flat space correlators

flat space correlators: integrate over shifted energies $\mathbf{X} \to \mathbf{x} + \mathbf{X}$

$\implies$ replace $S_i(\mathbf{X}, \mathbf{Y}) \to B_i(\mathbf{x}, \mathbf{X}, \mathbf{Y}) = S_i(\mathbf{x} + \mathbf{X}, \mathbf{Y})$

$u = (x_1 \cdots x_n)^{\varepsilon}$ universal m.v. fn. (twist)

$$\psi^{(\ell)}_{n,\mathrm{FRW}}(\mathbf{X}, \mathbf{Y}) \propto \int_0^\infty u \; \hat{\Omega}^{(\ell)}_n(\mathbf{x} + \mathbf{X}, \mathbf{Y})\, \mathrm{d}^n \mathbf{x}$$

s.v. diff. form ψ_{phys}

Too hard to directly integrate. Next best: canonical DEQ

$$\mathrm{d}_{(\mathbf{X},\mathbf{Y})}\, \boldsymbol{\varphi} = \varepsilon\, \underline{\boldsymbol{A}} \wedge \boldsymbol{\varphi} \qquad \psi_{\text{phys}} \in \text{Span}\{\varphi_a\}$$

goal

$$\implies \int_0^\infty u \varphi_a = \mathsf{bd}_a + \varepsilon \int A_{ab} \mathsf{bd}_b + \varepsilon^2 \int A_{ab} \int A_{bc} \mathsf{bd}_c + \mathcal{O}(\varepsilon^3)$$

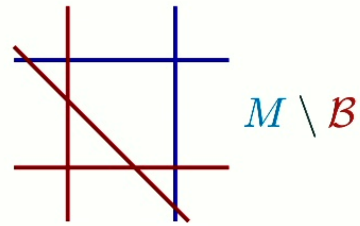
Twisted cohomology

[Mizera, Mastrolia, Frellesvig, Caron-Huot, AP, Weinzierl, Stieberger + many more]

$u = (x_1 \cdots x_n)^\varepsilon$ universal m.v. fn. (twist)

$$\psi^{(\ell)}_{n,\mathrm{FRW}}(\mathbf{X},\mathbf{Y}) = \int_0^\infty u \quad \psi_{\mathrm{phys}}$$

twisted cohomology class $\in H^n(M \setminus \mathcal{B}; \nabla)$



$$\mathbf{x} \in M = \mathbb{C}^n \setminus \; V(x_1 \cdots x_n)$$

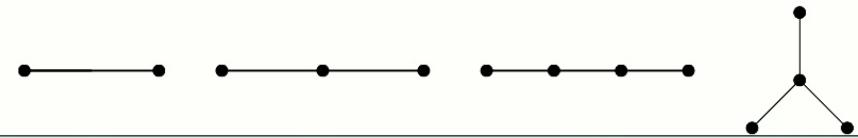
$$\mathcal{B} = \; V(B_1 \cdots B_m)$$

$$H^n_{\mathsf{dR}}(M \setminus \mathcal{B}; \nabla) := \frac{\nabla\text{-closed } n\text{-forms: } \{\varphi | \nabla\varphi = 0\}}{\nabla\text{-exact } (n-1)\text{-forms: } \{\nabla\psi\}}$$

$$\nabla = \mathrm{d} + \varepsilon \, \mathrm{d}\log(x_1 \cdots x_n)\wedge$$

\# of masters: $\dim H^n_{\mathsf{dR}}(M \setminus \mathcal{B}; \nabla) = |\chi(M \setminus \mathcal{B})|$

Severe overcounting



χ	4	25	213	312
4^{n-1}	4	16	64	64

Kinematic flow: choice of a 4^{n-1}-dimensional basis whose differential equation closes [Arkani-Hamed, Baumann, Hillman, Joyce, Lee, Pimentel '24; He, Jiang, Liu, Yang, Zhanga '24]

Elements of the DEQ predicted by emperical graphical rules

How do we define the physical sector invariantly? Why does this exist?

Goal: explain mechanism for splitting between physical and unphysical subspaces and how to find a basis for the physical subspace

What is relative twisted cohomology?

Dualize: turn untwisted singular surfaces $\mathcal{B}$ into boundaries [AP, Caron-Huot]

$$\check{H}^n := H^n(M, \mathcal{B}; \check{\nabla}) \subseteq \bigoplus_{p=0}^{n} \bigoplus_{J:|J|=p} \delta_J \left(H^{n-p}(M_J; \check{\nabla}) \right) ,$$

Sum over each boundary/cut cohomology

Boundaries = cut surfaces: $M_J = M \cap \mathcal{B}_J$ where $\mathcal{B}_J = \cap_{j \in J} V(B_j)$

How to represent dual forms?

$$\check{H}^n \ni \check{\varphi} = \sum_J \delta_J \left(\check{\phi}_J \right) \quad \text{with} \quad \check{\phi}_J \in H^{n-p}(M_J; \check{\nabla})$$

$$\delta_J \iff \circlearrowleft_J \colon \langle \delta_J(\check{\phi}) | \varphi \rangle = \langle \check{\phi} | \mathrm{Res}_J[\varphi] \rangle_J$$

$$\mathrm{d}\delta_J(\check{\phi}) = (-1)^{|J|} \Big(\delta_J(\mathrm{d}\check{\phi}) + \sum_{i \notin J} \delta_{Ji}(\check{\phi}|_i) \Big), \quad \delta_{\cdots ij \cdots} = -\delta_{\cdots ji \cdots}$$

Intuitive explanation of intersection theory

$\exists$ inner product on space of twisted differential forms

compactification step - skip

$$\langle \check{\varphi} | \varphi \rangle = \int_M [\check{\varphi}]_c \wedge \varphi = \sum_{\mathbf{z}^*} \mathrm{Res}_{\mathbf{z}=\mathbf{z}^*} [\, \check{\psi}_{\mathbf{z}^*} \, \varphi]$$

dual forms $\check{\varphi} \in \check{H}^n$ seeds for residue operators

FRW forms	dual FRW forms
mild twisted poles $V(x_1 \cdots x_n)$	mild twisted poles $V(x_1 \cdots x_n)$
dangerous un-twisted poles $V(S_1 \cdots S_m)$	boundaries $V(S_1 \cdots S_m)$
$\sim$ Feynman integrals	$\sim$ cuts of Feynman integrals

$$\text{Cuts: } \frac{1}{p^2 - m^2} \to \theta(p^0)\delta(p^2 - m^2)$$

Relative twisted cohomology and positive geometry

canonical form on cut/boundary M_J

Dual basis: $$\{\check{\varphi}_a\}_{a=1}^{|\chi(M\setminus\mathcal{B})|} = \bigcup_J \left\{ \delta_J(\, \Omega_{J,k}\,) \right\}_{k=1}^{|\chi(M_J)|}$$

$\tilde{\Omega}_{J,k}|_J = \Omega_{J,k}$

FRW basis: $$\{\varphi_a\}_{a=1}^{|\chi(M\setminus\mathcal{B})|} = \bigcup_J \left\{ \mathrm{d}\log_J \wedge\, \tilde{\Omega}_{J,k} \right\}_{k=1}^{|\chi(M_J)|}$$

$$\mathrm{d}\log_J := \bigwedge_{j\in J} \mathrm{d}\log B_j$$

$$\mathrm{dim} H^n(M\setminus\mathcal{B};\nabla) = |\chi(M\setminus\mathcal{B})| = \sum_J |\chi(M_J)| = \mathrm{dim} H^n(M,\mathcal{B};\check{\nabla})$$

Duality of bases

For FRW $\mathrm{d}\log$-forms, the intersection number simplifies to

$$\langle \delta_J(\check{\phi}) | \varphi \rangle = \sum_{\boldsymbol{z}^*_J \in \mathsf{Int}_J} \underbrace{\frac{\mathsf{Res}_{\boldsymbol{z}^*_J = \boldsymbol{0}}[\check{\phi}]}{\varepsilon^{n-|J|}}}_{\check{\psi}_{\boldsymbol{z}^*_J}} \mathsf{Res}_{\boldsymbol{z}^*_J = \boldsymbol{0}}[\underset{\downarrow}{\mathsf{Res}_J}\,[\varphi]]$$

generalized unitary cuts

Easy to check that the intersection matrix is block diagonal on cuts

$$\langle \delta_I(\Omega_{I,k}) | \mathrm{d}\log_J \wedge \tilde{\Omega}_{J,l} \rangle = \delta_{IJ} \langle \Omega_{J,k} | \Omega_{J,l} \rangle_J$$

Physical cuts have a 1-dimensional cohomology $\Longleftrightarrow$ physical sector of intersection matrix is diagonal

Compact formula for DEQ

Compact formula for DEQ

$$\mathrm{d}_{(\mathbf{X},\mathbf{Y})}\boldsymbol{\varphi} = \underline{\boldsymbol{A}} \wedge \boldsymbol{\varphi}$$
$$A_{ab} = C^{-1}_{bb'} \langle \check{\varphi}_{b'} | [\mathrm{d}_{(\mathbf{X},\mathbf{Y})}\varphi_a] \rangle$$
$$C_{ab} = \langle \check{\varphi}_a | \varphi_b \rangle$$

Reduces to residue calculation

$$\langle \delta_J(\check{\phi}) | [\mathrm{d}_{(\mathbf{X},\mathbf{Y})}\varphi_a] \rangle = \sum_{\boldsymbol{z}^*_J \in \mathrm{Int}_J} \frac{\mathrm{Res}_{\boldsymbol{z}^*_J = \mathbf{0}}[\check{\phi}]}{\varepsilon^{n-|J|}} \mathrm{Res}_{\boldsymbol{z}^*_J = \mathbf{0}}[\mathrm{Res}_J [\mathrm{d}_{(\mathbf{X},\mathbf{Y})}\varphi_a]]$$

Cut structure of DEQs

$$\mathrm{d}_{(\mathbf{X},\mathbf{Y})}\varphi_a \simeq \varepsilon\, \mathsf{D}\log(x_1\cdots x_n) \wedge \mathsf{D}\log_J \wedge \left(\tilde{\Omega}_{J,k}|_{\mathrm{d}\log\to\mathsf{D}\log}\right)\Big|_{\mathrm{d}X_a\wedge\mathrm{d}X_b,\cdots\to 0}$$

$$\mathsf{D} = \mathrm{d} + \mathrm{d}_{(\mathbf{X},\mathbf{Y})} \qquad \varphi_a = \mathrm{d}\log_J \wedge \tilde{\Omega}_{J,k}$$

$\mathrm{d}_{(\mathbf{X},\mathbf{Y})}$ cannot introduce new singularities on untwisted singular loci $\mathcal{B}$
$\implies \mathsf{Res}_K\varphi_a = 0 \implies \mathsf{Res}_K[\mathrm{d}_{(\mathbf{X},\mathbf{Y})}\varphi_a] = 0$

Recalling

$$\langle\delta_J(\check{\phi})|[\mathrm{d}_{(\mathbf{X},\mathbf{Y})}\varphi_a]\rangle = \sum_{\boldsymbol{z}^*_J\in\mathsf{Int}_J} \frac{\mathsf{Res}_{\boldsymbol{z}^*_J=\mathbf{0}}[\check{\phi}]}{\varepsilon^{n-|J|}} \mathsf{Res}_{\boldsymbol{z}^*_J=\mathbf{0}}[\mathsf{Res}_J[\mathrm{d}_{(\mathbf{X},\mathbf{Y})}\varphi_a]]$$

$\implies \psi_{\text{phys}}$ only couples to basis forms that share cuts

The answer?

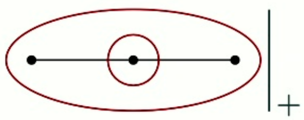
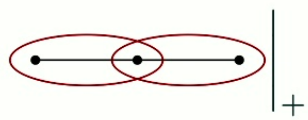
Answer seems to be: divide cuts into physical cuts ($\text{Res}_J[\psi_{\text{phys}}] \neq 0$) and unphysical ($\text{Res}_J[\psi_{\text{phys}}] = 0$) cuts

$$H^n_{\text{phys}} = \bigcup_{J:\text{Res}_J[\psi_{\text{phys}}]\neq 0} \left\{ \mathrm{d}\log_J \wedge \tilde{\Omega}_J \right\}$$

One more complication

Cosmological polytope $\iff$ arrangement of un-twisted hyperplanes B_i is degenerate

Linear relations among the B_i:

 $=$  $\iff B_2 + B_4 = B_5 + B_6$

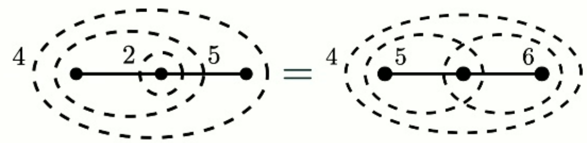
Degenerate cuts

$J \in \{(2,4,5),(2,4,6),(2,5,6),(4,5,6)\}$ all correspond to the same cut space $M_{245} = M_{246} = M_{256} = M_{456} = M_{2456}$

Residues do not necessarily anti-commute: $\text{Res}_{56i}[\psi_{\text{phys}}] = 0$ but $\text{Res}_{i56}[\psi_{\text{phys}}] \neq 0$ for $i = 2, 4$

Satisfy additional linear relations

$$\text{Res}_{4,5,2} = \text{Res}_{4,5,6} \iff$$



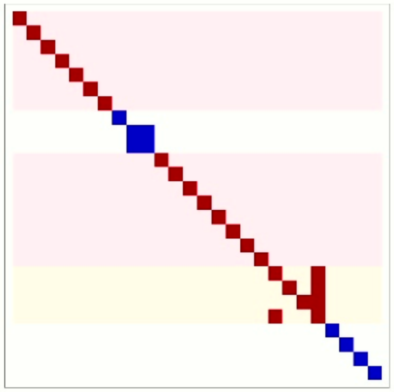
Detected by intersection matrix: $\langle \delta_J(\Omega_{J,j}) | \mathrm{d}\log_K \wedge \tilde{\Omega}_{K,k} \rangle$ does not have full rank

Orlik-Solomon algebra: after picking residue ordering (big to small), generates all residue relations

The pedestrian way forward

The intersection matrix is not full rank $\mathrm{rank}(\underline{\boldsymbol{C}}') = 25 = \chi(M \setminus \mathcal{B})$

$\underline{\boldsymbol{C}} =$



where $\underline{\boldsymbol{C}}_{\mathsf{degen}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$

Drop δ_{256} and $\mathrm{d}\log_{256}$ to get basis: $\check{\boldsymbol{\varphi}}' = (\check{\boldsymbol{\varphi}} \setminus \{\delta_{256}\})$ and $\boldsymbol{\varphi}' = (\boldsymbol{\varphi} \setminus \{\mathrm{d}\log_{256}\})$

Physical sector ∩ degenerate boundaries

Want to decompose degenerate block into a physical and unphysical sectors by constructing a gauge transformation $\check{\varphi}'' = \underline{\check{U}} \cdot \check{\varphi}'$

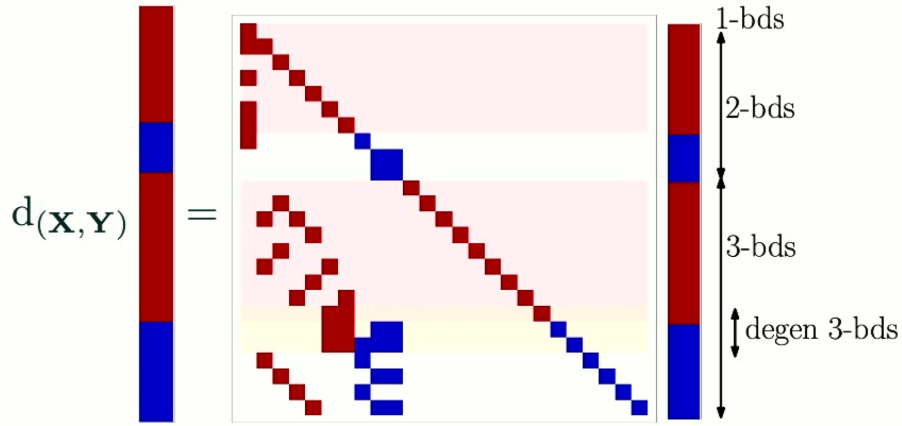
For the 3-site example:
$\mathbf{v} = (\mathsf{Res}_{452}[\psi_{\text{phys}}], \mathsf{Res}_{462}[\psi_{\text{phys}}], \mathsf{Res}_{562}[\psi_{\text{phys}}]) = (1, -1, 0)$

$$\underline{\check{U}} = \begin{pmatrix} \mathbb{1}_{18\times18} & & \\ & \mathbf{v}_{1\times3} & \\ & (\mathsf{Null}[\mathbf{v}])_{2\times3} & \\ & & \mathbb{1}_{4\times4} \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{18\times18} & & & & \\ & 1 & -1 & 0 & \\ & 0 & 0 & 1 & \\ & 1 & 1 & 0 & \\ & & & & \mathbb{1}_{4\times4} \end{pmatrix}$$

3-site DEQs

Apply gauge transformation to FRW forms: $\varphi'' = \underline{\boldsymbol{U}} \cdot \varphi'$ with $\underline{\boldsymbol{U}} = (\underline{\check{\boldsymbol{U}}} \cdot \underline{\boldsymbol{C}}')^{-1\top}$ so that $\underline{\boldsymbol{C}}'' = \mathbb{1}$

The DEQ for φ'':



Physical subspace has been isolated: $\mathrm{d}_{(\mathbf{X},\mathbf{Y})}\varphi_a^{\text{phys}} \in$ Span$\{\varphi_b^{\text{phys}}\}$

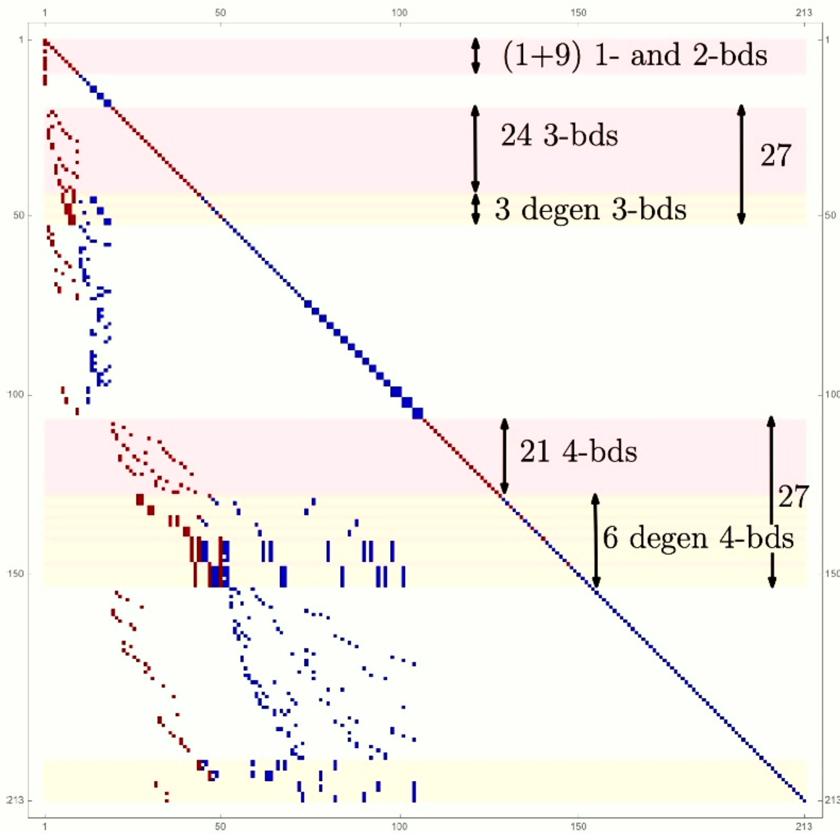
Couting the physical rows one finds that dim$H^n_{\text{phys}}(M \setminus \mathcal{B}; \nabla) = 16$

4-site chain: •—•—•—•

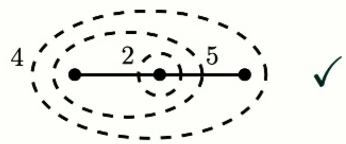
$\chi(M \setminus \mathcal{B}) = 213$ but

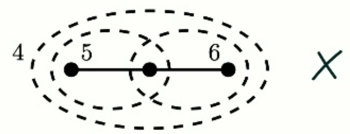
$\dim H^n_{\text{phys}} = 64 = 4^{4-1} = 1_{\text{1-cut}} + 9_{\text{2-cuts}} + 27_{\text{3-cuts}} + 27_{\text{4-cuts}}$

$\underline{\boldsymbol{A}}_{\text{4-chain}} =$



Graphical rules for physical cuts (at $\ell = 0$)

1) No crossed tubes: 



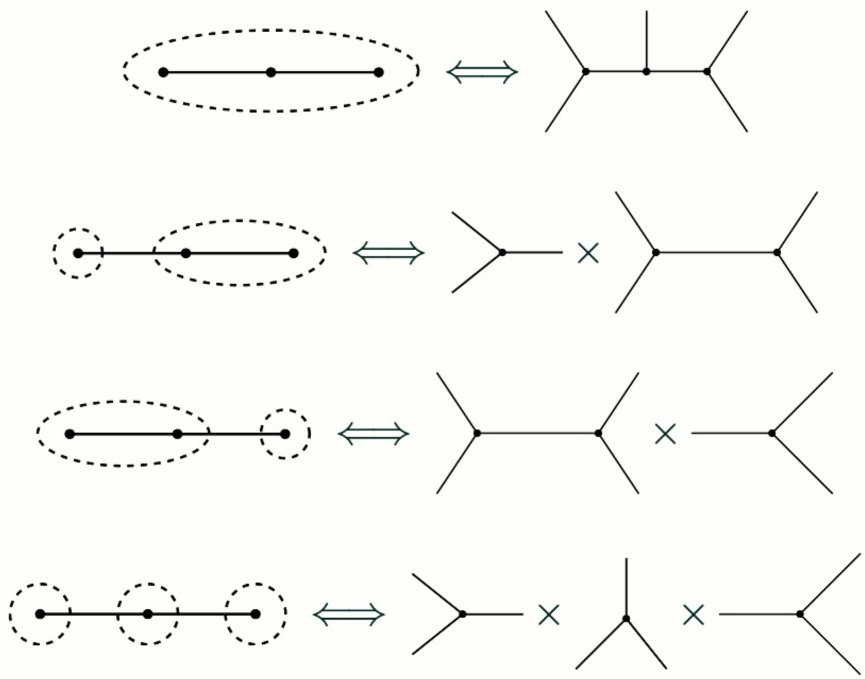
2) All vertices must be encircled at least once (for non-trivial cohomology)

3) $(k > 1)$-residue-tubes must contain one free vertex (for non-trivial cohomology)

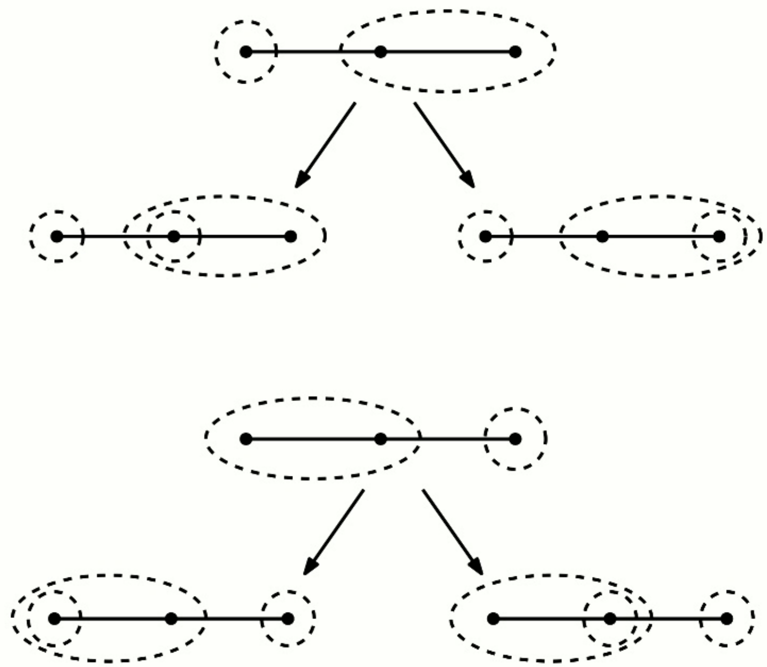
4) A cut is degenerate if 3 out of the 4 factors in a linear relation appear as cut tubes

5) Each block of degenerate cuts counts once

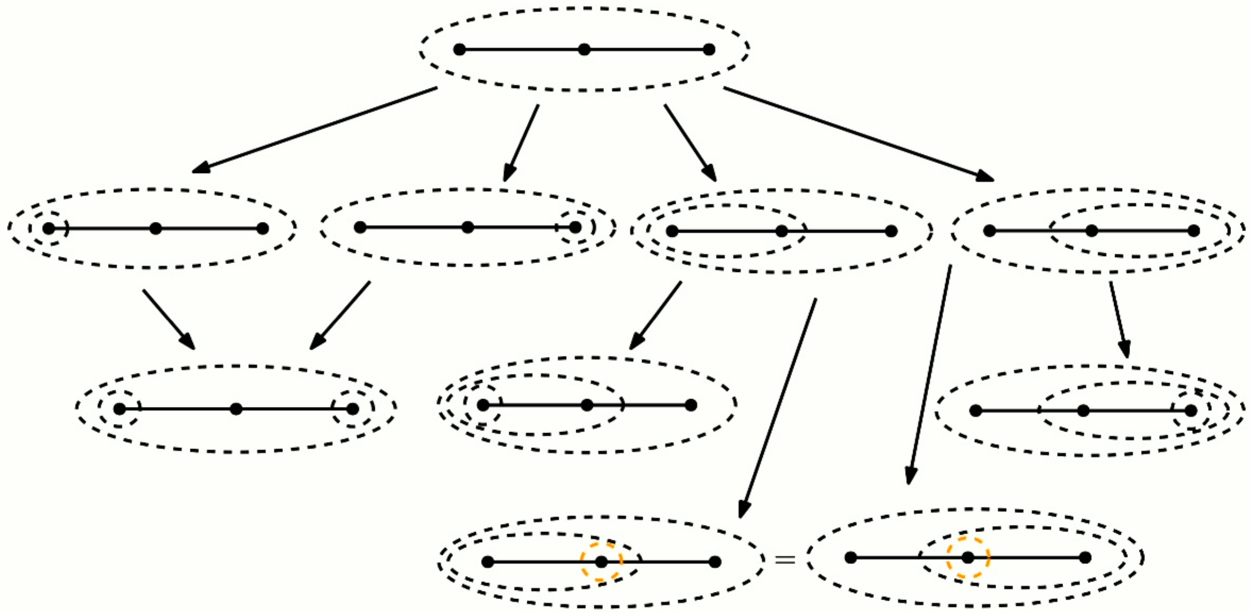
Graphical rules for physical cuts



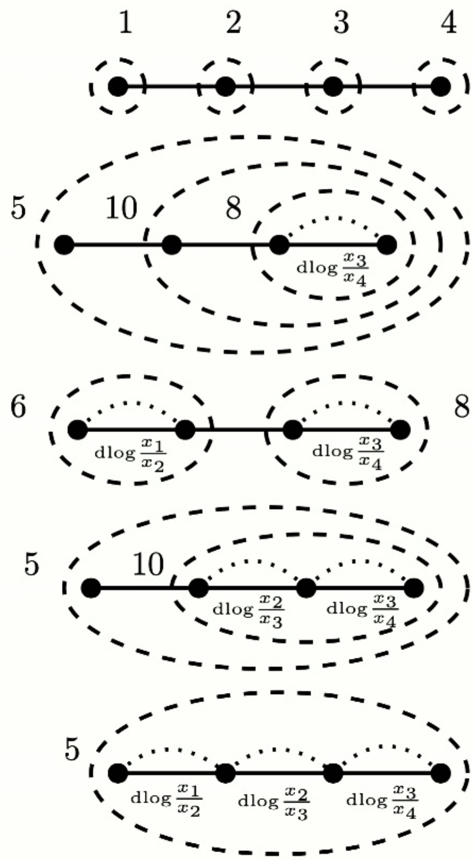
Graphical rules for physical cuts



Graphical rules for physical cuts



From physical cuts to forms



$$= \mathrm{d}\log_{1,2,3,4} = \mathrm{d}\log B_1 \wedge \cdots \wedge \mathrm{d}\log B_4$$

$$= \mathrm{d}\log_{5,10,8} \wedge \mathrm{d}\log \frac{x_3}{x_4}$$

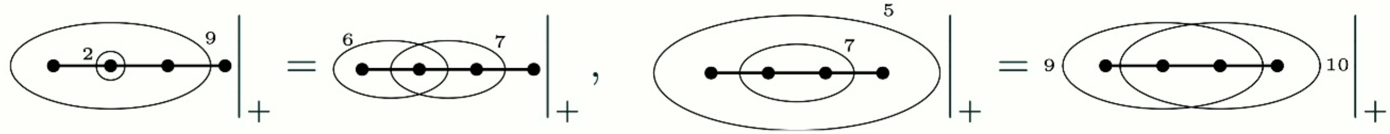
$$= \mathrm{d}\log_{6,8} \wedge \mathrm{d}\log \frac{x_1}{x_2} \wedge \mathrm{d}\log \frac{x_3}{x_4}$$

$$= \mathrm{d}\log_{5,10} \wedge \mathrm{d}\log \frac{x_2}{x_3} \wedge \mathrm{d}\log \frac{x_3}{x_4}$$

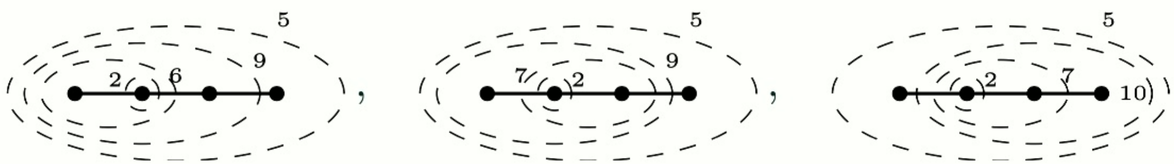
$$= \mathrm{d}\log_{5} \wedge \mathrm{d}\log \frac{x_1}{x_2} \wedge \mathrm{d}\log \frac{x_2}{x_3} \wedge \mathrm{d}\log \frac{x_3}{x_4}$$

From degenerate cuts to physical forms

Degenerate block of the 4-site chain generated by the linear relations



This block of 4-cuts has 3 that are non-crossing



Combine such that there is a relative sign between tubes that cross

$$
\begin{aligned}
&\mathrm{d}\log_{5,9,6,2} - \mathrm{d}\log_{5,9,7,2} + \mathrm{d}\log_{5,10,7,2} \\
&= \mathrm{d}\log B_5 \wedge \mathrm{d}\log \frac{B_6}{B_7} \wedge \mathrm{d}\log B_7 \wedge \mathrm{d}\log B_3 + \mathrm{d}\log_{5,10,7,3} \\
&= \mathrm{d}\log_{5,9,6,2} + \mathrm{d}\log B_5 \wedge \mathrm{d}\log \frac{B_{10}}{B_9} \wedge \mathrm{d}\log B_7 \wedge \mathrm{d}\log B_2
\end{aligned}
$$

Counting

$\ell = 0$	1-cuts	2-cuts	3-cuts	4-cuts	5-cuts	6-cuts	$\cdots$
	3^0	3^1	3^2	3^3	3^4	3^5	$\cdots$
$n=2$	1	1					
$n=3$	1	2	1				
$n=4$	1	3	3	1			
$n=5$	1	4	6	4	1		
$n=6$	1	5	10	10	5	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

$\ell = 0$	1-cuts	2-cuts	3-cuts	4-cuts	$\cdots$
$n=2$ (bubble)	3	7			
$n=3$ (triangle)	4	21	25		
$n=4$ (box)	5	42	100	79	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Independent of topology and agrees with [Arkani-Hamed, Baumann, Hillman, Joyce, Lee, Pimentel '24; He, Jiang, Liu, Yang, Zhanga '24]

Discussion and outlook

Relative twisted cohomology is a good mathematical framework for understanding FRW integrals: organizes space by cuts

Generalizes straightforwardly to loop-level time integrals

The physical cuts factor into flat space amplitudes (at tree level)

Interesting combinatorial formula for the counting of basis elements that agrees with empirical observations

Discussion and outlook

Relative twisted cohomology can be used to construct a coaction on these integrals [WIP with L. Ren and A. McLeod]

- connection to number theory
- insight into the kind of functions that can appear

Insights into the zeros of FRW integrals

[WIP with S. De, S. Paranjape, M. Spradlin, A. Volovich]

Predict zeros from

- limits of certain graph associahedra
- the factorizations of the adjoint polynomial
  (numerator of a canonical form with geometric interpretation)