Title: Running EFT-hedron with null constraints at loop level

Speakers: Long-Qi Shao

Collection/Series: Particle Physics

Subject: Particle Physics

Date: March 18, 2025 - 1:00 PM

URL: https://pirsa.org/25030169

Abstract:

I will introduce how to incorporate loop contributions into positivity bounds in a massless scalar theory with shift symmetry. The allowed region of Wilson coefficients is called the EFT-hedron. With loop contributions, the n=4 null constraint, which plays an important role in determining the allowed region for g3–g4 (the Wilson coefficients of dimension-10 and dimension-12 operators, respectively), is modified. This modification of the null constraint reveals new features of the EFT-hedron. We obtained a much stronger bound for g2 (the Wilson coefficient of dimension-8 operators) without using the full unitarity condition. Furthermore, we found a negative g4, which is not possible under the tree-level bound. It would be interesting to generalize our method to more complicated theory and test our results.

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Running EFT-hedron with null constraints at loop level

speaker： Long-Qi Shao

Based on arXiv： 2501.09717 in collaboration with Guangzhuo Peng, Yongjun Xu and Anna Tokareva

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Integrating out and UV completion

$$\int \mathcal{D}\phi_{\rm L}\,\mathcal{D}\phi_{\rm H}\,e^{iS(\phi_{\rm L},\phi_{\rm H})} = \int \mathcal{D}\phi_{\rm L}\,e^{iS_\Lambda(\phi_{\rm L})}, \qquad e^{iS_\Lambda(\phi_{\rm L})} = \int \mathcal{D}\phi_{\rm H}\,e^{iS(\phi_{\rm L},\phi_{\rm H})}.$$

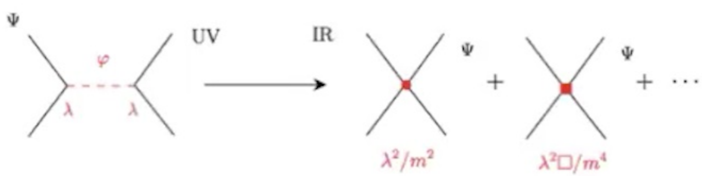
Integrating out

Make it renormalizable

$$\mathscr{L} = G_F\bar{\psi}\psi\bar{\psi}\psi \longrightarrow \mathscr{L} = G_F\bar{\psi}\psi\bar{\psi}\psi + a_1 G_F^2\bar{\psi}\psi\,\Box\,\bar{\psi}\psi + a_2 G_F^3\bar{\psi}\psi\,\Box^2\,\bar{\psi}\psi + \cdots$$

Breaks unitarity as $\Box$ increases, and perturbation theory becomes invalid.

It could be UV completed by adding a heavy particle φ into UV spectrum in the form $\mathscr{L} = \lambda\bar{\Psi}\Psi\varphi$



$$\frac{\lambda^2}{q^2 - m^2} = \frac{\lambda^2}{m^2} + \frac{\lambda^2\Box}{m^4} + \cdots$$

UV completion

Not very EFT is allowed

Positivity bounds

- The form of IR amplitude is usually known by symmetry. UV theory is unknown, but assuming UV theory has basic properties such as unitarity, causality, Lorentz invariance, crossing symmetry can give nontrivial bounds on IR Wilson coefficients through dispersion relation.

- Unitarity condition：

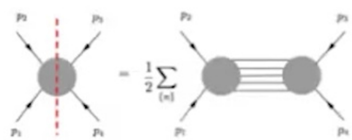
$$S^\dagger S = 1 \qquad S = 1 + iT$$

$$i(T^\dagger - T) = T^\dagger T$$

- Inserting it between the same |initial state> and |final state> — left hand side is imaginary part of scattering amplitude, right hand side is positive

$$\mathrm{Im}T \propto \mathrm{Im}A(s,t) \geq 0$$

$$A(s,t) = 16\pi \sum_{l\,\mathrm{even}} (2l+1) f_l(s)\, P_l\left(1 + \frac{2t}{s-4m^2}\right), \qquad 0 \leq |f_l|^2 \leq 2\mathrm{Im} f_l(s,t) \leq 4$$



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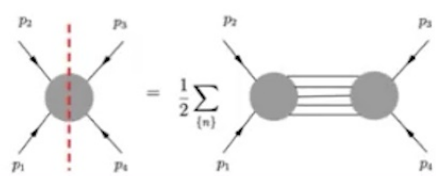
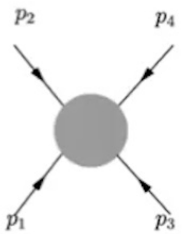


Scattering Amplitude

- For 2 to 2 scattering amplitude, there are 16 kinetic variables in 4D. 14 constraints: External particle on shell condition (4), energy momentum conservation (4), Lorentz invariance (6).

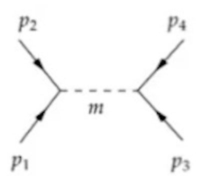


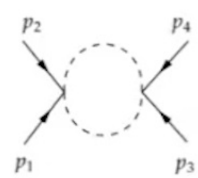
$$s = -(p_1 + p_2)^2, \quad t = -(p_1 + p_3)^2, \quad u = -(p_2 + p_3)^2$$

- Fix t, analytically continue the $A(s,t)$ into upper half plane. The causality condition implies $A(s,t)$ is analytic in upper half plane. Schwartz reflection principle

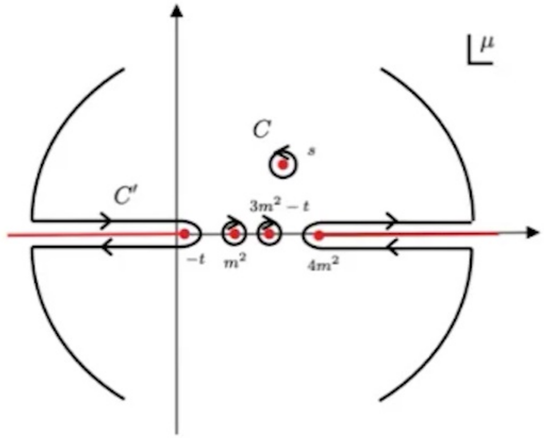
$$A(s^*, t) = A^*(s, t)$$

- Singularity appears only in real axis, a pole in $s = m^2$, a branch cut starts from $s = 4m^2$, where m is the mass of intermediate state.





Dispersion relation for massive theory



2 poles and 2 branch cut due to crossing symmetry $s + t + u = 4m^2$

$$A(s,t) = \frac{1}{2\pi i}\oint_C \frac{A(\mu,t)}{\mu - s}.$$

$$A(s,t) = \frac{Z}{m^2 - s} + \frac{Z}{m^2 - u} + \int_\infty \frac{d\mu}{2\pi i}\frac{A(\mu,t)}{\mu - s} + \int_{4m^2}^{+\infty}\frac{d\mu}{\pi}\frac{\mathrm{Disc}A(\mu,t)}{\mu - s} + \int_{-\infty}^{-t}\frac{d\mu}{\pi}\frac{\mathrm{Disc}A(\mu,t)}{\mu - s}$$

$$\mathrm{DiscA(s,t)} \equiv \frac{1}{2\mathrm{i}}[\mathrm{A(s + i\sigma, t) - A(s - i\sigma, t)}] = \mathrm{ImA(s,t)}$$

Froissart-Martin bounds： $A(s,t) \lesssim s \log^2 s$

Taking two subtractions makes infinite arc integral vanish

Phys. Rev., 1961, 123: 1053-1057
Phys. Rev., 1963, 129: 1432-1436
Phys. Rev., 1964, 135: B1375-B1377

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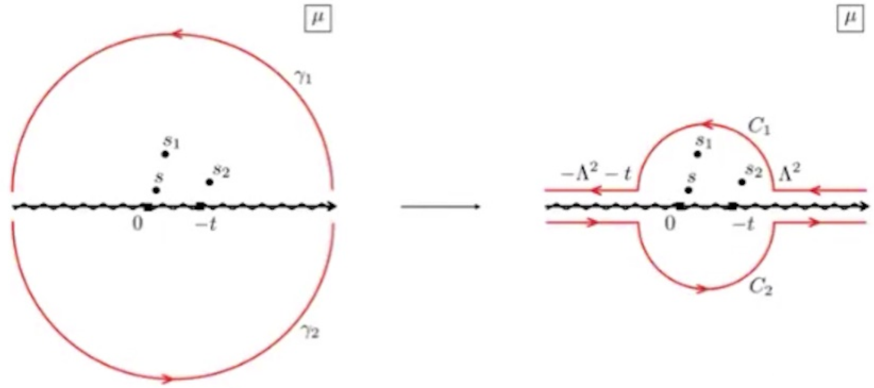
Dispersion relation in massless scalar theory



Froissart-Martin bounds：

$$A(s,t) \lesssim s \log^2 s$$

$$\int_{\gamma_1 \cup \gamma_2} \frac{d\mu}{2\pi i} \frac{A(\mu,t)}{(\mu - s_1)(\mu - s_2)(\mu - s)} = 0\,,$$

Branch cut integral

$$A_{\text{EFT}}(s,t) - sA'_{\text{EFT}}(0,t) - A_{\text{EFT}}(0,t) = s^2 \int_{\epsilon^2\Lambda^2}^{\infty} \frac{d\mu}{\pi} \frac{\text{Im}A(\mu,t)}{\mu^2(\mu - s)} + s^2 \int_{\epsilon^2\Lambda^2}^{\infty} \frac{d\mu}{\pi} \frac{\text{Im}A(\mu,t)}{(\mu+t)^2(\mu - u)}$$

Weak coupling limit assumed in addition

IR EFT

$$A_{\text{EFT}}(s,t) = A(s,t), \quad |s| < \epsilon^2\Lambda^2$$

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massless theory with shift symmetry

$$\mathscr{L}_{\mathrm{EFT}} = -\frac{1}{2}(\partial_\mu\phi)^2 + \frac{g_2}{2}[(\partial_\mu\phi)^2]^2 + \frac{g_3}{3}(\partial_\mu\partial_\nu\phi)^2(\partial_\sigma\phi)^2 + 4g_4[(\partial_\mu\partial_\nu\phi)^2]^2 + \cdots$$

$$A_{\mathrm{EFT}}(s,t) = g_2(s^2+t^2+u^2) + g_3 stu + g_4(s^2+t^2+u^2)^2 + \cdots$$

Substitute into l.h.s of dispersion relation

Do partial wave expansion on r.h.s of dispersion relation: $A(s,t) = 16\pi \sum_{\mathrm{even}\, l} (2l+1) f_l(s) P_l\left(1 + \frac{2t}{s-4m^2}\right),$

$$\langle F(\mu,t)\rangle = \sum_l 16\pi(2l+1)\int_{\epsilon^2\Lambda^2}^{+\infty} \frac{d\mu}{\pi} \mathrm{Im} f_l(\mu) F(\mu,t)$$

$$g_2 = \left\langle \frac{1}{\mu^3} \right\rangle, \quad g_3 = \left\langle \frac{3-2L^2}{\mu^4} \right\rangle, \quad g_4 = \left\langle \frac{1}{2\mu^5} \right\rangle$$

$g_2 > 0,\ g_4 > 0,$
g_3 has upper boundary

JHEP, 2006, 10: 014.

$$L^2 = l(l+1), \quad L \in \{0, 6, 20, \cdots\}$$

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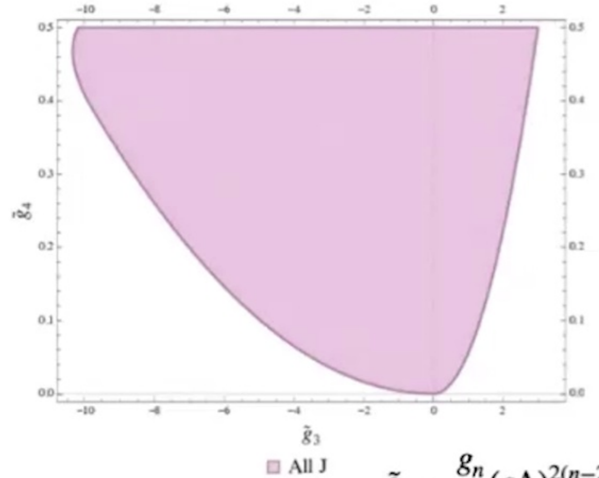
Null constraint

- s, t, u full crossing symmetry has:

$$\left.\frac{\partial^4 A}{\partial s^4}\right|_{s,t\to 0} \propto \left.\frac{\partial^4 A}{\partial s^2 \partial t^2}\right|_{s,t\to 0} \propto g_4$$

This leads to null constraint on the right hand side

$$\left\langle \frac{L^4 - 8L^2}{\mu^5} \right\rangle = 0, \quad L^2 = l(l+1)$$



$$\tilde{g}_n = \frac{g_n}{g_2}(\epsilon\Lambda)^{2(n-2)}$$

If there is spin 2 particles in the UV spectrum, there has to be higher spin particles

- Using null constraint, numerical results are given by Simon Caron-huot (SDPB), analytic result given by Yu-tin Huang etc. formed in terms of moment problem

JHEP, 2015, 06: 174
JHEP, 2021, 05: 280
JHEP, 2022, 03: 063

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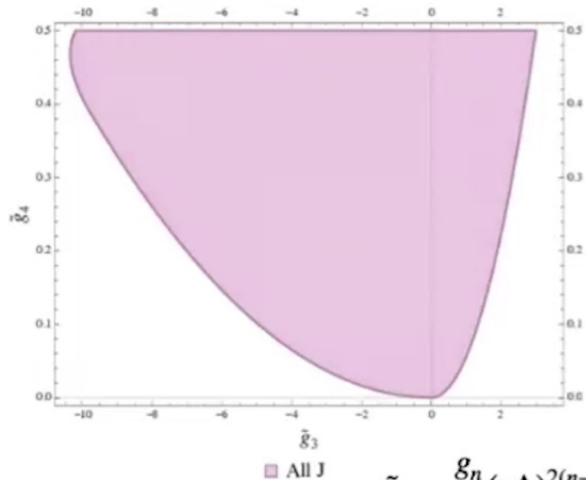

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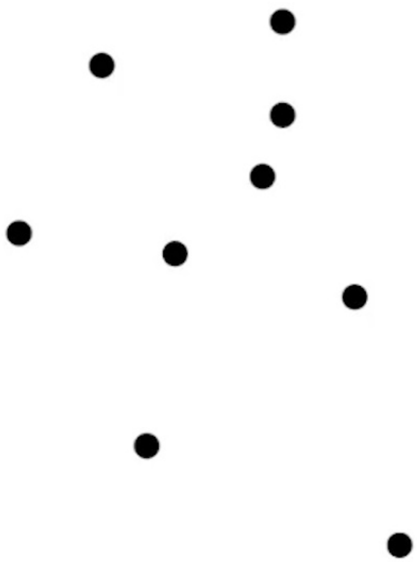
moment problem



$$a_1 = \sum_i p_i x_i\,, \quad a_2 = \sum_i p_i x_i^2 \quad \cdots\,, \quad a_n = \sum_i p_i x_i^n$$

$\{a_1, a_2, a_3, \cdots\}$ is called a moment sequence if it has a positive solution for p_i where a_i are called moments. The moment problem refers to: if the sequence is a moment sequence, what is the allowed range for moments a_i.

Analog:

Coordinate x	Integral variable $1/\mu$
Moments a_i	Wilson coefficients g_i
Mass p_i	Branch cut integral, $\mathrm{Im} f_l$

$$g_2 = \left\langle \frac{1}{\mu^3} \right\rangle, \quad g_3 = \left\langle \frac{3-2L^2}{\mu^4} \right\rangle, \quad g_4 = \left\langle \frac{1}{2\mu^5} \right\rangle$$

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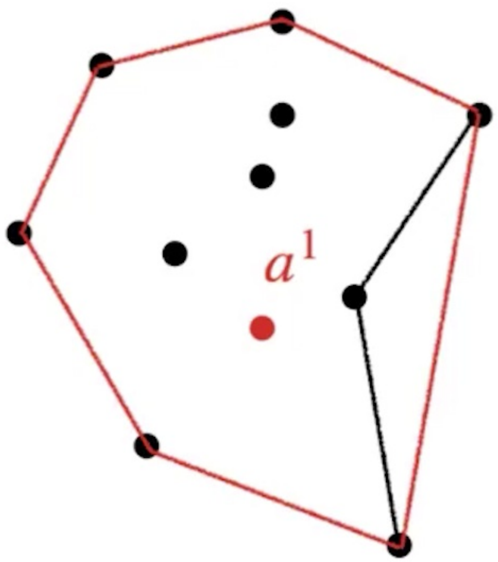
moment problem



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Hankel matrix condition

- The solution to moment problem is the Hankel matrix formed by $\{a_1, a_2, a_3, \cdots\}$ is positive semidefinite

$$H = \begin{pmatrix} a_0 & a_1 & a_2 & \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \\ & \vdots & & \ddots \end{pmatrix} \geq 0$$

$p_i > 0 \Rightarrow H$ is positive semidefinite

$$H(a) = \sum_i p_i \mathbf{x_i} \mathbf{x_i^T} = \sum_i p_i \begin{pmatrix} 1 \\ x_i \\ x_i^2 \\ \vdots \end{pmatrix} \left(1\ x_i\ x_i^2\ \cdots\right),$$

For any vector v :

$$\mathbf{v}^T H \mathbf{v} = \sum_i p_i (\mathbf{v}^T \mathbf{x}_i)^2 \geq 0 .$$

$x_i \in \mathbb{R}$

H is positive semidefinite $\Rightarrow p_i > 0$

Construct $(v^T x_i)^2$ so that it is a Dirac delta function

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Transformed Hankel matrix

$$1/\mu \in [0,1]$$

$$\begin{cases} \sum_i p_i(\mathbf{v}^T\mathbf{x}_i)^2 \geq 0 \\ 0 \leq x_i \leq 1 \end{cases} \Rightarrow \sum_i \mathbf{v}^T(p_i x_i(1-x_i)\mathbf{x}_i\mathbf{x}_i^T)\mathbf{v} \geq 0.$$

$$\sum_i p_i x_i(1-x_i)\begin{pmatrix} 1 \\ x_i \\ x_i^2 \\ \vdots \end{pmatrix}(1\ x_i\ x_i^2\ \cdots) = \begin{pmatrix} a_1-a_2 & a_2-a_3 & \cdots \\ a_2-a_3 & a_3-a_4 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \geq 0.$$

$$L^2 \in \{s_1, s_2, s_3, \cdots\}$$

$$(a)\quad \sum_i \mathbf{v}^T(p_i(x_i-s_1)\mathbf{x}_i\mathbf{x}_i^T)\mathbf{v} \geq 0 \geq 0 \Rightarrow \begin{pmatrix} (a_1-s_1a_0) & (a_2-s_1a_1) & \cdots \\ (a_2-s_1a_1) & (a_3-s_1a_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \geq 0,$$

$$(b)\quad \sum_i \mathbf{v}^T(p_i(x_i-s_i)(x_i-s_{i+1})\mathbf{x}_i\mathbf{x}_i^T)\mathbf{v} \cdots \geq 0$$

$$\Rightarrow \begin{pmatrix} (a_2-(s_i+s_{i+1})a_1+s_is_{i+1}a_0) & (a_3-(s_i+s_{i+1})a_2+s_is_{i+1}a_1) & \cdots \\ (a_3-(s_i+s_{i+1})a_2+s_is_{i+1}a_1) & (a_4-(s_i+s_{i+1})a_3+s_is_{i+1}a_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \cdots \geq 0,$$

$$(c)\quad \sum_i \mathbf{v}^T(p_i(s_n-x_i)\mathbf{x}_i\mathbf{x}_i^T)\mathbf{v} \geq 0 \Rightarrow \begin{pmatrix} (-a_1+s_na_0) & (-a_2+s_na_1) & \cdots \\ (-a_2+s_na_1) & (-a_3+s_na_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \geq 0.$$

Example

$$a_{k,q} = \sum_i p_i x_i^k y_i^q, \quad x \in [0,1],\ y \in \{s_i\}.$$

$$a_{k,q} = \left\langle \frac{L^{2q}}{\mu^k} \right\rangle \qquad 1/\mu \in [0,1], \quad L^2 \in \{0,6,20,\cdots\}$$

- Relaxing inequality:

$$\frac{1}{\epsilon^2}\left\langle \frac{1}{\mu^n} \right\rangle > \left\langle \frac{1}{\mu^{n+1}} \right\rangle. \qquad \Rightarrow \qquad a_{2,0} - \epsilon^2 a_{3,0} \geq 0, \quad a_{3,0} - \epsilon^2 a_{4,0} \geq 0, \quad a_{3,1} - \epsilon^2 a_{4,1} \geq 0, \ldots.$$

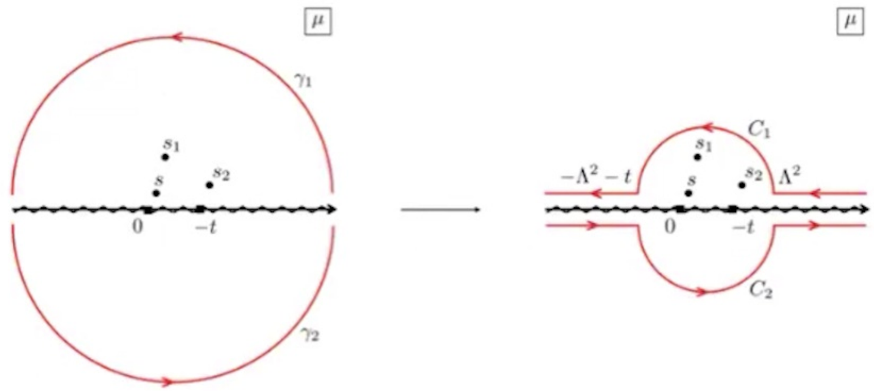
- Gram's inequality:

$$\begin{vmatrix} \int f_1^2 & \int f_1 f_2 & \int f_1 f_3 \\ \int f_2 f_1 & \int f_2^2 & \int f_2 f_3 \\ \int f_3 f_1 & \int f_3 f_2 & \int f_3^2 \end{vmatrix} \geq 0 \quad \Rightarrow \quad \begin{vmatrix} a_{2,0} & a_{3,0} & a_{3,1} \\ a_{3,0} & a_{4,0} & a_{4,1} \\ a_{3,1} & a_{4,1} & a_{4,2} \end{vmatrix} \geq 0, \qquad \begin{vmatrix} a_{2,0} & a_{3,0} \\ a_{3,0} & a_{4,0} \end{vmatrix} \geq 0, \quad \begin{vmatrix} a_{2,0} & a_{3,1} \\ a_{3,1} & a_{4,2} \end{vmatrix} \geq 0, \quad \begin{vmatrix} a_{4,0} & a_{4,1} \\ a_{4,1} & a_{4,2} \end{vmatrix} \geq 0.$$

- Polytope boundary condition:

$$\left\langle \frac{(L^2-6)(L^2-20)}{\mu^5} \right\rangle \geq 0 \qquad \Rightarrow \qquad a_{4,2} - 26 a_{4,1} + 120 a_{4,0} \geq 0$$

Dispersion relation in massless scalar theory



Froissart-Martin bounds：

$$A(s,t) \lesssim s \log^2 s$$

$$\int_{\gamma_1 \cup \gamma_2} \frac{\mathrm{d}\mu}{2\pi i} \frac{A(\mu,t)}{(\mu - s_1)(\mu - s_2)(\mu - s)} = 0\,,$$

$$A_{\mathrm{EFT}}(s,t) - sA'_{\mathrm{EFT}}(0,t) - A_{\mathrm{EFT}}(0,t) = s^2 \int_{\epsilon^2\Lambda^2}^{\infty} \frac{d\mu}{\pi} \frac{\mathrm{Im}A(\mu,t)}{\mu^2(\mu - s)} + s^2 \int_{\epsilon^2\Lambda^2}^{\infty} \frac{d\mu}{\pi} \frac{\mathrm{Im}A(\mu,t)}{(\mu + t)^2(\mu - u)}$$

Weak coupling limit assumed in addition

$$A_{\mathrm{EFT}}(s,t) = A(s,t), \quad |s| < \epsilon^2\Lambda^2$$

Result

$$a_{k,q} = \sum_i p_i x_i^k y_i^q, \quad x \in [0,1],\ y \in \{s_i\}.$$

$$a_{k,q} = \left\langle \frac{L^{2q}}{\mu^k} \right\rangle \qquad g_2 = \left\langle \frac{1}{\mu^3} \right\rangle, \quad g_3 = \left\langle \frac{3 - 2L^2}{\mu^4} \right\rangle, \quad g_4 = \left\langle \frac{1}{2\mu^5} \right\rangle$$

Collecting all constraint up to k=5:

$$0 < \tilde{a}_{3,1} < \sqrt{\frac{160}{3}\tilde{a}_{4,0}} \quad \text{for} \quad 0 < \tilde{a}_{4,0} < \frac{27}{40},$$
$$0 < \tilde{a}_{3,1} < \frac{20}{21}\left(6\tilde{a}_{4,0} + \sqrt{(21 - 20\tilde{a}_{4,0})\tilde{a}_{4,0}}\right) \quad \text{for} \quad \frac{27}{40} < \tilde{a}_{4,0} < 1.$$

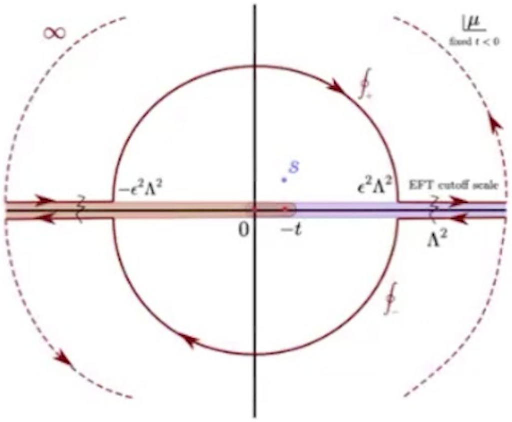
$$\tilde{a}_{k,q} \equiv \frac{a_{k,q}}{a_{2,0}}(\epsilon\Lambda)^{2(k-2)}$$

$$g_2 = a_{2,0} \quad g_3 = 3a_{3,0} - 2a_{3,1} \quad g_4 = \frac{1}{2}a_{4,0}.$$

$$-\sqrt{\frac{854}{3}\tilde{g}_4} < \tilde{g}_3 < 3\sqrt{2\tilde{g}_4} \quad \text{for} \quad 0 \le \tilde{g}_4 \le \frac{243}{854},$$
$$-\frac{120}{7}\tilde{g}_4 - \frac{74}{42}\sqrt{2\tilde{g}_4(21 - 40\tilde{g}_4)} < \tilde{g}_3 < 3\sqrt{2\tilde{g}_4} \quad \text{for} \quad \frac{243}{854} \le \tilde{g}_4 \le \frac{1}{2}.$$

$$\tilde{g}_n = \frac{g_n}{g_2}(\epsilon\Lambda)^{2(n-2)}$$

IR divergence



$$M_n(t) \equiv \frac{1}{2\pi i}\oint_{arc} \frac{A(\mu,t)}{\mu^{n+1}}$$

$$\begin{aligned} A_{\text{low}}(s,t,u) = {} & g_2\left(s^2+t^2+u^2\right) + g_3\, stu + g_4\left(s^2+t^2+u^2\right)^2 + g_5\, stu\left(s^2+t^2+u^2\right) \\ & + b_1\left(s^4\log(-s)+t^4\log(-t)+u^4\log(-u)\right) \\ & + b_2\left(s^2tu\log(-s)+st^2u\log(-t)+stu^2\log(-u)\right) \\ & + c_1\left(s^5\log(-s)+t^5\log(-t)+u^5\log(-u)\right) \\ & + c_2\left(s^3tu\log(-s)+st^3u\log(-t)+stu^3\log(-u)\right) + \ldots \end{aligned}$$

M_2, M_3, M_4 contain similar singular structures

$$\begin{aligned} M_2(t)\Big|_{t\to 0} = {} & \frac{1}{3}\left(6g_2+3b_1\epsilon^4+2c_1\epsilon^6\right)+\frac{1}{2}t\left(-2g_3-10b_1\epsilon^2-4b_2\epsilon^2-7c_1\epsilon^4-2c_2\epsilon^4\right) \\ & + t^2\left(7b_1+2b_2+12g_4+14c_1\epsilon^2+3c_2\epsilon^2+\boxed{6b_1\log(-t)}+\boxed{b_2\log(-t)}-b_2\log(\epsilon^2)\right) \end{aligned}$$

$$\begin{aligned} M_3(t)\Big|_{t\to 0} = {} & t\left(2b_1+8g_4+6c_1\epsilon^2+\boxed{4b_1\log(-t)}-b_2\left(\log(\epsilon^2)-\boxed{\log(-t)}\right)\right) \\ & + \frac{t^2\left(2b_1+2b_2-9c_1\epsilon^2-2c_2\epsilon^2-4g_5\epsilon^2-\boxed{10c_1\epsilon^2\log(-t)}-\boxed{3c_2\epsilon^2\log(-t)}-c_2\epsilon^2\log(\epsilon^2)\right)}{\epsilon^2} \end{aligned}$$

$$\begin{aligned} M_4(t)\Big|_{t\to 0} = {} & 4g_4+2c_1\epsilon^2+\boxed{b_1\log(-t)}+b_1\log(\epsilon^2) \\ & + \frac{t\left(3b_1+2b_2-2c_1\epsilon^2-2g_5\epsilon^2-\boxed{c_2\epsilon^2\log(-t)}-\boxed{5c_1\epsilon^2\log(t)}-c_2\epsilon^2\log(\epsilon^2)\right)}{\epsilon^2} \end{aligned}$$

Do combination of M_n to cancel the IR divergence

Bootstrapping tree level amplitude

$$A_{\text{tree}} = g_2(s^2+t^2+u^2) + g_3 stu + g_4(s^2+t^2+u^2)^2 + \cdots$$

$$f_l = \frac{1}{16\pi}\int_{-1}^{1} dx P_l(x) A_{\text{tree}}(s, -\frac{s}{2}(1-x), -\frac{s}{2}(1+x))$$

Orthogonality of Legengre polynomial

$$f_0(s) = \frac{5g_2 s^2}{48\pi} + \frac{g_3 s^3}{96\pi} + \frac{7g_4 s^5}{40\pi}, \qquad f_2(s) = \frac{g_2 s^2}{240\pi} - \frac{g_3 s^3}{480\pi} + \frac{g_4 s^5}{70\pi}.$$

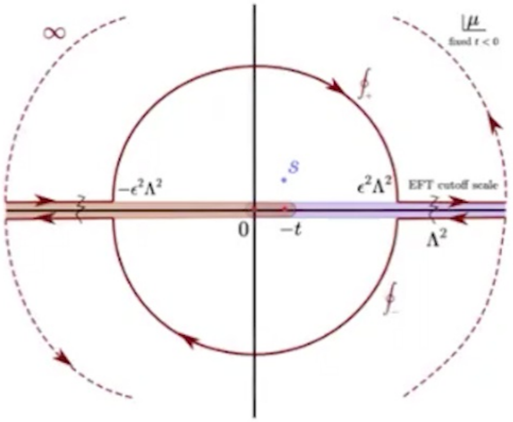
Saturate the unitarity condition $2\mathrm{Im} f_l = f_l^2$:

$$\mathrm{Im} f_0 = \left(\frac{5g_2}{48\pi}\right)^2 s^4, \qquad \mathrm{Im} f_2 = \left(\frac{g_2}{240\pi}\right)^2 s^4.$$

$$\begin{aligned} A_{\text{low}}(s,t,u) = {} & g_2\left(s^2+t^2+u^2\right) + g_3\, stu + g_4\left(s^2+t^2+u^2\right)^2 + g_5\, stu\left(s^2+t^2+u^2\right) \\ & + b_1\left(s^4\log(-s) + t^4\log(-t) + u^4\log(-u)\right) \\ & + b_2\left(s^2tu\log(-s) + st^2u\log(-t) + stu^2\log(-u)\right) \\ & + c_1\left(s^5\log(-s) + t^5\log(-t) + u^5\log(-u)\right) \\ & + c_2\left(s^3tu\log(-s) + st^3u\log(-t) + stu^3\log(-u)\right) + \ldots \end{aligned}$$

$$b_1 = -\frac{21g_2^2}{240\pi^2} \quad b_2 = \frac{g_2^2}{240\pi^2}. \quad c_1 = -\frac{g_2g_3}{60\pi^2} \quad c_2 = -\frac{g_2g_3}{240\pi^2}.$$

IR divergence



$$A_{\text{low}}(s,t,u) = g_2\left(s^2+t^2+u^2\right) + g_3\, stu + g_4\left(s^2+t^2+u^2\right)^2 + g_5\, stu\left(s^2+t^2+u^2\right)$$
$$+ b_1\left(s^4\log(-s) + t^4\log(-t) + u^4\log(-u)\right)$$
$$+ b_2\left(s^2tu\log(-s) + st^2u\log(-t) + stu^2\log(-u)\right)$$
$$+ c_1\left(s^5\log(-s) + t^5\log(-t) + u^5\log(-u)\right)$$
$$+ c_2\left(s^3tu\log(-s) + st^3u\log(-t) + stu^3\log(-u)\right) + \ldots$$

$$M_n(t) \equiv \frac{1}{2\pi i}\oint_{arc}\frac{A(\mu,t)}{\mu^{n+1}}$$

M_2, M_3, M_4 contain similar singular structures

$$M_2(t)\Big|_{t\to 0} = \frac{1}{3}\left(6g_2+3b_1\epsilon^4+2c_1\epsilon^6\right) + \frac{1}{2}t\left(-2g_3-10b_1\epsilon^2-4b_2\epsilon^2-7c_1\epsilon^4-2c_2\epsilon^4\right)$$
$$+ t^2\left(7b_1+2b_2+12g_4+14c_1\epsilon^2+3c_2\epsilon^2+\boxed{6b_1\log(-t)}+\boxed{b_2\log(-t)}-b_2\log(\epsilon^2)\right)$$

$$M_3(t)\Big|_{t\to 0} = t\left(2b_1+8g_4+6c_1\epsilon^2+\boxed{4b_1\log(-t)}-b_2\left(\log(\epsilon^2)-\boxed{\log(-t)}\right)\right)$$
$$+\frac{t^2\left(2b_1+2b_2-9c_1\epsilon^2-2c_2\epsilon^2-4g_5\epsilon^2-\boxed{10c_1\epsilon^2\log(-t)}-\boxed{3c_2\epsilon^2\log(-t)}-c_2\epsilon^2\log(\epsilon^2)\right)}{\epsilon^2}$$

$$M_4(t)\Big|_{t\to 0} = 4g_4+2c_1\epsilon^2+\boxed{b_1\log(-t)}+b_1\log(\epsilon^2)$$
$$+\frac{t\left(3b_1+2b_2-2c_1\epsilon^2-2g_5\epsilon^2-\boxed{c_2\epsilon^2\log(-t)}-\boxed{5c_1\epsilon^2\log(t)}-c_2\epsilon^2\log(\epsilon^2)\right)}{\epsilon^2}$$

Do combination of M_n to cancel the IR divergence

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Modified null constraint

$$\frac{1}{12b_1+2b_2}\frac{\partial^2 M_2(t)}{\partial t^2}-\frac{1}{4b_1+b_2}\frac{\partial M_3(t)}{\partial t}=\left\langle\frac{(4b_1L^2(L^2-8)+b_2(4+L^2(L^2-8)))}{2(4b_1+b_2)(6b_1+b_2)\mu^5}\right\rangle$$

$$\frac{1}{4b_1+b_2}\frac{\partial M_3(t)}{\partial t}-\frac{M_4(t)}{b_1}=\left\langle-\frac{2(2b_1+b_2)}{b_1(4b_1+b_2)\mu^5}\right\rangle.$$

Leading order:

$$n_4=\left\langle\frac{L^2(L^2-8)}{\mu^5}\right\rangle=\frac{28b_1^2+28b_1b_2+5b_2^2}{2b_1+b_2}+\frac{(20b_1c_1+12b_1c_2+12b_2c_1+6b_2c_2)}{2b_1+b_2}\epsilon^2,$$

Only shifted by a constant

$$g_4=\left\langle\frac{1}{2\mu^5}\right\rangle+\frac{b_1(6b_1+b_2)}{4(2b_1+b_2)}-\frac{c_1(b_1+b_2)}{2(2b_1+b_2)}\epsilon^2-\frac{b_1}{2}\log\epsilon^2,$$

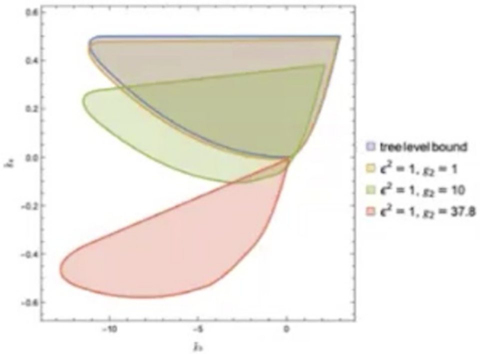
$\Lambda=1$

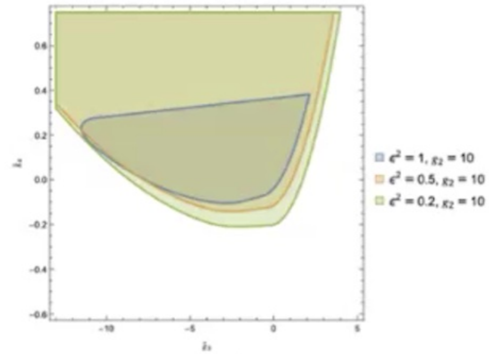
Next leading order:

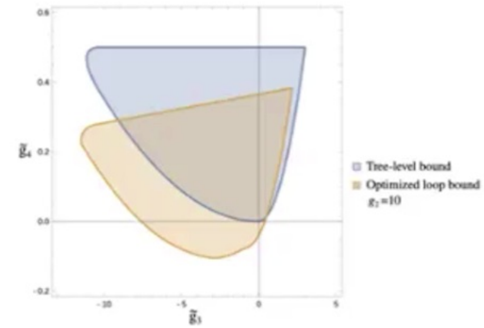
$$n_5=\left\langle\frac{(L^2(150+L^2(-43+2L^2))}{\mu^6}\right\rangle=-480c_1-264c_2-12(16b_1+11b_2)\frac{1}{\epsilon^2},$$

$$g_5=\left\langle\frac{5-2L^2}{2\mu^6}\right\rangle+\frac{b_1(20c_1+9c_2)-b_2(7c_1+c_2)}{2b_2}+\frac{8b_1^2+7b_1b_2+2b_2^2}{2b_2}\frac{1}{\epsilon^2}-$$
$$-\frac{(5b_2c_1+2b_2c_2)}{2b_2}\log\epsilon^2.$$

Result



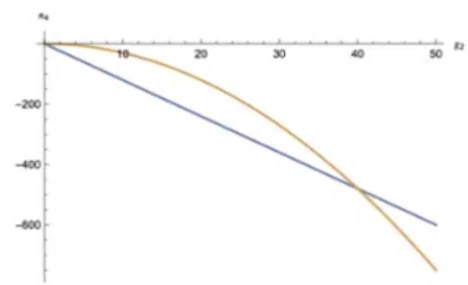




$$\tilde{g}_4 \approx \left\langle \frac{1}{2\mu^5} \right\rangle - g_2 \tilde{g}_3$$

$$\tilde{g}_4 \leq \frac{1}{2(\epsilon\Lambda)^4}$$

$$L^2 = l(l+1) \in \{0,6,20,\cdots\} \qquad n_4 = \left\langle \frac{L^2(L^2-8)}{\mu^5} \right\rangle \geq \left\langle \frac{-12}{\mu^5} \right\rangle \geq -12 g_2$$



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massless theory with shift symmetry

$$\mathcal{L}_{\rm EFT} = -\frac{1}{2}(\partial_\mu\phi)^2 + \frac{g_2}{2}[(\partial_\mu\phi)^2]^2 + \frac{g_3}{3}(\partial_\mu\partial_\nu\phi)^2(\partial_\sigma\phi)^2 + 4g_4[(\partial_\mu\partial_\nu\phi)^2]^2 + \cdots$$

$$A_{\rm EFT}(s,t) = g_2(s^2+t^2+u^2) + g_3 stu + g_4(s^2+t^2+u^2)^2 + \cdots$$

Substitute into l.h.s of dispersion relation

Do partial wave expansion on r.h.s of dispersion relation: $A(s,t) = 16\pi \sum_{\text{even } l} (2l+1) f_l(s) P_l\left(1+\frac{2t}{s-4m^2}\right),$

$$\langle F(\mu,t)\rangle = \sum_l 16\pi(2l+1)\int_{\epsilon^2\Lambda^2}^{+\infty}\frac{d\mu}{\pi}\text{Im} f_l(\mu)F(\mu,t)$$

$$g_2 = \left\langle \frac{1}{\mu^3}\right\rangle, \quad g_3 = \left\langle \frac{3-2L^2}{\mu^4}\right\rangle, \quad g_4 = \left\langle \frac{1}{2\mu^5}\right\rangle$$

$g_2 > 0,\ g_4 > 0,$
g_3 has upper boundary

JHEP, 2006, 10: 014.

$$L^2 = l(l+1), \quad L \in \{0,6,20,\cdots\}$$

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Conclusion

- Negative $\tilde{g}_4$ is related to loop diagram that characterizes finite-range interaction, this is a reminiscent of the non-local UV completion which also gives negative $\tilde{g}_4$.

- Loop effect breaks projective nature of some positivity bounds, therefore relates positivity condition $\text{Im}f_l \geq 0$ with full unitarity condition $0 \leq \text{Im}f_l \leq 2$. Moreover, this non-projective nature sets a bound on g_2 which enters the running of g_4, therefore, this gives a bound on β function of g_4. How to implement the unitarity condition fully is also worth exploring.

- The best bounds including loops is not given by the bound on a fixed scale, but by an overlap of bounds with different scale.

- More informations can be implemented in the UV spectrum, asymptotic free, asymptotic safety… Also There is a reverse problem, restoring UV from moments in IR. More informations (low energy theorem) are known from the IR side. Low energy theorem could have constraints on UV completion.

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Thanks for listening!

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