Title: QED in a Box

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Abstract:

The measurement of the electron magnetic moment (g-2)_e has reached the spectacular precision of sub-parts-per-trillion, and is currently the highest-precision measurement of a property of a fundamental particle. The next iteration of the measurement is expected to improve the precision by almost an order of magnitude, at which point the leading systematic uncertainty will be the “cavity shift”: the modification of the electron energy levels due to the conducting walls of the cavity in which the electron is trapped. This quantity cannot be directly measured at the required precision, and must be calculated. I will review the setup of the (g-2)_e experiment and present a new calculation of the cavity shift using the full machinery of quantum mechanics and quantum electrodynamics, which improves upon previous classical calculations with a consistent renormalization scheme while allowing for individual quality factors for each cavity mode.

QED in a Box

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Perimeter Institute Particle Physics Seminar, 3/11/25

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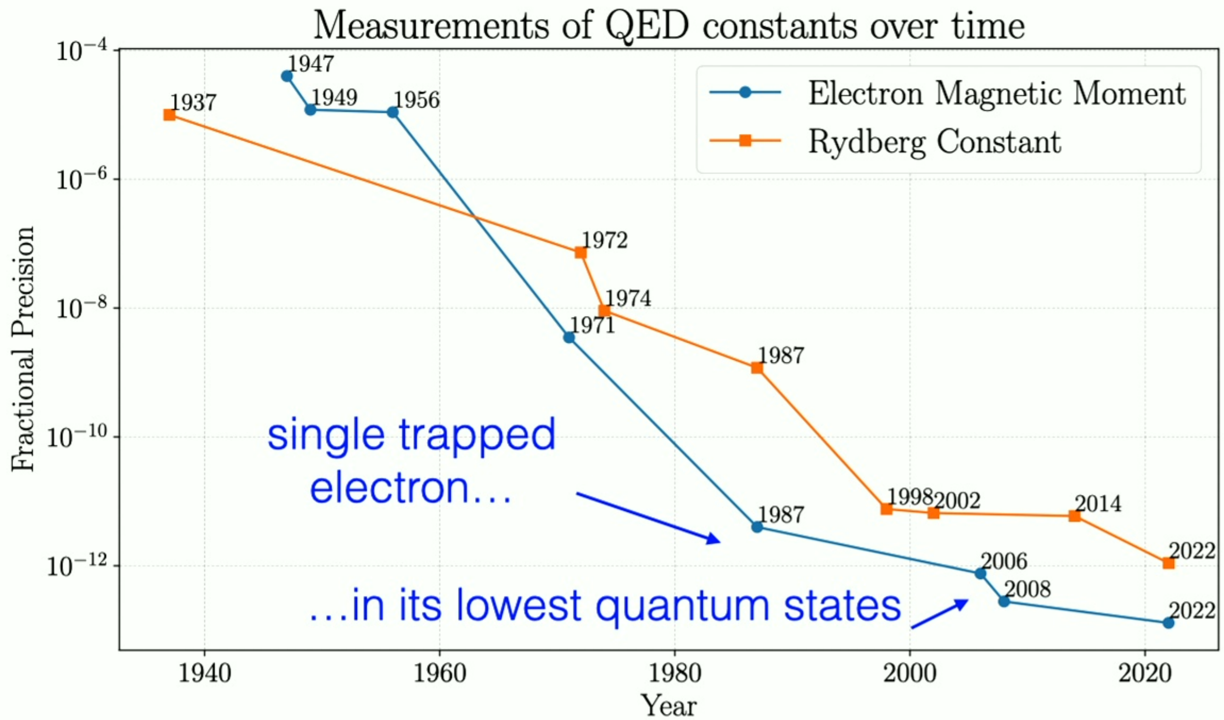
The frontiers of precision measurement

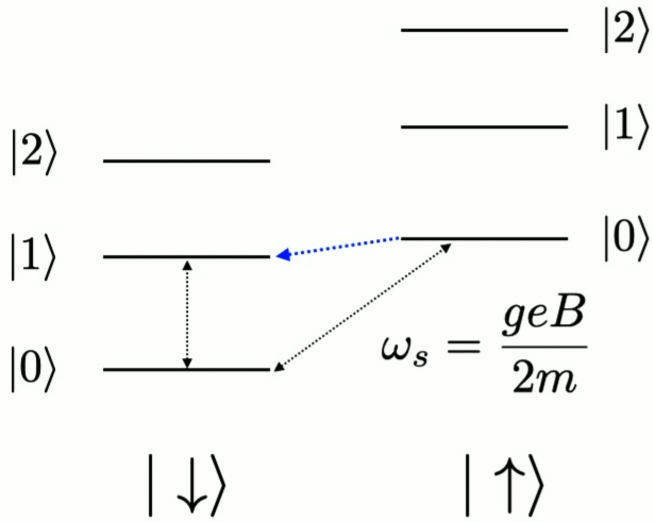


Most precise theory/experiment comparison humanity has ever made!
How do we do even better?

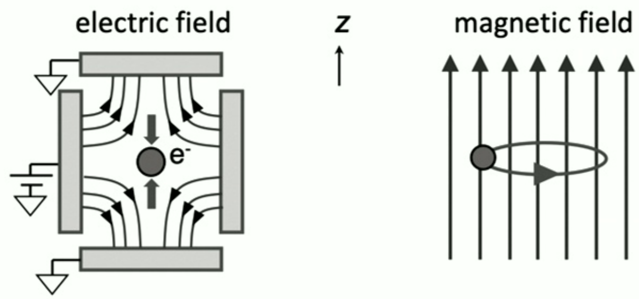
Measuring the g-factor

An single electron in a B-field is its own magnetometer

$$\omega_c = \frac{eB}{m}$$



$$\omega_s = \frac{geB}{2m}$$



$$\frac{g-2}{2} = \frac{\omega_a}{\omega_c}$$

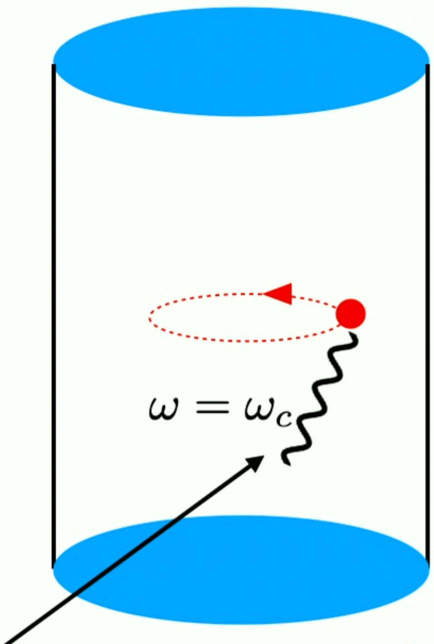
enormous reduction of systematics

$$\omega_a = |0, \uparrow\rangle - |1, \downarrow\rangle$$

Doesn't get much more quantum than this!

$$-\frac{\mu}{\mu_B} = \frac{g}{2} = 1.001\,159\,652\,180\,59(13) \quad [0.13 \text{ ppt}]$$

Cavity inhibits spontaneous emission

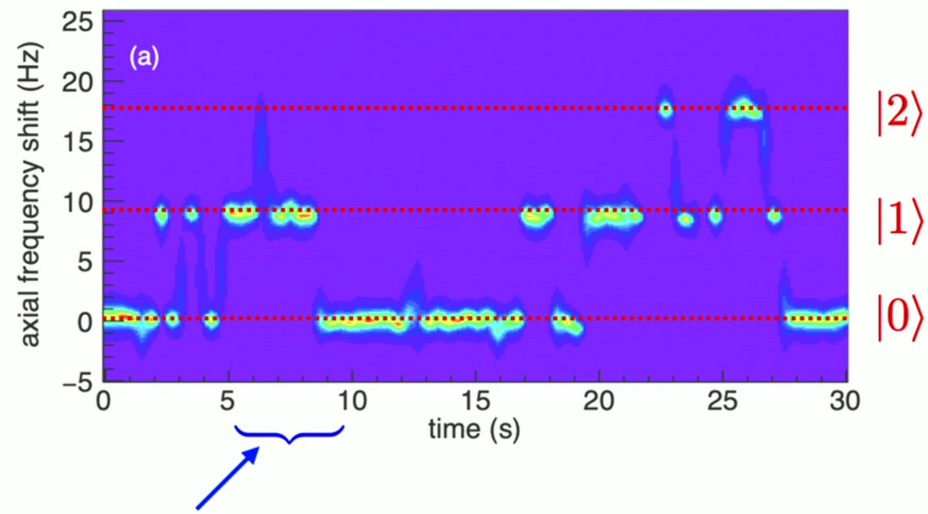


synchotron
(electric dipole)
radiation

$$\Gamma_c^{\text{free}} = \frac{4\alpha}{3}\frac{\omega_c^2}{m} \approx (0.1\ \text{s})^{-1}$$

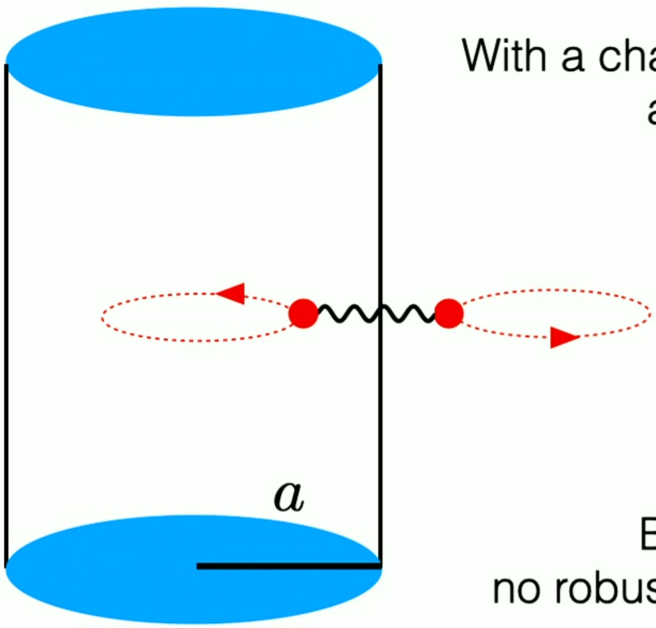
for B = 5.3 T

But if no cavity mode at $\boldsymbol{\omega_c}$, electron can't radiate!



for Q ~ 3000, can increase cyclotron state lifetime to >5s, crucial for precision

The (real part of the) cavity shift



With a change in the lifetime (imaginary part) unavoidably comes a change in the cyclotron frequency (real part)

Classical E+M: image charge also radiates, perturbs trapped electron orbit

$$\frac{\Delta\omega_c}{\omega_c} \simeq \frac{\alpha}{4ma} \simeq 10^{-13} \times \frac{1\text{cm}}{a}$$

But this estimate is multiplied by a divergent sum: no robust way to deal with electron self-energy in classical E+M!

Leading systematic uncertainty for next iteration of the experiment

[Boulware, Brown, Lee, PRD 1985; Brown, Gabrielse, Helmerson, Tan, PRA 1985]

Cavity shift in quantum mechanics

For now, forget the electric field and consider only a constant B-field with vector potential $\mathbf{A}_c$

$$(E+\delta E)\Psi = \left[\frac{1}{2m}\big(\underbrace{\mathbf{p}+e\mathbf{A}_c}_{\boldsymbol{\pi}}+e\mathbf{A}_q\big)^2\right]\Psi \quad \Longrightarrow \quad \delta H = \frac{e}{m}\left(\mathbf{A}_q\cdot\boldsymbol{\pi}\right)$$

creates or destroys exactly 1 photon

unperturbed Hamiltonian (Landau levels)

quantized photon field

matrix element between 0- and 1-photon states

Second-order perturbation theory:

$$\delta E_N = \sum_{\sigma=1,2}\sum_s\sum_{N'}\frac{|\langle N;0|\delta H|N';1_{s\sigma}\rangle|^2}{E_N-(E_{N'}+\omega_{s\sigma})+i\epsilon}.$$

polarizations

mode numbers

electron states

electron, photon energies

The Lamb shift

$$\delta E_N = \sum_{\sigma=1,2} \sum_s \sum_{N'} \frac{|\langle N; 0|\delta H|N'; 1_{s\sigma}\rangle|^2}{E_N - (E_{N'} + \omega_{s\sigma}) + i\epsilon}.$$

In an atom with a Coulomb potential, this is called the Lamb shift. Famously, it diverges:

$$\delta E_{2s} \sim \frac{2\alpha}{3\pi} \int_0^\infty dk = \infty$$

Relativistic effects are important, so matching this low-energy calculation with a relativistic quantum electrodynamics calculation at energies close to the electron mass to get a finite answer is crucial. (Even Feynman and Schwinger got the matching wrong!)

Cavity shift as an IR Lamb shift

Fortunately, the cavity shift is explicitly a low-energy (infrared) effect.
When we take the box away, the shift must vanish: this is our renormalization condition

$$\delta E_N^{\text{cav.}} = \sum_{\sigma=\text{TE,TM}} \sum_{m\mu p} \sum_{n'l'q'} \frac{|\langle n01;0|\delta H|n'l'q';1_{\sigma,m\mu p}\rangle|^2}{E_{01} - (E_{n'q'} + \omega_{\sigma,m\mu p}) + i\epsilon}$$

cavity modes

Both will diverge (linearly and logarithmically), but their difference must be finite:

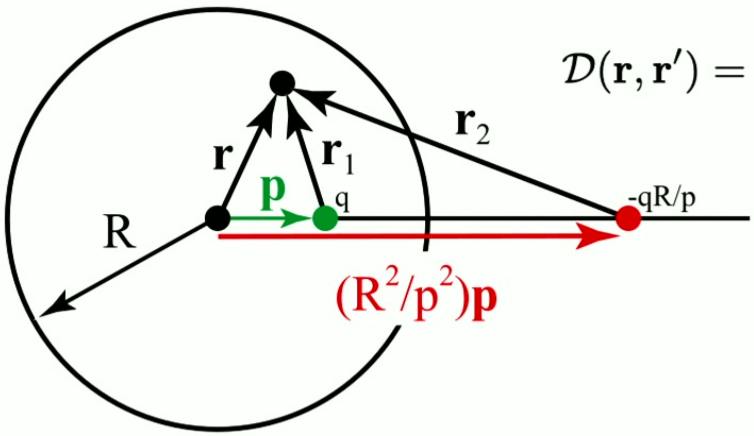
$$\delta E_N^{\text{free}} = \sum_\lambda \int \frac{d^3k}{(2\pi)^3} \sum_{n'l'} \int dq'_z \frac{|\langle n01;0|\delta H|n'l'q'_z;1_{\lambda\mathbf{k}}\rangle|^2}{E_{01} - (E_{n'q'_z} + k) + i\epsilon}$$

plane waves

$$\Delta\omega_c \equiv (\delta E_1^{\text{cav.}} - \delta E_1^{\text{free}}) - (\delta E_0^{\text{cav.}} - \delta E_0^{\text{free}})$$

Warm-up: spherical cavity

Classic Jackson problem: Green's function for a spherical cavity has a closed form by method of images



$$\mathcal{D}(\mathbf{r}, \mathbf{r}') = -\frac{R}{\sqrt{\mathbf{r}^2(\mathbf{r}')^2 - 2R^2\mathbf{r}\cdot\mathbf{r}' + R^4}}$$

$$\rightarrow \quad \Delta\omega_c = \frac{2\alpha}{3}\frac{\omega_c^2}{m}\left(\frac{(1-z^2)\cos z + z\sin z}{(1-z^2)\sin z - z\cos z} + \frac{3}{2z^3}\right)$$

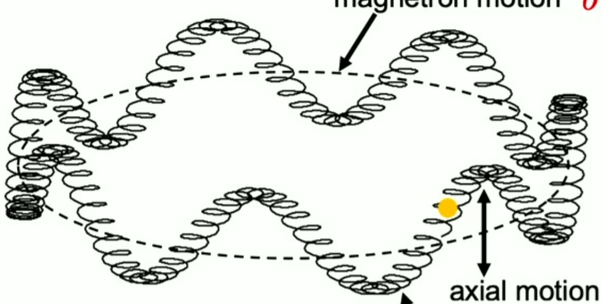
$$z = \omega_c R$$

Can easily compare quantum calculation to this classical calculation, which does not use a mode sum and thus doesn't require renormalization

[Brown, Helmerson, Tan, PRA 1986]

Electron eigenstates

Spherical boundary means we need to consider quadrupole confinement of electron



Three harmonic oscillators:

$$H = \omega_c' \left(a^\dagger a + \frac{1}{2}\right) + \omega_z \left(a_z^\dagger a_z + \frac{1}{2}\right) - \omega_m \left(b^\dagger b + \frac{1}{2}\right)$$

metastable (1000 yr lifetime)

$$\omega_c' \approx \omega_c \sim 100 \text{ GHz}, \qquad \omega_z \sim 100 \text{ MHz}, \qquad \omega_m \sim 50 \text{ kHz}$$

$$|nlq\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} \frac{(b^\dagger)^l}{\sqrt{l!}} \frac{(a_z^\dagger)^q}{\sqrt{q!}} |0\rangle, \qquad E_{nlq} \approx \left(n + \frac{1}{2}\right) \omega_c$$

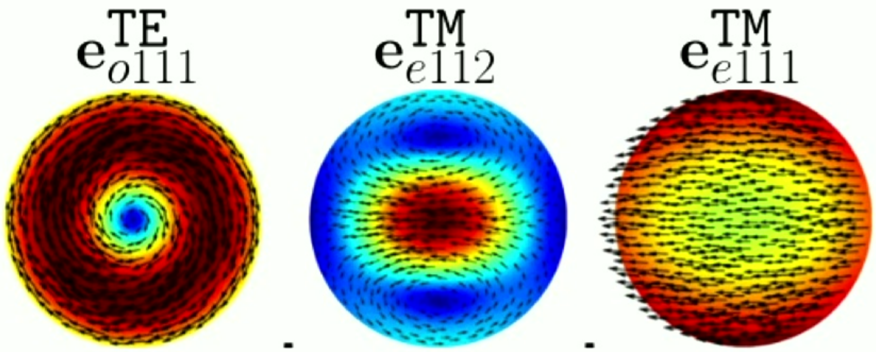
With this (standard) hierarchy of frequencies, $\omega_m \ll \omega_c$, we have

$$\pi_x \approx \sqrt{\frac{m\omega_c}{2}} \left(a + a^\dagger\right), \qquad \pi_y \approx i\sqrt{\frac{m\omega_c}{2}} \left(a - a^\dagger\right), \qquad \pi_z = -i\sqrt{\frac{m\omega_z}{2}} (a_z - a_z^\dagger)$$

[Brown and Gabrielse, Rev. Mod. Phys. 1986]

Photon eigenstates

Spherical cavity:



$$\mathbf{A}_q(\mathbf{x}) = \sum_{s,\sigma} \frac{1}{\sqrt{2\omega_{s\sigma}}} \left[\mathbf{u}_{s\sigma}(\mathbf{x}) a_{s\sigma} + \mathrm{h.c.}\right]$$

$$\mathbf{u}_{\mathrm{TE},mnp} = c_{mnp}^{(\mathrm{TE})} \frac{1}{\omega_{\mathrm{TE},np} r} \hat{J}_n(\omega_{\mathrm{TE},np} r) e^{im\phi} \left[-\frac{im}{\sin\theta} P_n^m(\cos\theta)\hat{\boldsymbol{\theta}} + \frac{d}{d\theta} P_n^m(\cos\theta)\hat{\boldsymbol{\phi}}\right]$$

$$\hat{J}_n(x) \equiv x j_n(x)$$

Free space:

$$\mathbf{A}_q(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k}} \sum_{\lambda=1,2} \left[\boldsymbol{\epsilon}_\lambda(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} a_{\lambda\mathbf{k}} + \mathrm{h.c.}\right]$$

$$\boldsymbol{\epsilon}_\pm(\mathbf{k}) = \frac{1}{\sqrt{2}} \left[(\cos\theta_k \cos\phi_k \mp i \sin\phi_k)\hat{\mathbf{x}} + (\cos\theta_k \sin\phi_k \pm i\cos\phi_k)\hat{\mathbf{y}} - \sin\theta_k \hat{\mathbf{z}}\right]$$

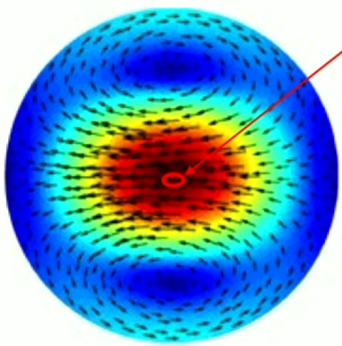
[Pictures from Pascale et al., Nature Sci. Rept. 2019]

One-photon matrix elements: sphere

For sufficiently strong B-field, electron is confined at the center, can use dipole approximation:

$$\ell = \frac{1}{\sqrt{eB}} \sim 10 \text{ nm}$$

$\mathbf{e}^{\mathtt{TM}}_{e112}$



Two subsets of modes survive:

$$\mathbf{u}_{\mathrm{TM},11p}(0) = -\frac{2}{3} c^{(\mathrm{TM})}_{11p} (\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \quad \text{(transverse)}$$

$$\langle nlq | \mathbf{u}_{\mathrm{TM},m1p} \cdot \boldsymbol{\pi} | n'l'q' \rangle = -\frac{2}{3} c^{(\mathrm{TM})}_{11p} \sqrt{2eB}\, \delta_{ll'}\, \delta_{qq'} \times \begin{cases} \sqrt{n}\, \delta_{n',n-1}, & m = +1 \\ \sqrt{n+1}\, \delta_{n',n+1}, & m = -1, \end{cases}$$

$$\mathbf{u}_{\mathrm{TM},01p}(0) = \frac{2}{3} c^{(\mathrm{TM})}_{01p} \hat{\mathbf{z}} \qquad \text{(longitudinal)}$$

$$\langle nlq | \mathbf{u}_{\mathrm{TM},01p} \cdot \boldsymbol{\pi} | n'l'q' \rangle \sim \sqrt{\frac{\omega_z}{\omega_c}} \times \langle nlq | \mathbf{u}_{\mathrm{TM},11p} \cdot \boldsymbol{\pi} | n'l'q' \rangle$$

longitudinal squared matrix element suppressed by $\sim 10^{-3}$, will neglect from now on

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[Pictures from Pascale et al., Nature Sci. Rept. 2019]

Un-renormalized cavity shift: sphere

$$\delta E_n^{\text{cav.}} = -\frac{e^2}{2m^2}\frac{8eB}{9}\sum_{p=0}^{p_{\max}}\frac{(c_{11p}^{(\text{TM})})^2}{\omega_{\text{TM},np}}\left(\frac{n}{\omega_{\text{TM},np}-\omega_c}+\frac{n+1}{\omega_{\text{TM},np}+\omega_c}\right)$$

unlike in Lamb shift, only transitions between neighbouring states

$$\Delta\omega_c^{\text{cav.}} \equiv \delta E_{n+1}^{\text{cav.}} - \delta E_n^{\text{cav.}} = -\frac{8\alpha}{3\pi}\frac{\omega_c}{mR}\sum_{p=0}^{p_{\max}}\frac{(\zeta'_{1p})^3}{(\zeta'_{1p})^2-2}\frac{1}{J_{3/2}^2(\zeta'_{1p})}\frac{1}{(\zeta'_{1p})^2-R^2\omega_c^2}$$

ζ'_{1p} : p-th zero of $\frac{d}{dx}[xj_1(x)] \propto x\cos x + (x^2-1)\sin x$

For large p, asymptotic behavior of Bessel functions and zeros gives

$$\Delta\omega_c \sim -\frac{4\alpha}{3}\frac{\omega_c}{mR}p_{\max}$$

Linear divergence with radial mode number cutoff, as expected

Matching and renormalizing: sphere

One-photon matrix elements in free space:

$$\langle nlq|\boldsymbol{\epsilon}_\pm(\mathbf{k})\cdot\boldsymbol{\pi}|n'l'q'\rangle = \sqrt{\frac{eB}{2}}\langle nlq|\epsilon^x_\pm(a+a^\dagger)+i\epsilon^y_\pm(a-a^\dagger)|n'l'q'\rangle = \frac{\sqrt{eB}}{2}\delta_{ll'}\delta_{qq'}\left(\sqrt{n+1}e^{i\phi_k}(\cos\theta_k\mp 1)\delta_{n',n+1}+\sqrt{n}e^{-i\phi_k}(\cos\theta_k\pm 1)\delta_{n',n-1}\right)$$

$$\sum_\lambda\sum_{l'q'}|\langle nlq|\boldsymbol{\epsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\pi}|n'l'q'\rangle|^2 = \frac{eB}{2}(1+\cos^2\theta_k)((n+1)\delta_{n',n+1}+n\delta_{n',n-1})$$

Sum over mode numbers becomes integral over continuous wavenumber:

$$\Delta\omega_c^{\rm free} = \delta E^{\rm free}_{n+1}-\delta E^{\rm free}_n = -\frac{2\alpha}{3\pi}\frac{eB}{m^2}\int_0^{k_{\max}} dk\; k\left(\frac{1}{k-\omega_c}+\frac{1}{k+\omega_c}\right)$$

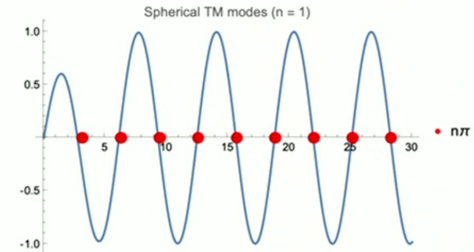
Principal part gives the real part of the frequency shift:

$$\mathrm{P}\left(\Delta\omega_c^{\rm free}\right) = -\frac{4\alpha}{3\pi}\omega_c\left(\frac{k_{\max}}{m}+\frac{\omega_c}{2m}\log\left(\frac{k_{\max}-\omega_c}{k_{\max}+\omega_c}\right)\right)$$

$\zeta'_{1p}\sim p\pi \implies \omega_{\rm TM,1p}\sim p\pi/R$

match by choosing

$k_{\max} = \pi p_{\max}/R$



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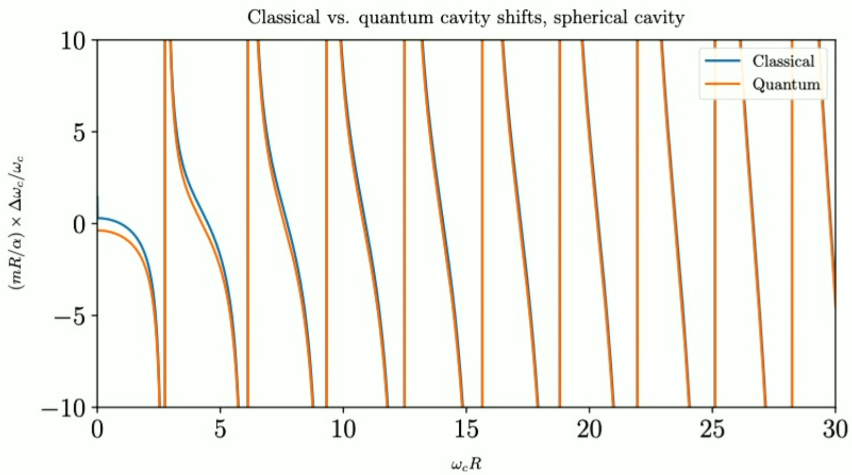
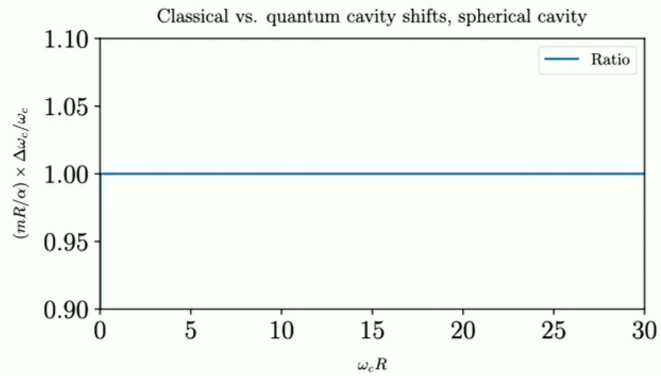
Comparison to classical calculation: sphere



If we add a constant shift by hand…



$$k_{\max} \to \left(\pi + \frac{1}{2}\right) p_{\max}$$

Perfect agreement! This suggests a bizarre formula:

$$\lim_{p_{\max}\to\infty} \left[2p_{\max} + 1 + \frac{z}{\pi} \log\left(\frac{\pi p_{\max} - z}{\pi p_{\max} + z}\right) - \frac{4}{\pi} \sum_{p=1}^{p_{\max}} \frac{(\zeta'_{1p})^3}{(\zeta'_{1p})^2 - 2} \frac{1}{J^2_{3/2}(\zeta'_{1p})} \frac{1}{(\zeta'_{1p})^2 - z^2} \right] = \frac{z(1-z)^2 \cos z + z^2 \sin z}{(1-z)^2 \sin z - z \cos z} + \frac{3}{2z^2}$$

Where the heck does this come from?? Why the +1? Which calculation is right?

Matching and renormalizing: sphere

One-photon matrix elements in free space:

$$\langle nlq|\boldsymbol{\epsilon}_\pm(\mathbf{k})\cdot\boldsymbol{\pi}|n'l'q'\rangle = \sqrt{\frac{eB}{2}}\langle nlq|\epsilon^x_\pm(a+a^\dagger)+i\epsilon^y_\pm(a-a^\dagger)|n'l'q'\rangle = \frac{\sqrt{eB}}{2}\delta_{ll'}\delta_{qq'}\left(\sqrt{n+1}e^{i\phi_k}(\cos\theta_k\mp 1)\delta_{n',n+1}+\sqrt{n}e^{-i\phi_k}(\cos\theta_k\pm 1)\delta_{n',n-1}\right)$$

$$\sum_\lambda\sum_{l'q'}|\langle nlq|\boldsymbol{\epsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\pi}|n'l'q'\rangle|^2 = \frac{eB}{2}(1+\cos^2\theta_k)((n+1)\delta_{n',n+1}+n\delta_{n',n-1})$$

Sum over mode numbers becomes integral over continuous wavenumber:

$$\Delta\omega_c^{\rm free} = \delta E^{\rm free}_{n+1}-\delta E^{\rm free}_n = -\frac{2\alpha}{3\pi}\frac{eB}{m^2}\int_0^{k_{\rm max}}dk\;k\left(\frac{1}{k-\omega_c}+\frac{1}{k+\omega_c}\right)$$

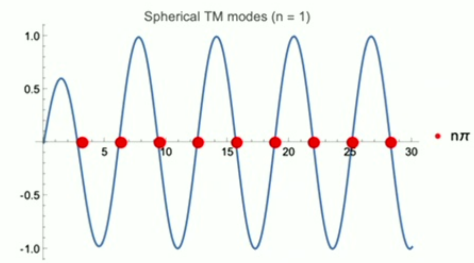
Principal part gives the real part of the frequency shift:

$$\mathrm{P}\left(\Delta\omega_c^{\rm free}\right) = -\frac{4\alpha}{3\pi}\omega_c\left(\frac{k_{\rm max}}{m}+\frac{\omega_c}{2m}\log\left(\frac{k_{\rm max}-\omega_c}{k_{\rm max}+\omega_c}\right)\right)$$

$$\zeta'_{1p}\sim p\pi \implies \omega_{\rm TM,1p}\sim p\pi/R$$

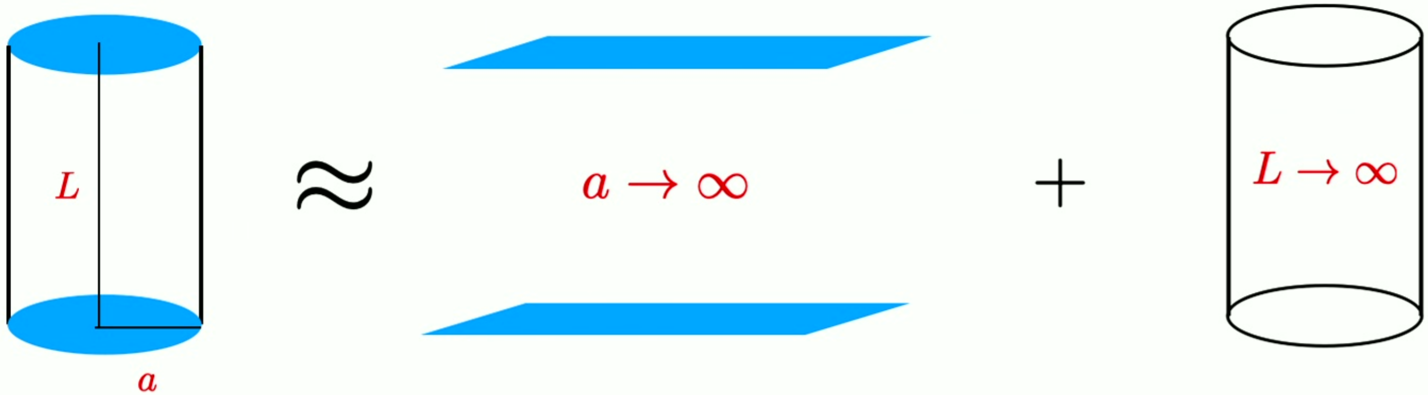
match by choosing

$$k_{\rm max} = \pi p_{\rm max}/R$$



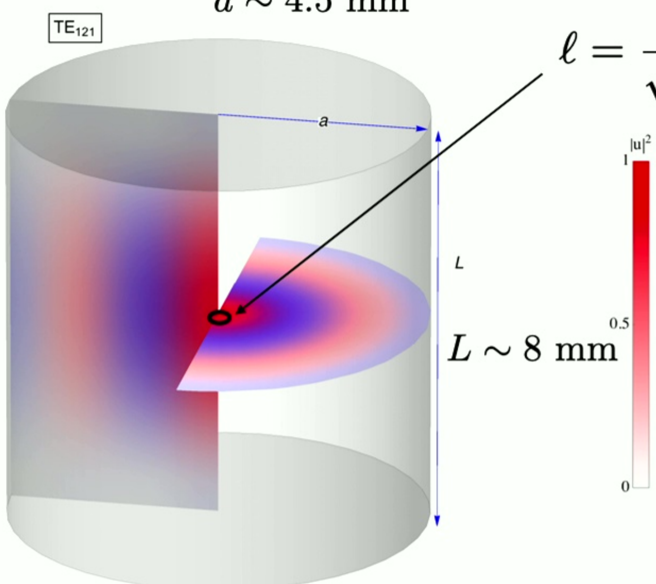
Challenges with a cylindrical cavity

No exact classical calculation possible. The best one can do:



But this misses effects of order L/a!
Need to do a renormalized mode sum calculation;
“boundary-only” calculation doesn’t allow for individual Q factors for each mode

One-photon matrix elements: cylinder



$a \sim 4.5$ mm

$$\ell = \frac{1}{\sqrt{eB}} \sim 10 \text{ nm}$$

$L \sim 8$ mm

Dipole approximation in radial direction:

$\mathbf{u}_{s\sigma}(\mathbf{x}) \approx \mathbf{u}_{s\sigma}(\rho = 0)$ picks out $\mathbf{TE/TM}_{1\mu p}$

$$\langle nlq|\mathbf{u}_{m,\mu,2p'+1} \cdot \boldsymbol{\pi}|n'l'q'\rangle = -u^{\perp}_{\mu,2p+1}(0)\sqrt{2eB}\,\delta_{ll'}\,\delta_{qq'} \times \begin{cases} \sqrt{n}\,\delta_{n',n-1} & m = +1 \\ \sqrt{n+1}\,\delta_{n',n+1} & m = -1 \end{cases}$$

Same matrix element structure as for spherical cavity, but this time, 2 free mode indices and both TE and TM modes contribute

Un-renormalized cavity shift: cylinder

$$\Delta\omega_c^{\text{cav.}} = -8\pi\alpha\frac{eB}{m^2}\sum_{\sigma=\text{TE,TM}}\sum_{\mu=1}^{\mu_{\max}}\sum_{p'=0}^{\infty}\frac{(u^{\perp}_{\sigma,\mu,2p'+1}(0))^2}{\omega^2_{\sigma,\mu,2p'+1}-\omega_c^2}$$

radial index   longitudinal index

$$u^{\perp}_{\text{TE},\mu,p}(0) = \sqrt{\frac{1}{2\pi a^2 L}}\frac{1}{\sqrt{J_1^2(\chi'_{1\mu})-J_2^2(\chi'_{1\mu})}}$$

roots of $J_1'(x)$

$$u^{\perp}_{\text{TM},\mu,p}(0) = \sqrt{\frac{\pi}{2L^3}}\frac{p}{J_2(\chi_{1\mu})\sqrt{{\chi_{1\mu}}^2+\frac{p^2\pi^2a^2}{L^2}}}$$

roots of $J_1(x)$

Longitudinal sum can be performed analytically (Poisson summation):

$$\Delta\omega_c^{\text{cav.}} = \sum_{\mu=1}^{\mu_{\max}}\left[\frac{\tanh\left(\frac{L}{2a}\sqrt{(\chi'_{1\mu})^2-(a\omega_c)^2}\right)}{(J_1^2(\chi'_{1\mu})-J_2^2(\chi'_{1\mu}))\sqrt{(\chi'_{1\mu})^2-(a\omega_c)^2}} - \frac{\sqrt{\chi_{1\mu}^2-(a\omega_c)^2}\tanh\left(\frac{L}{2a}\sqrt{\chi_{1\mu}^2-(a\omega_c)^2}\right)-\chi_{1\mu}\tanh\left(\frac{L\chi_{1\mu}}{2a}\right)}{(a\omega_c)^2J_2^2(\chi_{1\mu})}\right]$$

Not obvious, but this is also linearly divergent

Matching and renormalizing: cylinder

Redo the free-space calculation but in cylindrical coordinates:

$$\Delta\omega_c^{\text{free}} = -\frac{\alpha}{2\pi}\frac{eB}{m^2}\int_0^{k_\perp^{\max}} dk_\perp \int_{-\infty}^{\infty} dk_z \left[k_\perp \left(1 + \frac{k_z^2}{k_\perp^2 + k_z^2}\right)\left(\frac{1}{k_\perp^2 + k_z^2 - \omega_c^2}\right)\right]$$

As in the cavity, can do the longitudinal (k_z) integral analytically. Principal part gives

$$\mathrm{P}(\Delta\omega_c^{\text{free}}) = -\frac{\alpha}{2\pi}\frac{eB}{m^2}\left\{\frac{\pi}{\omega_c^2}\left[\int_0^{k_\perp^{\max}} dk_\perp\, k_\perp^2\right] - \frac{\pi}{\omega_c^2}\left[\int_{\omega_c}^{k_\perp^{\max}} \frac{k_\perp(k_\perp^2 - 2\omega_c^2)}{\sqrt{k_\perp^2 - \omega_c^2}}\right]\right\} = -\frac{3\alpha}{4}\frac{eB}{m^2}k_\perp^{\max} + \mathcal{O}\left(\frac{1}{k_\perp^{\max}}\right)$$

Coefficient of linear divergence matches mode sum divergence. But how to match cutoffs?

$$\chi_{1\mu} \sim \left(\mu + \frac{1}{4}\right)\pi, \qquad \chi'_{1\mu} \sim \left(\mu - \frac{1}{4}\right)\pi$$

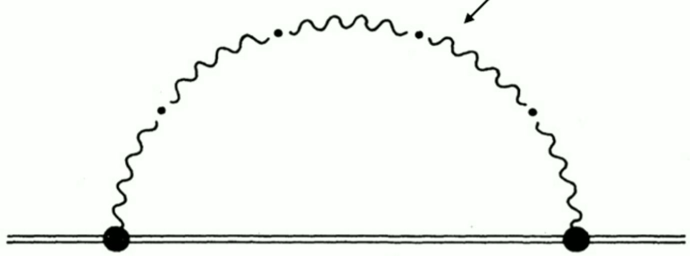
Finite parts really matter here!

Next steps: check w/relativistic QED

Exploit the fact that the relativistic Landau level problem has exact solutions:

$$\delta E_N = \int d^3x\, d^3y\, \overline{u}_N(\mathbf{x})\Sigma_A(\mathbf{x},\mathbf{y};E_N)u_N(\mathbf{y})$$ ← 4-component Dirac spinors

Coulomb-gauge photon propagator in cavity: $D^{ij}(\mathbf{x},\mathbf{y};\omega) = i\sum_{s\sigma}\frac{u^i_{s\sigma}(\mathbf{x})u^{*j}_{s\sigma}(\mathbf{y})}{\omega^2-\omega^2_{s\sigma}+i\epsilon}.$

$\Sigma_A(\mathbf{x},\mathbf{y};E_N) =$  $= -e^2\int d(t-t')e^{iE_N(t-t')}\gamma^\mu S(\mathbf{x},t,\mathbf{y},t')\gamma^\nu D_{\mu\nu}(\mathbf{x},t,\mathbf{y},t')$

Exact propagator for electron in a constant B-field (Schwinger):

$$S(X,Y) = e^{i\Phi(X,Y)}\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(X-Y)}\int_0^\infty\frac{d\tau}{\cos(eB\tau)}\exp\left[i\tau\left(k_\parallel^2 - k_\perp^2\frac{\tan(eB\tau)}{eB\tau} - m^2 + i\epsilon\right)\right]\left\{(\not{k}_\parallel + m)[\cos(eB\tau)+\gamma^1\gamma^2\sin(eB\tau)] + \frac{\not{k}_\perp}{\cos(eB\tau)}\right\}$$

[Schwinger, PRL 1951; Kuznetsov and Mikheev, Springer 2013]

Next steps: check w/Lorentz-invariant boundary propagator

Can isolate the boundary-only part of the photon propagator by inserting delta functions:

$$Z = \int \mathcal{D}A\, \delta(n^\nu \widetilde{F}_{\mu\nu}|_{\text{bdry}}) \exp\left[i \int d^4x\, \mathcal{L}[A]\right] \sim \int \mathcal{D}A\, \mathcal{D}B \exp\left[i \left(\int d^4x\, \mathcal{L}[A] + \int dS\, \delta\mathcal{L}[A, B]\right)\right]$$

conducting boundary conditions

additive correction to action on surface

$$\longrightarrow \quad D_{\mu\nu}(X, X') = D^{(\text{free})}_{\mu\nu}(X - X') + \overline{D}^{(\text{bdry})}_{\mu\nu}(X, X')$$

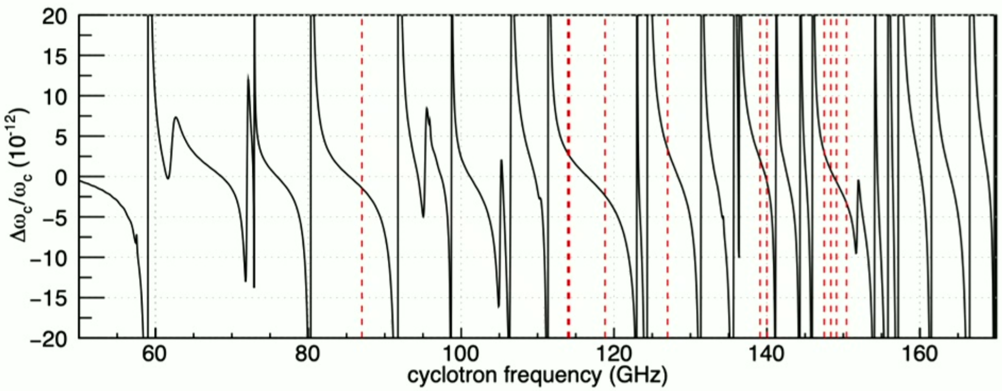
Technical complications: cylinder has sharp edges, QED in curved space is tricky… probably most useful for comparing with classical calculation in spherical cavity

[Bordag, Robaschik, Wieczorek, Ann. Phys. 1985]

Next steps: Q-factors and systematic uncertainty

$$\delta E_N = \sum_{\sigma=1,2} \sum_{s} \sum_{N'} \frac{|\langle N;0|\delta H|N';1_{s\sigma}\rangle|^2}{E_N - (E_{N'} + \omega_{s\sigma}) + i\Gamma_{s\sigma}}.$$

UV divergences independent of finite Q-factors for low-lying cavity modes:
classical calculation requires assuming uniform Q-factor for all TE and TM modes



reproduce this plot using renormalized quantum calculation, check systematic uncertainty as number of included modes is varied

[X. Fan, Ph.D. thesis 2022]

Conclusion and perspectives

Lamb Schwinger

Shelter Island conference, 1947

Bethe's notes

Lamb

0.365 cm^{-1} = 11,000 Mc

Notes on Shelter Island Conference. 1

Kemble & Present

to explain 0.0[illegible] cm^{-1} raising ~~shift~~ of S level

cutting Coulomb potential off flat, or replace by infinite

[3] The first use of the renormalization concept was in a non-relativistic calculation of the Lamb shift by Bethe (1947). Schwinger was the first to treat the problem of the Lamb shift and the electron anomaly using a relativistic formulation. The theory of QED was also developed independently by Tomonaga (1946) and Feynman (1948).

From the Lamb shift to g-2 and back again: pushing QED to its ultimate quantum limits