Title: Matroids and the Moduli Space of Abelian Varieties

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Abstract:

Inspired by recent work calculating the top weight cohomology of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g for small values of g, I will discuss a connection between matroids and compactifications of $\mathcal{A}_g$ that is anlogous to the connection between graphs and compactifications of the moduli space of curves. Given time I will also discuss recent work computing the homology of various matroid complexes.

Matroids & the moduli of Abelian Varieties

~ Juliette Bruce (Dartmouth)

M_g = moduli space of genus g curves

$\cong \left\{ \begin{matrix} \text{Compact Riemann} \\ \text{Surfaces genus } g \end{matrix} \right\} / \cong \;\cong\; \mathcal{T}(S_g) / \mathrm{Mod}(S_g)$

A_g = moduli of p.p. abelian Varieties of dim g.

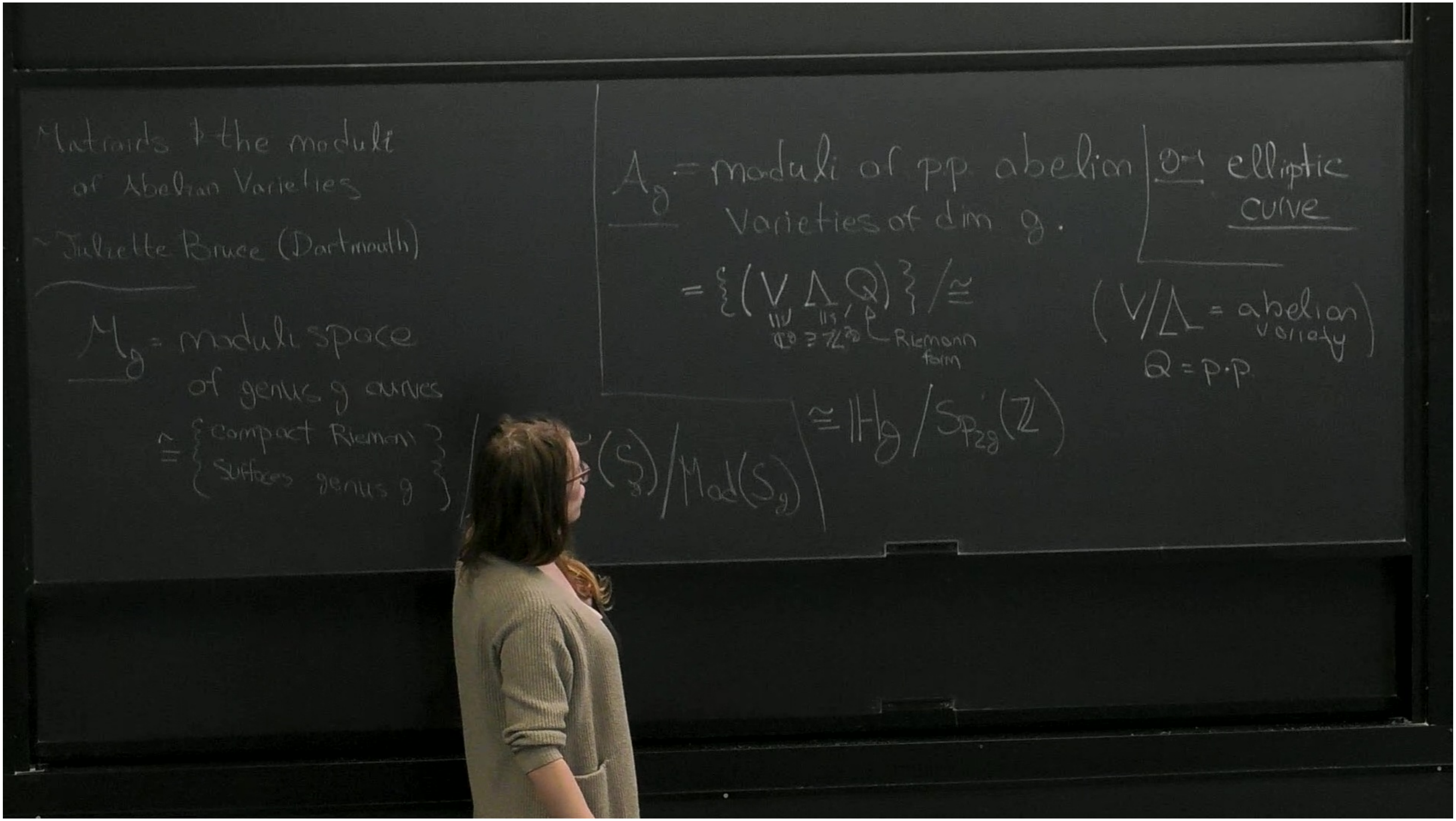
$g=1$ elliptic curve

$= \{ (V, \Lambda, Q) \} / \cong$

$V \cong \mathbb{C}^g$, $\Lambda \cong \mathbb{Z}^{2g}$, Q — Riemann form

(V/Λ = abelian variety

Q = p.p.)


Matroids & the moduli
of Abelian Varieties
~ Juliette Bruce (Dartmouth)
M_g = moduli space
of genus g curves
≅ { compact Riemann surfaces genus g }
(S_g)/Mod(S_g)
A_g = moduli of p.p. abelian
Varieties of dim g.
= {(V, Λ, Q)}/≅
ℂ^g ≅ ℤ^2g
Riemann form
≅ H_g / Sp_2g(ℤ)
g=1 elliptic curve
(V/Λ = abelian variety)
Q = p.p

of genus g curves

$\cong \left\{ \begin{array}{l} \text{compact Riemann} \\ \text{surfaces genus } g \end{array} \right\} / \cong \;\cong \mathcal{T}(S_g) / \mathrm{Mod}(S_g)$

form

$\cong \mathbb{H}_g / Sp_{2g}(\mathbb{Z})$

$Q = P \cdot P$

$\mathbb{H}_g = \{ \tau \in M^{sym.}_{g \times g}(\mathbb{C}) \mid \mathrm{Im}(\tau) > 0 \}$

$Sp_{2g}(\mathbb{Z}) \circlearrowright \mathbb{H}_g$

$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau = (A\tau + B)(C\tau + D)^{-1}$

Q: What is cohomology of M_g & A_g

$H^*_c(M_g; \mathbb{Q}) \cong H^*(\mathrm{Mod}(S_g), \mathbb{Q})$

$H^*_c(A_g, \mathbb{Q}) \cong H^*(Sp_{2g}(\mathbb{Z}); \mathbb{Q})$

$$\mathbb{H}_g = \left\{ \tau \in M^{\text{sym.}}_{g\times g}(\mathbb{C}) \;\middle|\; \operatorname{Img}(\tau) > 0 \right\}$$

$$Sp_{2g}(\mathbb{Z}) \circlearrowright \mathbb{H}_g$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau = (A\tau + B)(C\tau + D)^{-1}$$

<u>Q</u>: What is co

$$H^*_c(\mathcal{M}_g; \mathbb{Q}) \cong H$$

$$H^*_c(\mathcal{A}_g; \mathbb{Q}) \cong$$

Known Results

+ Completely know H_c^*

$\mathcal{M}_g$: $g=0$, $g=1$, $g=2$ (?)

$\mathcal{A}_g$: $g=0$, $g=1$, $g=2$ (Igusa '62), $g=3$ (Hain '02)

+ Stable Cohomology $H^i_c(\mathcal{M}_g;\mathbb{Q})$ & $H^i_c(\mathcal{A}_g;\mathbb{Q})$

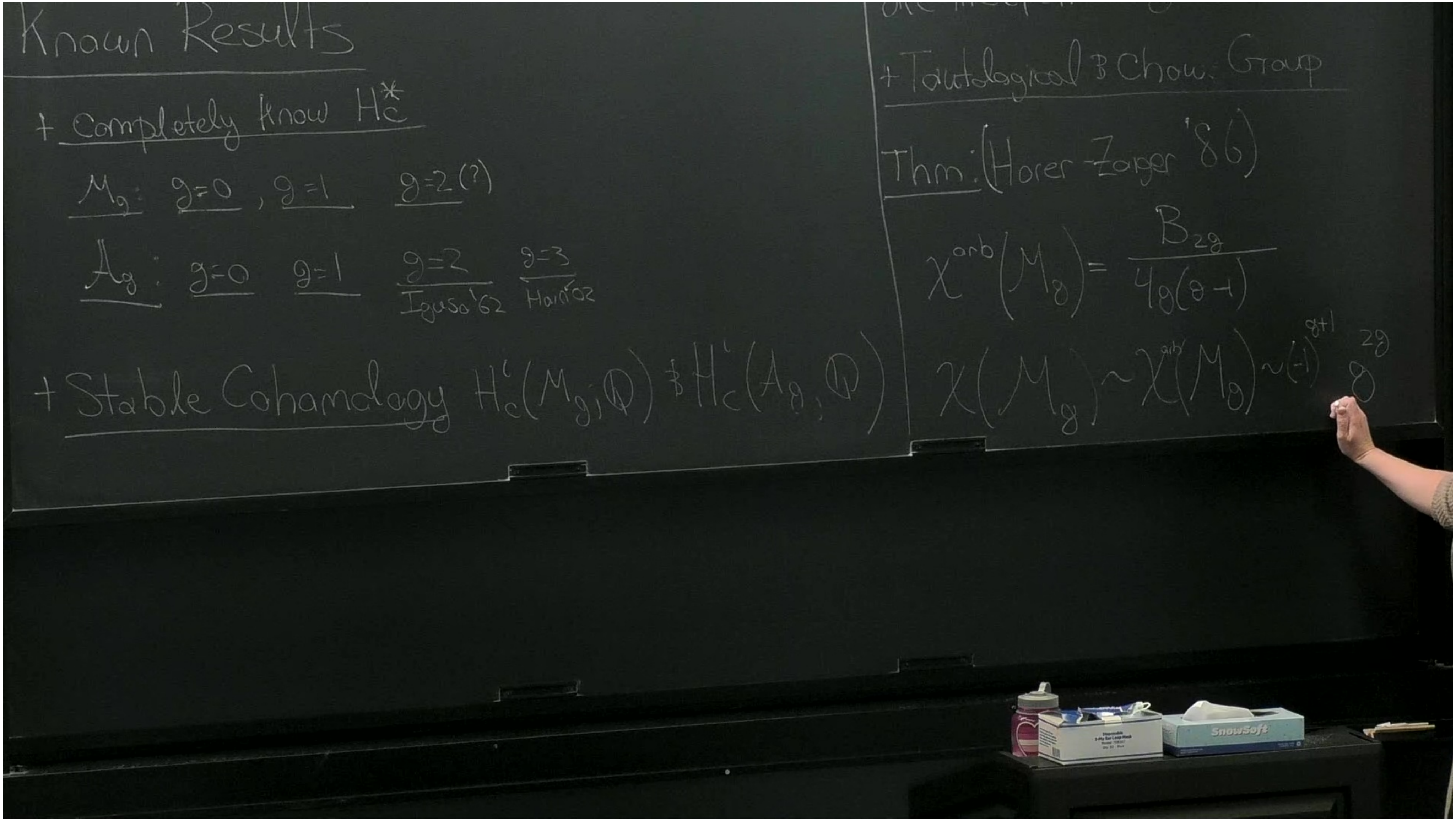
are indep of g if $i \ll g$

+ Tautological & Chow Group

Thm: (Harer-Zagier '86)

$$\chi^{orb}(\mathcal{M}_g) = \frac{B_{2g}}{4g(g-1)}$$

$$\chi(\mathcal{M}_g)$$


Known Results
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$\mathcal{A}_g$: $g=0$ $g=1$ $g=2$ $g=3$
Igusa '62
Hain '02
+ Stable Cohomology $H_c^i(\mathcal{M}_g, \mathbb{Q})$ & $H_c^i(\mathcal{A}_g, \mathbb{Q})$
+ Tautological & Chow Group
Thm: (Harer-Zagier '86)
$\chi^{orb}(\mathcal{M}_g) = \frac{B_{2g}}{4g(g-1)}$
$\chi(\mathcal{M}_g) \sim \chi^{orb}(\mathcal{M}_g) \sim (-1)^{g+1} g^{2g}$

+ Stable Cohomology $H^i_c(\mathcal{M}_g;\mathbb{Q}) \not\cong H^i_c(A_g;\mathbb{Q})$ | $\chi(\mathcal{M}_g) \sim \chi^{orb}(\mathcal{M}_g) \sim (-1)^{g}\, g^{2g}$

Fact only odd cohomology class

$$H^5(\mathcal{M}_4;\mathbb{Q}) \cong \mathbb{Q}$$

[Tommasi '04]

Thm [Chan-Galatius-Payne '21]

1) $\dim_{\mathbb{Q}} H^{2g}_c(\mathcal{M}_g;\mathbb{Q}) \geq \beta^g + C \quad (\beta \approx 1.3247\ldots)$

2) $H^{15}(\mathcal{M}_6,\mathbb{Q}) \neq 0$, $H^{23}(\mathcal{M}_8,\mathbb{Q}) \neq 0$, $H^{27}(\mathcal{M}_{10};\mathbb{Q}) \neq 0$

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}\cdot\tau = (A\tau+B)(C\tau+D)^{-1}$$

$$H^*_c(\mathcal{A}_g;\mathbb{Q}) \cong H^*(Sp_{2g}(\mathbb{Z});\mathbb{Q})$$

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Thm [Brandt-Bruce-Chan-Melo-Moreland-Wolfe '23]

$$H^0_c(\mathcal{A}_0,\mathbb{Q}) \quad H^6_c(\mathcal{A}_3;\mathbb{Q}) \quad H^{10}_c(\mathcal{A}_5;\mathbb{Q})$$

$$\underline{H^{15}_c(\mathcal{A}_5;\mathbb{Q})} \quad H^{12}_c(\mathcal{A}_6,\mathbb{Q}) \quad H^{14}_c(\mathcal{A}_7,\mathbb{Q})$$

$$\underline{H^{19}_c(\mathcal{A}_7,\mathbb{Q})} \quad \underline{H^{23}_c(\mathcal{A}_7,\mathbb{Q})} \quad H^{28}_c(\mathcal{A}_7,\mathbb{Q})$$

$$\neq 0$$

- Melo - Moreland - Walte '23]

$H^{14}_c(A_7, \mathbb{Q})$ $H^{23}_c(A_7, \mathbb{Q})$ $H^{28}_c(A_7, \mathbb{Q})$

$\neq 0$

<u>Fact</u> (Deligne '71, '74)

If X is variety

$$\underbrace{W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{2i}}_{\text{weight filtration}} = H^i_c(X, \mathbb{Q})$$

<u>Lemma</u> Suppose X a non-compact variety and $X \subseteq \overline{X}$ a compactification

A) X & $\overline{X}$ are smooth

B) $D = \overline{X} \setminus X$ has codim 1 (divisor)

C) $D = D_1 \cup D_2 \cup \ldots D_r$ ← irred components & they meet transversal

2) $H(M_6, \mathbb{Q}) \neq 0$, $H(M_8, \mathbb{Q}) \neq 0$, $H(M_{10}, \mathbb{Q})$

1) there is a chain complex $D_I = \bigcap_{i \in I} D_i$

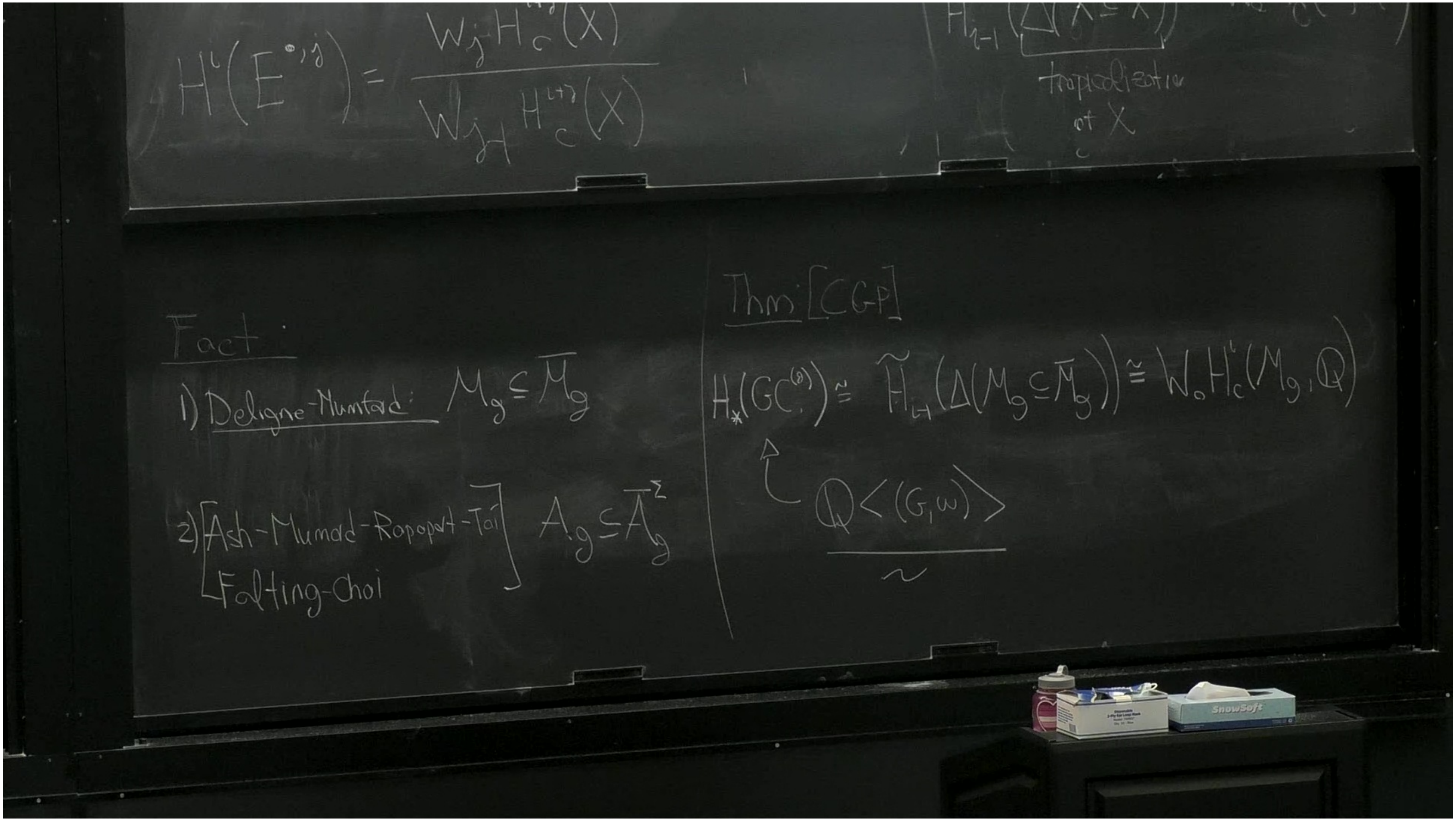
$$0 \to H^j(\overline{X}, \mathbb{Q}) \to \bigoplus_{i=1}^{r} H^j(D_i, \mathbb{Q}) \to \bigoplus_{\substack{|I|=2 \\ I \subseteq \{1, \dots, r\}}} H^j(D_I; \mathbb{Q})$$

$$\to \cdots \qquad {}^{\text{II}}E^{\bullet, j}$$

$$H^i(E^{\bullet, j}) = \frac{W_j H_c^{i+j}(X)}{W_{j-1} H_c^{i+j}(X)}$$

2) $j = 0$! There is a simplicial complex $\Delta(X \subseteq \overline{X})$ on $\{1, \dots, r\}$

$$\widetilde{H}_{i-1}(\Delta(X \subseteq \overline{X})) \cong W_0 H_c^i(X; \mathbb{Q})$$



$H^i(E^{\bullet,j}) = \dfrac{W_j H_c^{i+j}(X)}{W_{j-1} H_c^{i+j}(X)}$
$H_{i-1}(\Delta(X \subseteq \overline{X}))$
tropicalization of X
Fact:
1) Deligne-Mumford: $M_g \subseteq \overline{M}_g$
2) Ash-Mumford-Rapoport-Tai / Falting-Chai $A_g \subseteq \overline{A}_g^{\Sigma}$
Thm [CGP]
$H_*(GC_0^{(g)}) \cong \widetilde{H}_{i-1}(\Delta(M_g \subseteq \overline{M}_g)) \cong W_0 H_c^i(M_g, \mathbb{Q})$
$\mathbb{Q}\langle (G,\omega) \rangle / \sim$

2) Ash-Mumford-Rapoport-Tai, Falting-Chai $\quad A_g \subseteq \bar{A}_g$

$\mathbb{Q}\langle (G,\omega)\rangle$

$\sim$

<u>Thm</u> [BBCMMW]: The complex $P_\bullet^{\infty}$ is filtered

$$P_\bullet^0 \subseteq P_\bullet^1 \subseteq \cdots \subseteq P_\bullet^{\infty}$$

such that $H_\bullet(P_\bullet^k / P_\bullet^{k-1}) \cong H^*(GL_k(\mathbb{Z}); \mathbb{Q})$

& the spectral sequences converges on E^2.