Title: Complexity of Fermionic 2-SAT

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Abstract:

In this talk, I will discuss the complexity of a fermionic analogue of Quantum k-SAT. In this Fermionic k-SAT problem, one is given the task to decide whether there is a fermionic state in the null-space of a collection of fermionic, parity-conserving, projectors on n fermionic modes, where each fermionic projector involves at most k fermionic modes. We prove that this problem can be solved efficiently classically for k = 2. In addition, we show that deciding whether there exists a satisfying assignment with a given fixed particle number parity can also be done efficiently classically for Fermionic 2-SAT: this problem is a quantum-fermionic extension of asking whether a classical 2-SAT problem has a solution with a given Hamming weight parity. We also prove that deciding whether there exists a satisfying assignment for particle-number-conserving Fermionic 2-SAT for some given particle number is NP-complete. Complementary to this, we show that Fermionic 9-SAT is QMA_1-hard.

Complexity of Fermionic 2-SAT

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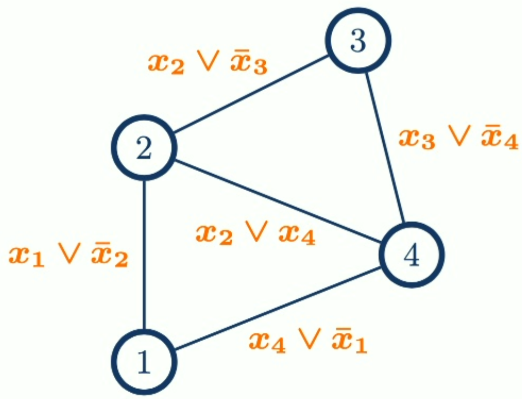
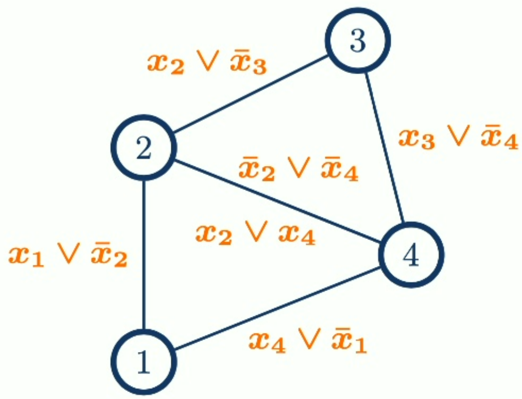
What are *satisfiability* problems?

k-Satisfiability: *Is there an assignment (a state) that satisfies a given collection of constraints (each acting on k degrees of freedom)?*

Classical 2-satisfiability: Is there an assignment to n boolean variables s.t. $(x_1 \vee \bar{x}_4) \wedge \ldots \wedge (x_{12} \vee x_{n-3})$ is true?



Satisfiable



Not satisfiable

What do we know?

(Classical) k-SAT:

Decide if there is an assignment of n Boolean variables $\{x_i\}_{i=1}^n$ such that a formula of type

$$\underbrace{(x_1 \vee \overline{x}_3 \vee \ldots \vee x_6)}_{k \text{ variables}} \wedge \ldots \wedge (x_4 \vee \overline{x}_{n-3} \vee \ldots \vee \overline{x}_{n-1})$$

is true.

Can be solved in linear time for $k = 2$ [APT79] and is NP-complete for $k \geq 3$.

Quantum k-SAT:

Decide if there is a state in the null-space of a collection of projectors, each acting on k qubits.

Can be solved in linear time for $k = 2$ [Bra11, dBG16, ASSZ18] and is QMA_1-hard for $k \geq 3$ [Bra11, GN13].

Why *Fermionic* 2-SAT?

- Satisfying assignments are fundamentally different from those of Quantum 2-SAT.
- The fermionic nature of the problem lets us investigate the complexity of adding global constraints:

  $\rightarrow$ Particle number constraint.
  $\rightarrow$ Particle number parity constraint.

Defining fermionic systems

Fermionic system:

- Operators $a_j^\dagger$ and a_j for $j = 1, \ldots, n$ on a 2^n-dim. Hilbert space such that $\{a_i, a_j^\dagger\} = \delta_{ij}$ and $\{a_i, a_j\} = 0$.

  Via Jordan-Wigner transformation: $a_j^\dagger \to Z_1 \ldots Z_{j-1} |1\rangle \langle 0|_j I_{j+1} \ldots I_n$.

- A *vacuum* or *empty* state $|\mathrm{vac}\rangle$ s.t. $\forall j \; a_j |\mathrm{vac}\rangle = 0$.
- Particle number $\hat{N} = \sum_{j=1}^n a_j^\dagger a_j$ and parity $\hat{P} = (-1)^{\hat{N}}$.

States are of the form

$$|\psi\rangle = \sum_{S \subseteq [n]} \alpha_S \; a_S^\dagger |\mathrm{vac}\rangle ,$$

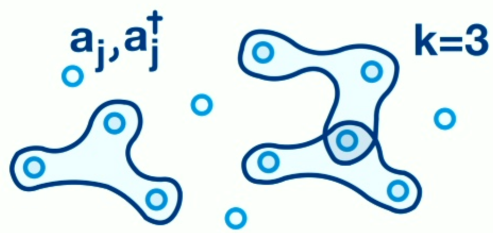
with $a_S^\dagger |\mathrm{vac}\rangle$ a *classical* state.

The problem that we want to solve: **Fermionic satisfiability**

Fermionic k-SAT:

Given n fermionic modes and projectors $\{\Pi_S\}$ with $S \subset [n]$ and $|S| = k$, where each projector is a polynomial in a_j and $a_j^\dagger$ with $j \in S$. Decide whether

1. there exists $|\psi\rangle$ s.t. $\forall S\ \Pi_S |\psi\rangle = 0$, or
2. for all $|\psi\rangle$, $\sum_S \langle\psi| \Pi_S |\psi\rangle > 1/\text{poly}(n)$.



Furthermore, we take each Π_S to be **parity preserving**.

Or, moreover, **particle number conserving** → PNC Fermionic k-SAT.

Characterization of projectors

Projectors are **parity preserving**, and so we have two types of rank-1 projectors:

- Π_e^1 projects onto a 1-particle state on edge $e = (j, k)$.

  $\longrightarrow \ \alpha \left|01\right\rangle_{jk} + \beta \left|10\right\rangle_{jk}.$

  For example: $\Pi^1_{(1,4)} = \frac{1}{2}\big(\left|10\right\rangle_{14} - Z_2 Z_3 \left|01\right\rangle_{14} \big)\big(\left\langle 10\right|_{14} - Z_2 Z_3 \left\langle 01\right|_{14} \big).$

- Π_e^{02} projects onto a (0+2)-particle state on edge $e = (j, k)$.

  $\longrightarrow \ \alpha \left|00\right\rangle_{jk} + \beta \left|11\right\rangle_{jk}.$

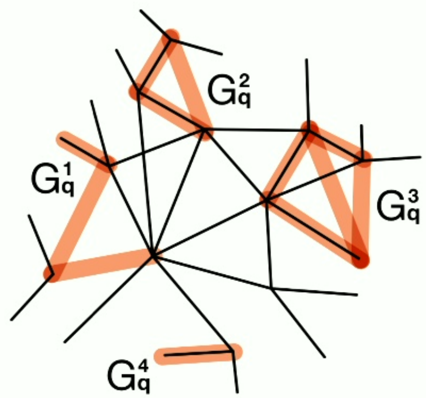
Higher-rank ($\leq$3) projectors are (particular) sums of these rank-1 projectors or *clauses*.

We distinguish between *classical clauses* $\Pi_e^{1,c}$ and $\Pi_e^{02,c}$, and genuinely *quantum clauses* $\Pi_e^{1,q}$ and $\Pi_e^{02,q}$.

Quantum clusters

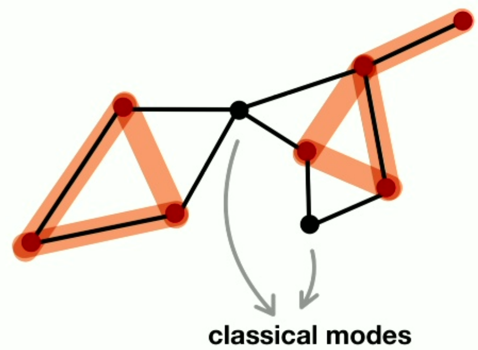
Quantum clusters → The sub-graphs that remain when taking away all classical clauses and classical modes.

Come in handy when characterizing the satisfying assignments.



Satisfying assignments come in a **cluster-product form**

It turns out to be sufficient to look for assignments in *cluster-product form.*



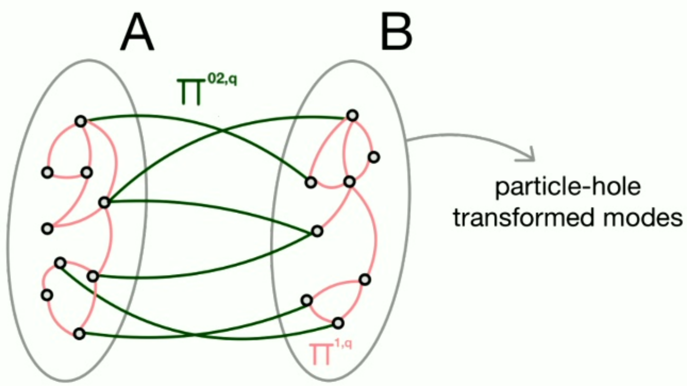
$$\Big[\sum_{S\subseteq \text{cluster 1}} \alpha_S\, a_S^\dagger\Big]\Big[\sum_{S\subseteq \text{cluster 2}} \alpha_S\, a_S^\dagger\Big]\, a^\dagger_{\text{Class 1}}\, a^\dagger_{\text{Class 2}}\, |\text{vac}\rangle$$

Understanding the satisfying assignments on **quantum clusters**

If a given cluster contains only $\Pi^{1,q}$-type clauses, then we call it a **particle-number-conserving (PNC) cluster**.

PNC clusters have **at most** one satisfying assignment per cluster particle number $N_q = 0, 1, \ldots, n_{\text{cluster}} - 1, n_{\text{cluster}}$.

Some clusters are PNC *in disguise*. $\rightarrow$ there is a particle-hole transformed basis in which they are PNC.



Understanding the satisfying assignments on **quantum clusters**

Any cluster is either **hidden PNC (hPNC)** or **non-hPNC**.

Clusters that are non-hPNC have **at most** one satisfying assignment per cluster parity $P_q = \pm 1$.

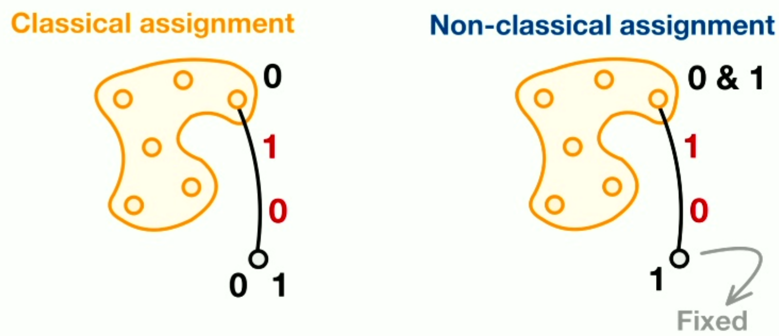
These non-hPNC satisfying assignments are always **non-classical**. $\rightarrow$ very restrictive.

The **non-classical** satisfying assignments on hPNC clusters (at $0 < N_q < n_{\text{cluster}}$) are similarly restrictive.

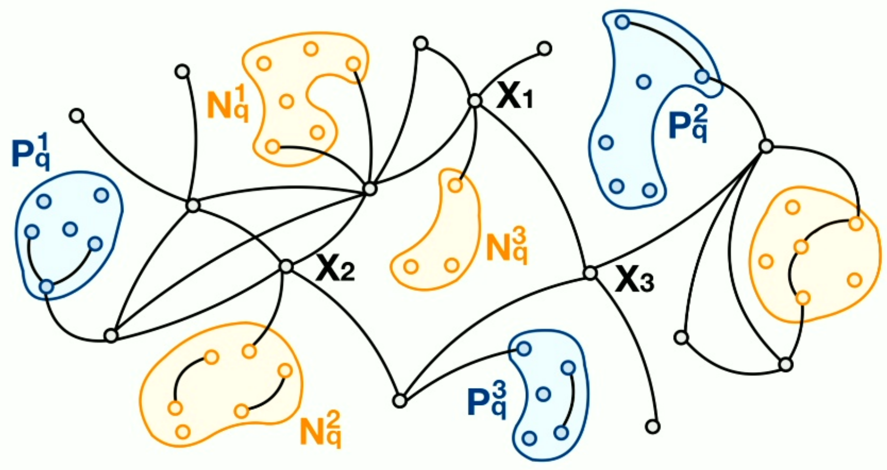
Distinguishing two types of quantum clusters

- **hPNC clusters:** at most $n_{\text{cluster}} + 1$ satisfying assignments, labeled by $N_q \in \{0, 1, 2, \ldots, n_{\text{cluster}} - 1, n_{\text{cluster}}\}$.
- **non-hPNC clusters:** at most 2 satisfying assignments, labeled by $P_q \in \{-1, +1\}$.

hPNC clusters allow for **classical assignments**. $\rightarrow$ **Crucially, can be enforced using just classical clauses.**



Solving Fermionic 2-SAT



Solve Fermionic 2-SAT by checking (1) whether clusters are satisfiable, and whether (2) a particular classical 2-SAT instance is satisfiable.

Which N_q's or P_q's are actually allowed?

	hPNC clusters	**non-hPNC clusters**
degree $\geq$ 3*	$N_q = 0, 1, n_q - 1, n_q$ allowed, all Gaussian.	Only $n_q = 4$, **non-Gaussian!**
lines and loops	Some N_q's allowed, only Gaussian ones required.	Some P_q's allowed, all Gaussian.

*Fermionic simplification.

Bottom line is, which N_q's or P_q's are allowed on a given cluster can be checked in time $O(n_q + m_q)$.

Solving Fermionic 2-SAT with fixed parity P

Naively solving this problem would take exponential time.

Two ingredients allow it to be solved efficiently:

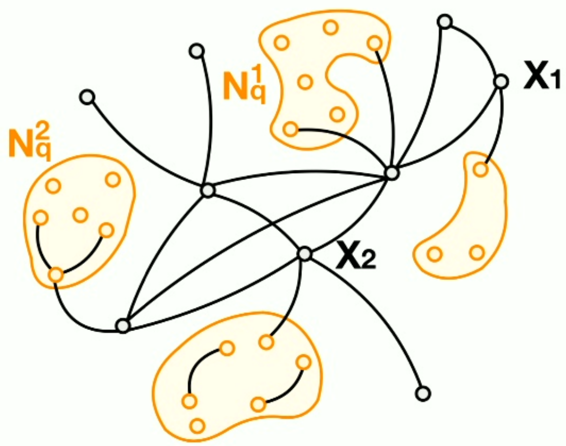
- We can perform checks of **allowed parities on clusters** efficiently.
- Our classical $O(nm)$-time algorithm for solving classical 2-SAT with fixed Hamming weight parity.

Complexity of fixing particle number to N

The *efficiently checkable* cluster-product form of satisfying assignments implies **containment in NP**.

→ Witness is a list of classical assignments (on classical modes) and cluster particle numbers.

NP-hardness follows from the NP-hardness of the ***N*-vertex cover problem**.



Conclusions

- Fermionic 2-SAT can be solved in linear time.
- When also fixing the particle number parity of the problem, we can solve it in polynomial time.
- When, moreover, fixing the particle number, the problem becomes NP-complete.
- What is the complexity of the fixed "particle number" problem for Quantum 2-SAT? $\rightarrow$ Not clearly in NP.