Title: The (quantum) life of pi

Speakers: Aninda Sinha

Collection/Series: Quantum Fields and Strings

Subject: Quantum Fields and Strings

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Abstract:

In 1914, Ramanujan wrote down 17 intriguing formulas for 1/pi, which motivated the modern-day machinery of computing trillions of digits of pi. Most of these formulas lay unproven until the 1980s. The Canadian Borwein brothers wrote a comprehensive treatise proving these formulas in the 1980s. We can now ask, “What is the physics behind Ramanujan’s pi”? I will argue that the physics connection can be found via logarithmic conformal field theories, for instance, those studied in the fractional quantum hall effect, polymers, and percolation. The CFT connection gives rise to an infinite number of new formulas for pi. In contrast, the myriad Ramanujan formulas provoke us into looking into faster converging basis for conformal correlators, which is, in turn, provided by the stringy dispersion relation mentioned above.

The (quantum) life of pi

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$\pi @ PI, 2025$

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Madhava's pi

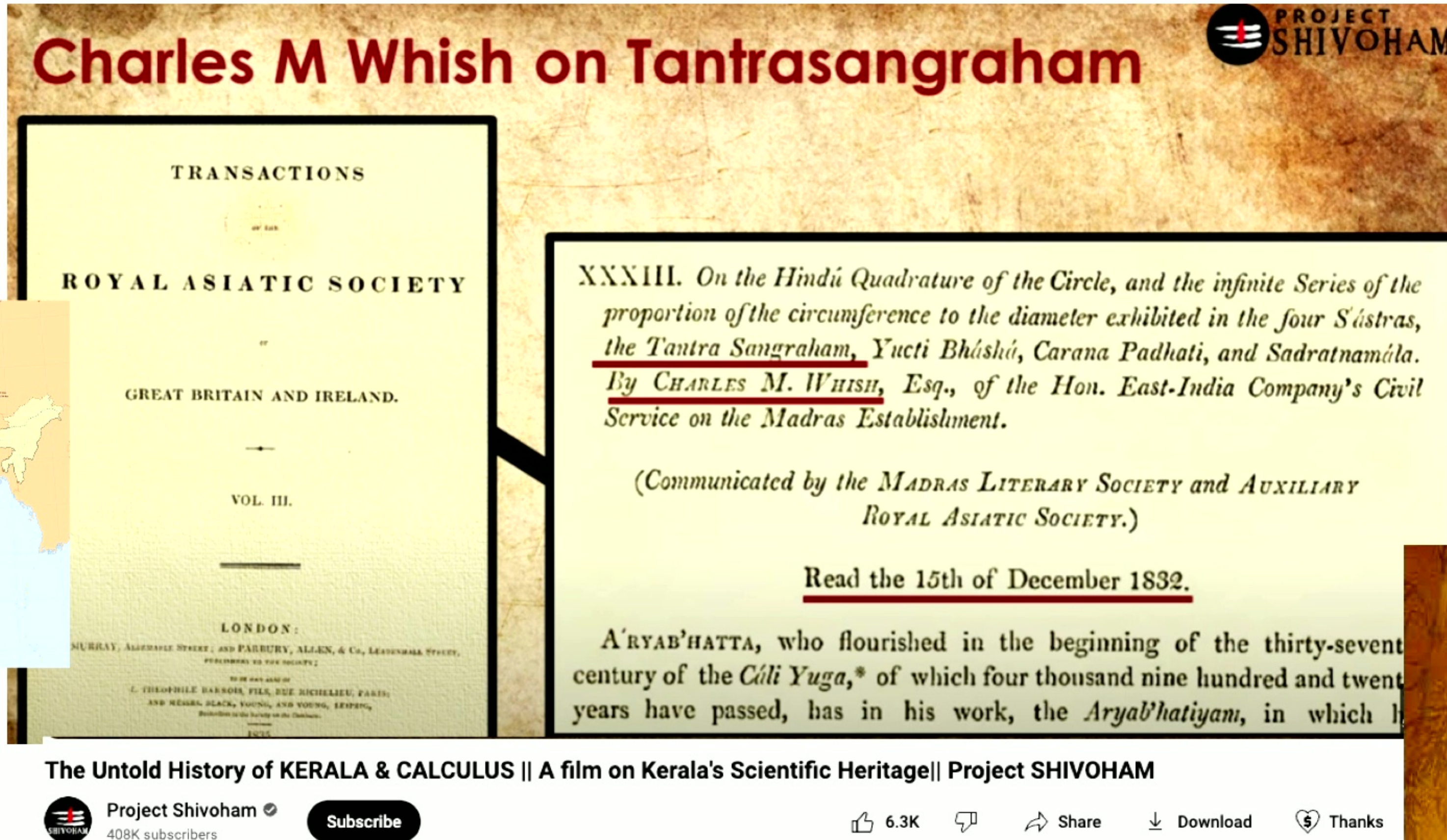
Madhava 1340-1425

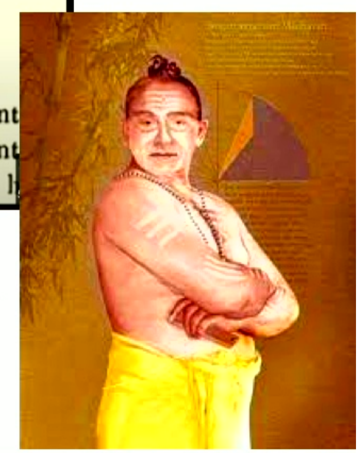
$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Madhava-Leibniz series

Notoriously slow to converge. 10 decimal place needs billions of terms.

Kerala school of mathematics





Astronomical treatise by Nilakantha Somayaji of the Kerala school of astronomy and mathematics, 1501. Indians were uniformly referred to as the Hindus by the British!

The Foundation of Calculus

PROJECT SHIVOHAM

Vyásé varidhinihité rúpa hṛité vyásaságarábhihate
Triṣarádhi vishama sanchyá bhactamṛiṇamswam prithacramát curyát
Yatsanchyayátra haraṇé critérnivrittá hṛitistujámitayá
Tasyá úrdhwa gatáyássamasanchyá taddalamgunóntésyát.
Jadvarggó rúpayutó háró vyásábdhighátacah prágwat.
Tabhyámáptam swamṛiṇé critédhané ṣódhananchacaraṇíyám
Súcshmah paridhissasyát bahucritwóharaṇatóti súcshmascha.

" Multiply the diameter by 4, and from it subtract and add alternately the quotients
" obtained by dividing four times the diameter by the odd numbers 3, 5, 7, 9, 11, &c.,
" do thus to the extent required; and having fixed a limit, take half the even number
" next less than the last odd divisor for a multiplier, and its square *plus* one for a divisor.
" Multiply four times the diameter by the multiplier, and divide the product b
" divisor, and add it or subtract it, according to the sign of the last quote in the
" from the sum of the series· thus the circumference of the given diameter w
" obtained very correctly."

The Hindu Quadrature of the Circle

- by Mr.Whish, 1835 Page 512-513

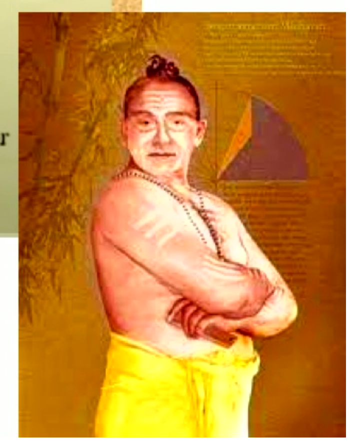
Mr. Whish on the Hindú Quadrature of the Circle. 513

If we proceed according to the rule, we have an infinite series of the following form :

$$C = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \frac{4d}{9} - \frac{4d}{11} + \&c. \pm \frac{4d \times \frac{1}{2}p}{p^2 + 1}.$$

where C=circumference, d diameter, and p the last odd divisor
nished by unity. When $d=1$ the series becomes

$$C = 4 \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \pm \&c. \frac{\frac{1}{2}p}{p^2+1}\right)$$



Euler: pi, Beta, Zeta……

- Euler invented a huge amount of mathematics.
- e, $e^{i\pi} + 1 = 0$, $\Gamma(x)$, $B(x, y)$, $\zeta(n)$…also popularised the notation π
- Went blind towards the end of his life, lost house in a fire, lost wife, but productivity increased!
- 25,000 pages of math—his work continued to be published 50 years after his death!
- B(x,y) plays an important role in our story.

Ramanujan's pi

Srinivasa Ramanujan Aiyangar[a] (22 December 1887 – 26 April 1920) was an Indian mathematician. Often regarded as one of the greatest mathematicians of all time, though he had almost no formal training in pure mathematics, he made substantial contributions to mathematical analysis, number theory, infinite series, and continued fractions, including solutions to mathematical problems then considered unsolvable.

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3 (6n+1)}{4^n n!^3} = \frac{4}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3 (42n+5)}{64^n n!^3} = \frac{16}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2})_n^3 (4n+1)}{n!^3} = \frac{4}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{3}{4})_n (\frac{1}{4})_n}{n!^3} (26390n+1103) \frac{1}{99^{4n}} = \frac{9801\sqrt{2}}{\pi}$$

The first two formulas, (1.2) and (1.3), appeared in the Walt Disney film *High School Musical*, starring Vanessa Anne Hudgens, who plays an exceptionally bright high school student named Gabriella Montez. Gabriella points out to her teacher that she had incorrectly written the left-hand side of (1.3) as $8/\pi$ instead of $16/\pi$ on the blackboard. After first claiming that Gabriella is wrong, her teacher checks (possibly Ramanujan's *Collected Papers*?) and admits that Gabriella is correct. Formula (1.2) was correctly recorded on the blackboard.

Pi and the AGM

A Study in Analytic Number Theory and Computational Complexity

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- **Fast converging formulas. General proofs found by (Canadian) Borwein brothers in the 1980s. Motivates many modern methods to calculate trillions of digits of pi.**
- **Is there a physics story? What is n? What do these numbers mean? Are these just isolated formulas?**

Work in progress with F. Bhat

Ramanujan, Modular Equations, and Approximations to Pi or How to Compute One Billion Digits of Pi

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Preface. The year 1987 was the centenary of Ramanujan's birth. He died in 1920 Had he not died so young, his presence in modern mathematics might be more immediately felt. Had he lived to have access to powerful algebraic manipulation software, such as MACSYMA, who knows how much more spectacular his already astonishing career might have been.

∞-ly new formulas for π from string/QFT

$$\pi = 4 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{4}{2n+1} + \frac{1}{n+\lambda} \right) \left(-n + \frac{(2n+1)^2}{4(n+\lambda)} \right)_{n-1}$$

Arises from QFT considerations

$Re(\lambda) > -1$, free parameter

$(a)_n = a(a+1)(a+2)\cdots(a+n-1)$
$(1)_n = n! = 1 \cdot 2 \cdot 3 \cdots n$

Rising factorial/ Pochhammer: polynomial

This formula does not exist in the mathematics literature. If $\lambda \gg 1$ then summand is exactly the Madhava series! So this formula interpolates the Madhava series to a faster converging QFT-inspired one!

Something very interesting. Between $-1/4 < \lambda \leq 1$, each $n \geq 1$ term in the summand is negative. UNLIKE Madhava's series.

Bizarre fact: Put $\lambda = n\pi$ and the formula should still work! Truncation gives an approximate polynomial equation for π.

2401.05733 with Arnab Priya Saha;
2409.07529 H. Rosengren (proofs and extensions for mathematicians).

Ramanujan pi and CFT

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\sigma)_n (1-\sigma)_n}{n!^3} (a+bn) z^n = \frac{1}{\pi}$$

- All Ramanujan formulas are of this form. $\sigma \in \{1/2, 1/3, 1/4, 1/6\}$

- I will argue that σ is related to the central charge of a 2d log-CFT: $c \in \{-2, -7, -25/2, -24\}$. z is the conformal cross-ratio in the "diagonal" limit. These formulas arise from 4pt correlation functions of eg. twist operators.

- Different Ramanujan formulas have different convergence rate. In the language of CFT we will ask what is the minimum number of operators needed to give 1/pi. The answer (very surprisingly) will turn out to be 1.

New formulas for π from CFT

$$\frac{4}{\pi} \approx -\frac{56496149211311 + 674161005879900 \log^2(2) - 557725361939340 \log(2)}{3435973836800\sqrt{2}}.$$

$$\frac{1}{\pi} \approx \frac{-1103136 + 954217 \log(4)}{786432\sqrt[4]{3}} + \frac{(4800909 - 3186838 \log(4))\log\left(\frac{4}{3}\right)}{786432\sqrt[4]{3}}$$

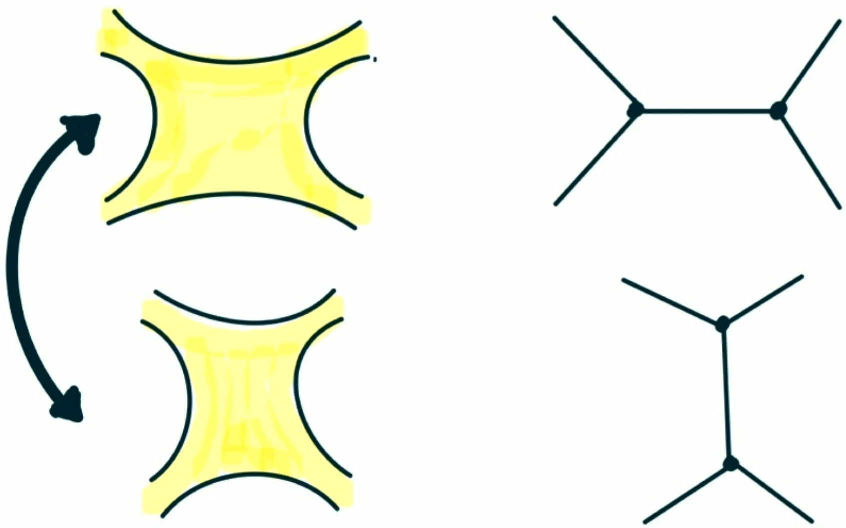
These bizarre looking formulas come from log-CFTs.

Work in progress with F. Bhat

The underlying motivating question is:
—What is the best basis to expand an amplitude/correlator?
—Feynman/Witten diagrams, string world sheet picture, partial wave expansion or something else?
—Something else: Stringy dispersion relation

Motivational question: string theory

$$M(s,t) \sim \int_{s_0}^{\infty} d\sigma \frac{\mathrm{disc}_s M(\sigma,t)}{\sigma - s}$$



Witten review, 2001, Snowmass

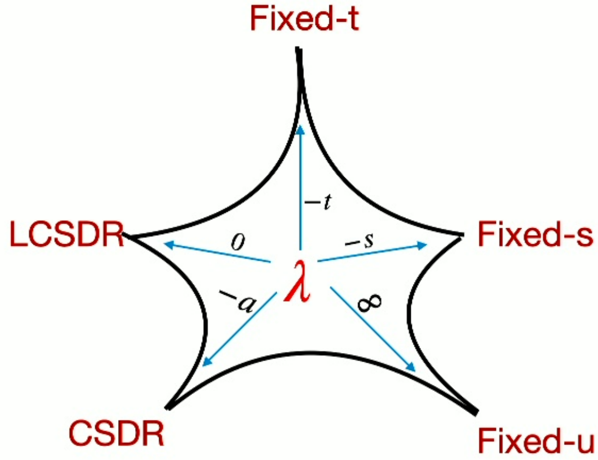
https://www.slac.stanford.edu/econf/C010630/papers/P337.PDF

- We are familiar with fixed-t or fixed-s dispersion relations
- Is there a way to deform the fixed-t into fixed-s (what is the deformation parameter??)?
- In other words, can we come up with a stringy flavoured dispersion relation?

The answer is yes!

Based on

1. Saha, AS, 2401.05733, Phys. Rev. Lett. ’24 ;
2. Bhat, Chowdhury, Saha, AS, 2409.18259, Phys. Rev. D ’25;
3. In progress with F. Bhat



Uses heavily:

4. AS, Zahed, 2012.04877, Phys. Rev. Lett. ’21;
5. Raman, AS, 2107.06559, JHEP ’22.

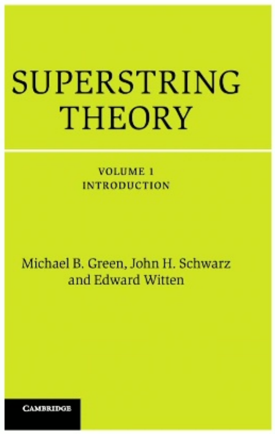
This line of research originated from exploring critical phenomena—
see 4. Gopakumar, Kaviraj, Sen, AS, 1609.00572 Phys. Rev. Lett. ’17;
5. Gopakumar, AS, Zahed, 2101.09017, Phys. Rev. Lett. ’21.

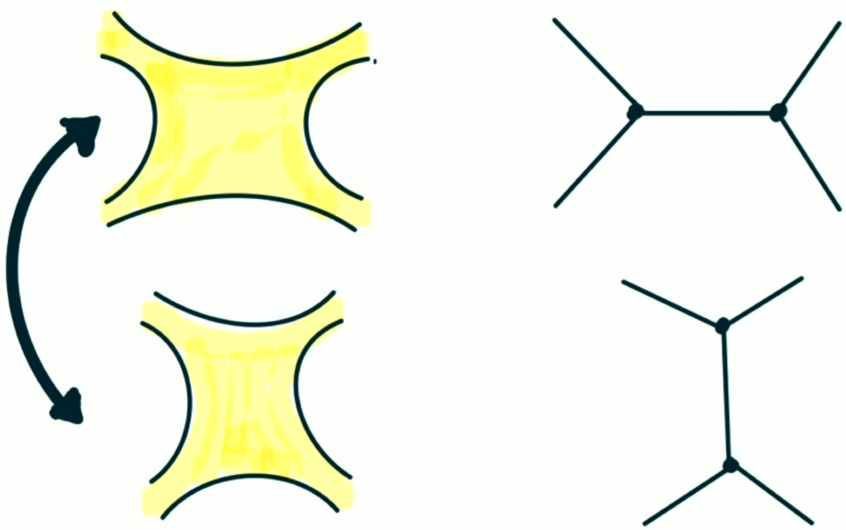
Outline

- The bootstrap origins of string theory
- Stringy dispersion relation and pi
- Application: Stringy bootstrap (breeze through, ask later)
- Ramanujan’s pi and log-CFTs

Bootstrap origins of string theory

Famous textbook description



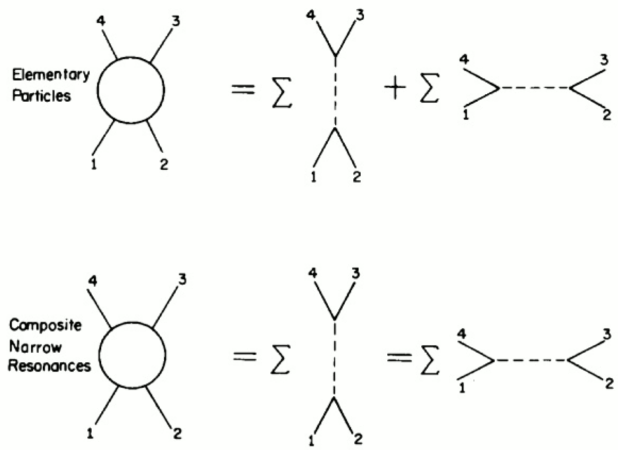


- Apparently different from QFT. There s and t channels need to be added together!!
- But modern understanding based on string field theory gives an alternative picture. Let me illustrate using the closed string case.

to something important were probably even worse than the prospects in 1900 that study of experimental data on black body radiation would be the first clue to a completely new theory. The duality hypothesis never had more than slender experimental support, and the Veneziano model was merely an *ad hoc* way of satisfying this not-so-well motivated hypothesis.

Confusing stmts never clarified

S. Mandelstam, Dual-resonance models



The difference between amplitudes with elementary particles and dual-resonance amplitudes.

The bootstrap idea had a precise formulation in the narrow resonance approximation, which was called 'duality'. This is the statement that a scattering amplitude can be expanded in an infinite series of s-channel poles, and this gives the same result as its expansion in an

The Birth of String Theory, Eds Cappelli et al

40 *John H. Schwarz*

infinite series of t-channel poles. To include both sets of poles, as usual Feynman diagram techniques might suggest, would amount to double counting.

37.2 Connection of dual models to field theories

From these early days of my studies on dual models, there had been a basic question which was increasingly occupying my mind. That was on the relationship between dual models and ordinary field theory. In characterizing the Veneziano amplitude, it had been emphasized that the amplitude could be expanded into a sum over an infinite set of either

Yoneya

Gravity from strings: personal reminiscences 461

s-channel poles or t-channel poles, but not of both. Adding both would amount to double counting. The unitarization programme started by the KSV paper was to be regarded as

Confusing stmts never clarified

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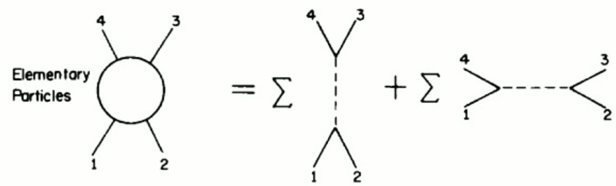

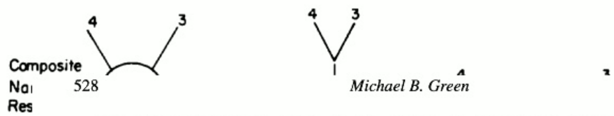

The differe

The most memorable event in my period as a graduate student in Cambridge (1967–1970) was the arrival of the preprint of Veneziano's 1968 paper [Ven68]. This was a period in which Cambridge was under the spell of the bootstrap ideas promoted most notably by Chew [Che66]. This philosophy had been born out of the apparent failure of quantum field theory to describe the strong force. It was hoped that the S-matrix for the strong interactions could be determined in a self-consistent manner by incorporating a minimal set of assumptions, most notably analyticity, crossing symmetry and unitarity. Quantum field theory was used merely as a model for how various aspects of the strong force, notably Regge pole asymptotic behaviour, might emerge in such a framework. These ideas were summarized in a book (Eden, Landshoff, Olive and Polkinghorne [ELOP66]) that became the basic text for graduate students in Cambridge.

Despite the limitations of the original bootstrap programme, it promoted inventive ways of analyzing hadronic data. Together with the advent of current algebra sum rules, this eventually led to the development of finite energy sum rules (see Igi and Matsuda [IM67], Ademollo, Rubinstein, Veneziano and Virasoro [ARVV67, ARVV69], Dolen, Horn and Schmid [DHS68] and Mandelstam [Man68]), which provided evidence for a deep connection between hadronic resonances and the Regge pole description of high energy scattering. This implied that the sum of the resonance contributions to a scattering amplitude is equivalent to the sum of Regge pole contributions, which are themselves determined by the sum of resonance exchanges in the crossed channel. In other words, one should not add the resonance poles in the s-channel to those in the t-channel as one would with Feynman diagrams in conventional field theory, since that would be double counting. This was known as the 'duality' between resonances and Regge poles and subsequently became known as 'world-sheet duality'.

The bootstrap idea had a precise formulation in the narrow resonance approximation, which was called 'duality'. This is the statement that a scattering amplitude can be expanded in an infinite series of s-channel poles, and this gives the same result as its expansion in an

The Birth of String Theory, Eds Cappelli et al

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Yoneya

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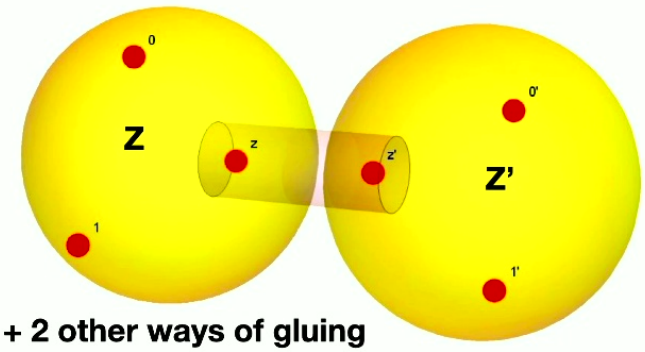
Plumbing Fixture

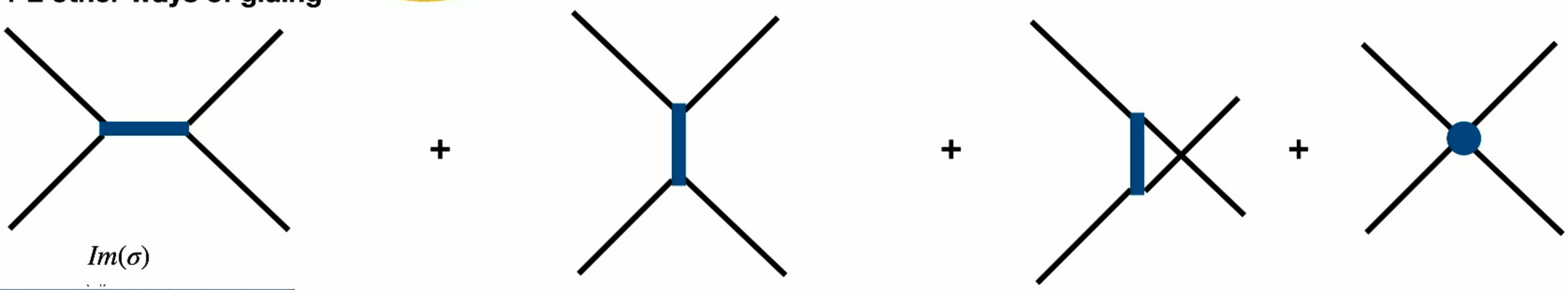
Polchinski 1

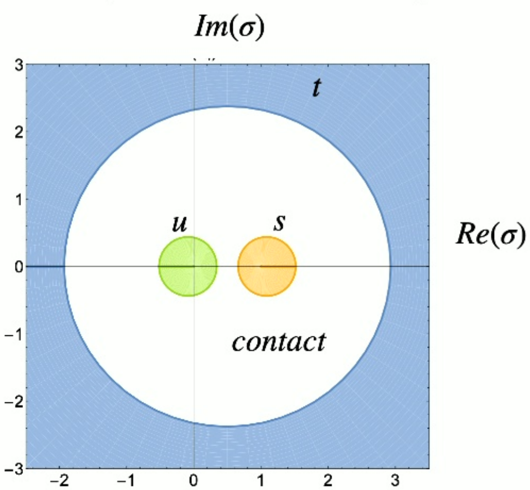


$$z^{(0)} = \frac{\lambda z}{z-2},\ z^{(1)} = \frac{\lambda(z-1)}{(z+1)},\ z^{(\infty)} = \frac{\lambda}{2z-1}$$

λ : Free parameter > 3

+ 2 other ways of gluing





- Only singularities are poles for tree level
- White region needs contact terms
- Sen, '19 showed how contact terms can be obtained numerically and depend on lambda.

String Field Theory

Series reps that are "known"

$$B(-s_1,-s_2)=\frac{\Gamma(-s_1)\Gamma(-s_2)}{\Gamma(-s_1-s_2)}$$

Rep1

Fixed-t or s

$$B(-s_1,-s_2)=-\sum_{n=0}^{\infty}\frac{1}{n!}\frac{(s_2+1)_n}{s_1-n}=-\sum_{n=0}^{\infty}\frac{1}{n!}\frac{(s_1+1)_n}{s_2-n}$$

$Re(s_2)<0$ $\qquad$ $Re(s_1)<0$

Rep2

Fixed-u

$$B(-s_1,-s_2)=\sum_{n=0}^{\infty}\left(\frac{1}{n-s_1}+\frac{1}{n-s_2}\right)\frac{(s_1+s_2-n+1)_n}{n!},$$

$Re(s_1+s_2)>-1$

$(a)_n=a(a+1)(a+2)\cdots(a+n-1)$

- Last one is rarely mentioned in string literature
- Neither meets the SFT expectations (only non-analyticity at poles) and neither exhibit the parametric ambiguity.

Resolution: A new series for the Euler-Beta function

$$B(-s_1,-s_2)=\frac{\Gamma(-s_1)\Gamma(-s_2)}{\Gamma(-s_1-s_2)}=-\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{s_1-n}+\frac{1}{s_2-n}+\frac{1}{\lambda+n}\right)\left(1-\lambda+\frac{(s_1+\lambda)(s_2+\lambda)}{n+\lambda}\right)_n \qquad \text{Re}\,\lambda>0$$

Can truncate and still retain all the stringy features

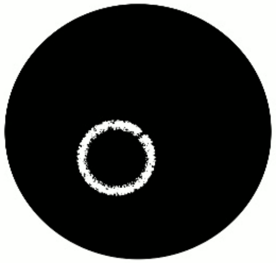
- Manifests poles in both channels
- Each piece analytic on its own away from poles
- Has a parametric form (which we will reinterpret in the language of field redefinition ambiguity)

- Does not exist in the literature. (Generalisation of hypergeometric series)
- Reason: Uses a local crossing symmetric dispersion relation in 2 channels (which did not exist before our work)
- Similar formula for open superstring, denominator is $\Gamma(1-s_1-s_2)$. Pi and Zeta(n) formulas in 2401.05733 come from here but there is a family of these formulas as is evident.

2401.05733 PRL with Arnab Priya Saha;
2409.07529 H. Rosengren (proofs and extensions for mathematicians).

Other reasons to desire a truncated representation

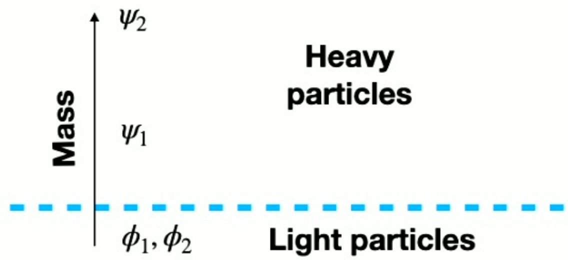
- String perturbation theory will break down when $R_S > \ell_{string}$ or for



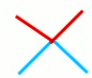
$$E_{CM} > \frac{1}{g_s^2 \ell_{string}}$$

- To study this using the bootstrap, we should match perturbation theory up to some energy scale to a nonperturbative representation and search for signatures of black holes as intermediate states.

Exploiting field redefinitions



- Say one of the interactions is: $\phi_1\phi_2\psi_2$

- Now redefine: $\psi_2 \to \psi_2 + \lambda\phi_1\phi_2$ and then drop all ψ_2 terms. The interaction term gives an extra piece $\lambda\phi_1\phi_2\phi_1\phi_2$.  **"Contact" interaction**

- This new term only depends on the light fields! So here is the imprint of the heavy fields. This is the important lesson we learn:

If we truncate the heavy fields, then the observables involving only light fields will know about the heavy guys through an ambiguity. This is the parametric ambiguity in the series !!

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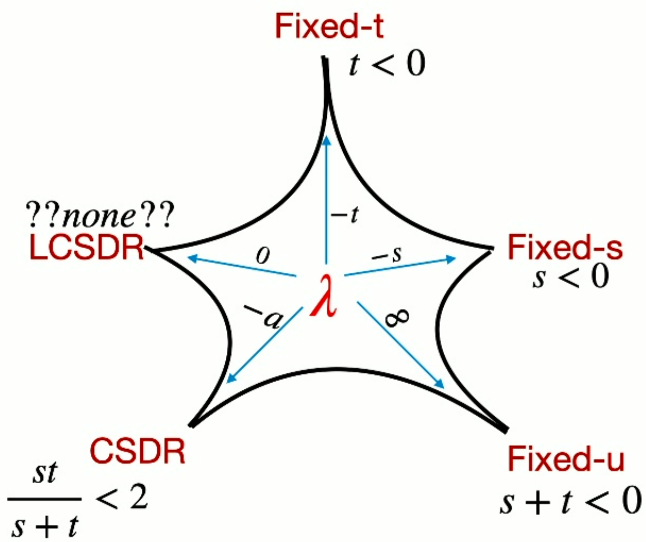
"Web of world sheet dualities"

Bhat, Chowdhury, Saha, AS, 2409.18259

$$M(s_1,s_2)=\frac{1}{\pi}\int_{s_0}^{\infty}\mathrm{d}\sigma\left(\frac{1}{\sigma-s_1}+\frac{1}{\sigma-s_2}-\frac{1}{\sigma+\lambda}\right)\mathscr{A}^{(s_1)}\left(\sigma,\frac{(s_1+\lambda)(s_2+\lambda)}{\sigma+\lambda}-\lambda\right)$$



$$a=\frac{s_1 s_2}{s_1+s_2}$$

- Parameter interpolates various pictures

- Tempting to relate to world sheet

Auberson, Khuri '73; AS, Zahed '21; Gopakumar, AS, Zahed '21; Raman, AS '22; Song '23; Saha, AS '24

A unifying series formula for The Euler Beta function

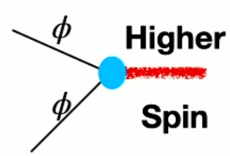
2401.05733 with Arnab Priya Saha

If $\lambda = -s_1$ or s_2 then recover Rep1.
If $\lambda \gg 1$ then interpolates to Rep2

New rep

$$B(-s_1,-s_2) = \frac{\Gamma(-s_1)\Gamma(-s_2)}{\Gamma(-s_1-s_2)} = -\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{s_1-n}+\frac{1}{s_2-n}+\frac{1}{\lambda+n}\right)\left(1-\lambda+\frac{(s_1+\lambda)(s_2+\lambda)}{n+\lambda}\right)_n$$

$$\mathscr{L} \overset{?}{=} \sum_{J,m}$$



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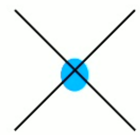
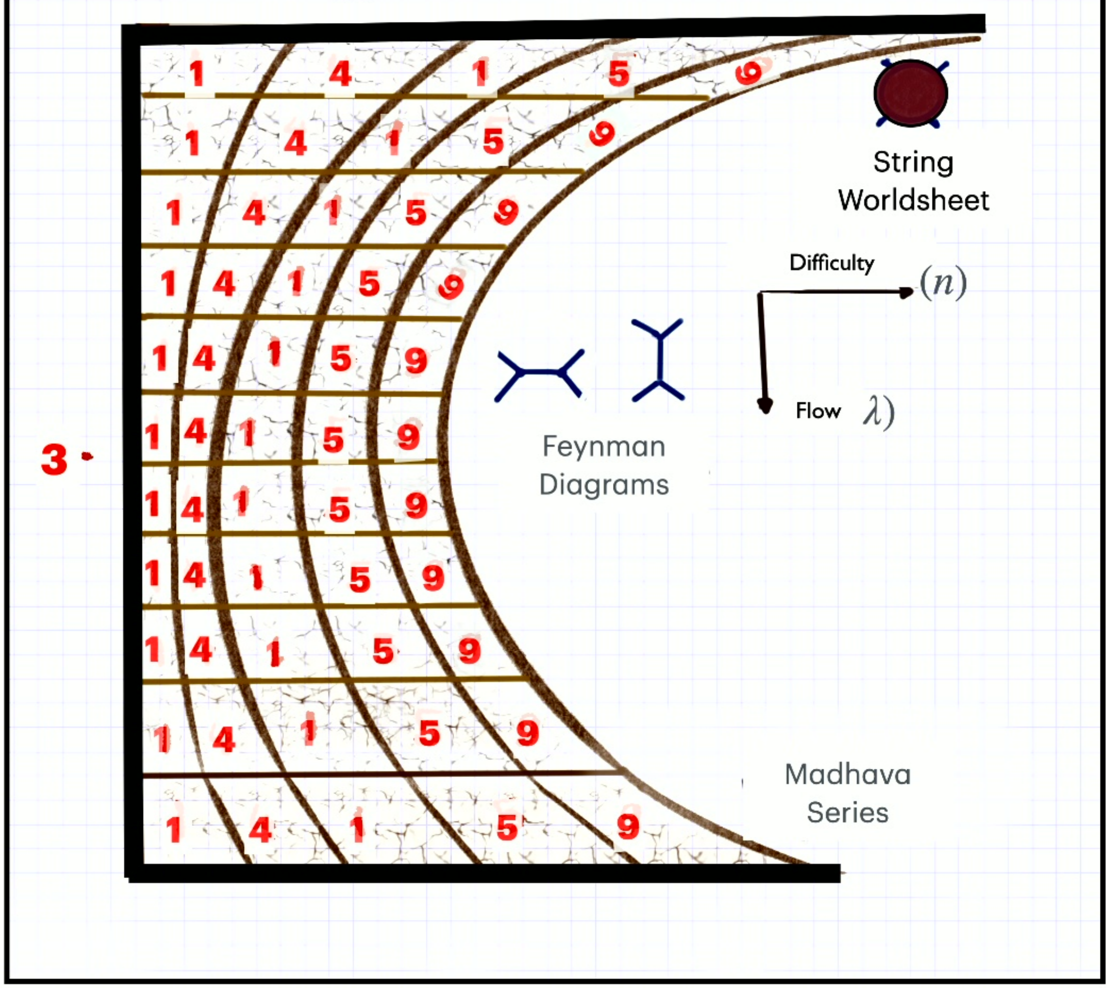
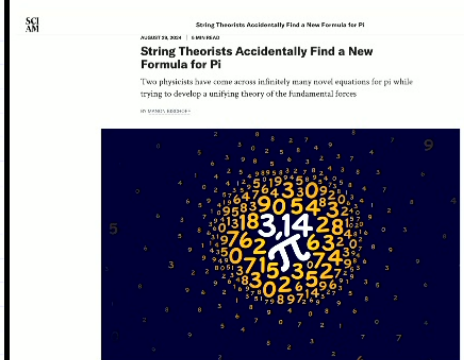


π − flow

$$\pi = 4 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{4}{2n+1} + \frac{1}{n+\lambda}\right)\left(-n + \frac{(2n+1)^2}{4(n+\lambda)}\right)_{n-1}$$



"Holographic"
pi?



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Field Theory Expansions of String Theory Amplitudes

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More

Generalised string models

- With Faizan Bhat, Debapriyo Chowdhury and Arnab Saha, in 2409.18259 we enlarged the class of "string-like" models using linear programming and Machine Learning.

- We want to consider a general class of amplitudes that arise using the dispersion relation with integer poles but with field redefinition ambiguity built-in.

- $$M(s_1, s_2) = \frac{1}{s_1 s_2}$$ Unitarity gives $c_\ell^{(n)} \geq 0$

$$+\sum_{n=1}^{\infty}\sum_{\ell} c_\ell^{(n)} \left[\frac{1}{s_1-n}+\frac{1}{s_2-n}+\frac{1}{\lambda+n}\right] \mathcal{G}_\ell^{\left(\frac{D-3}{2}\right)} \left[1+\frac{2}{n}\left(\frac{(s_1+\lambda)(s_2+\lambda)}{\lambda+n}-\lambda\right)\right]$$

How to constrain?

$$\bar{M} \equiv M(s_1, s_2) - \frac{1}{s_1 s_2}$$

- Find $c_\ell^{(n)} \geq 0$ such that $\bar{M}(s_1, s_2)$ is λ- independent.
- Subject to $\partial_{s_1} \bar{M}|_{s_1=0,s_2=0} = W_{10} \quad \partial_{s_1}\partial_{s_2}\bar{M}|_{s_1=0,s_2=0} = W_{01}$
- Minimize $\bar{M}(0,0)$ [physically, we are minimising the first finite moment of entangling power]
  Bose, Haldar, AS, Sinha, Tiwari 2006.12213; Aoude, Chung, Huang, Machato, Tam 2007.09486; Aoude, Alor, Remmen, Sumensari 2402.16956
- For the superstring values of W_{10}, W_{01} we recover the $c_\ell^{(n)}$s of the superstring numerically.
- Here Machine-Learning helps. This is because, numerically we have to work with level truncation and decide what it means for M to be lambda-independent.
- In other words impose $|1 - \frac{M_{\lambda_1}}{M_{\lambda_2}}| \leq \epsilon$ which is a nonlinear condition on $c_\ell^{(n)}$s.

n	Exact	Bootstrap
1	1	1.000
2	0.0714	0.0717
3	0.0119	0.0121
4	0.00289	0.00297
5	0.000867	0.000898
6	0.000300	0.000311
7	0.000114	0.000119
8	0.0000469	0.0000486
9	0.0000204	0.0000211
10	9.26×10^{-6}	9.59×10^{-6}
11	4.37×10^{-6}	4.52×10^{-6}
12	2.12×10^{-6}	2.19×10^{-6}
13	1.06×10^{-6}	1.09×10^{-6}
14	5.41×10^{-7}	5.57×10^{-7}
15	2.81×10^{-7}	2.90×10^{-7}
16	1.49×10^{-7}	1.53×10^{-7}
17	7.99×10^{-8}	8.21×10^{-8}
18	4.35×10^{-8}	4.47×10^{-8}
19	2.39×10^{-8}	2.46×10^{-8}
20	1.33×10^{-8}	1.37×10^{-8}

Leading

n	Exact	Bootstrap
3	0	0
4	0.00108	0.000367
5	0.000636	0
6	0.000325	0.000123
7	0.000163	0.000118
8	0.0000831	0.0000752
9	0.0000430	0.0000427
10	0.0000227	0.0000235
11	0.0000122	0.0000130
12	6.64×10^{-6}	7.20×10^{-6}
13	3.67×10^{-6}	4.01×10^{-6}
14	2.05×10^{-6}	2.25×10^{-6}
15	1.16×10^{-6}	1.27×10^{-6}
16	6.65×10^{-7}	7.23×10^{-7}
17	3.84×10^{-7}	4.14×10^{-7}
18	2.23×10^{-7}	2.39×10^{-7}
19	1.31×10^{-7}	1.39×10^{-7}
20	7.73×10^{-8}	8.05×10^{-8}

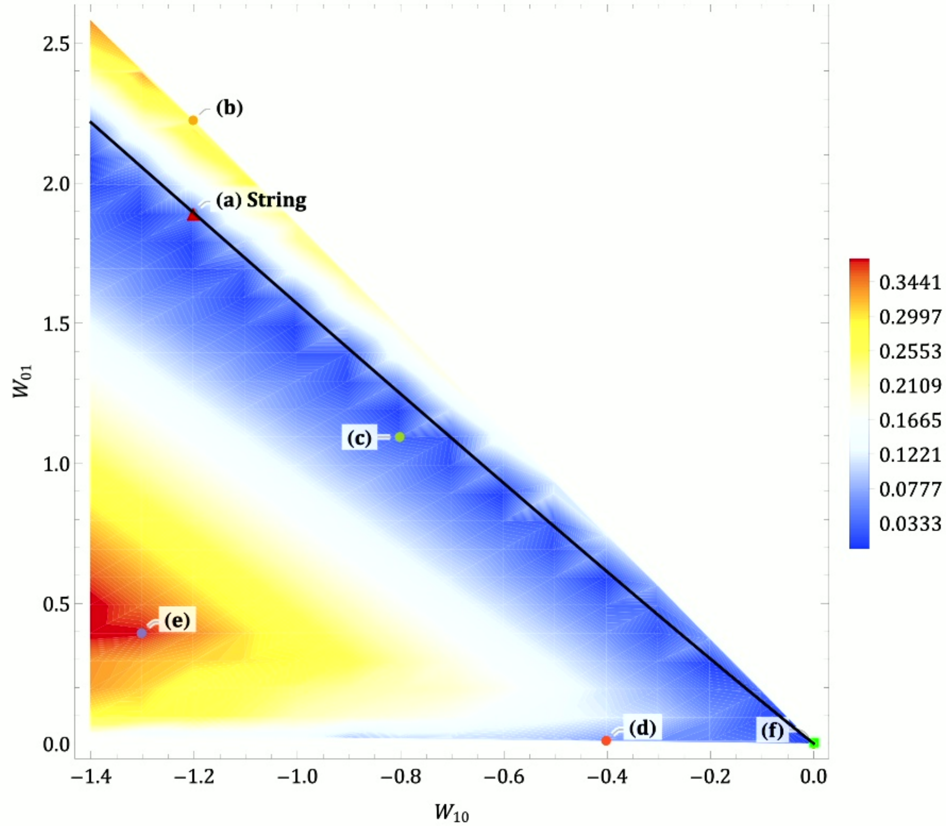
1st daughter

n	Exact	Bootstrap
5	0.0000842	0
6	0.000133	0
7	0.000104	0
8	0.0000698	0.0000491
9	0.0000444	0
10	0.0000277	0
11	0.0000170	4.93×10^{-6}
12	0.0000105	5.68×10^{-6}
13	6.44×10^{-6}	4.22×10^{-6}
14	3.96×10^{-6}	2.76×10^{-6}
15	2.44×10^{-6}	1.83×10^{-6}
16	1.51×10^{-6}	1.27×10^{-6}
17	9.39×10^{-7}	8.74×10^{-7}
18	5.84×10^{-7}	5.79×10^{-7}
19	3.65×10^{-7}	3.73×10^{-7}
20	2.29×10^{-7}	2.52×10^{-7}

2nd daughter

SDPB with 10^{-9} tolerance

String amplitude seems to be an extremal function



▲ Open superstring

■ Theory with only a massless pole = $1/(s_1 s_2)$

— Hypergeometric deformations

Mansfield, Spradlin, '24

Closed string/KLT

$$\frac{\Gamma(-s_1)\Gamma(-s_2)\Gamma(-s_3)}{\Gamma(1+s_1)\Gamma(1+s_2)\Gamma(1+s_3)} = -\frac{1}{s_1 s_2 s_3} + \sum_{n=1}^{\infty}\frac{1}{(n!)^2}\left[\frac{1}{s_1-n}+\frac{1}{s_2-n}+\frac{1}{s_3-n}+\frac{1}{\lambda+n}\right]\left(1-\frac{n}{2}+\frac{n-2\lambda}{2}\sqrt{1-\frac{4(s_1+\lambda)(s_2+\lambda)(s_3+\lambda)}{(n+\lambda)(n-2\lambda)^2}}\right)^2_{n-1}$$

KLT

$$\mathcal{M}_{\text{cl}}(s_1,s_2,s_3) = \mathcal{M}_{\text{op}}(s_1,s_2)\frac{\sin(\pi s_1)\sin(\pi s_2)}{\pi\sin(\pi(s_1+s_2))}\mathcal{M}_{\text{op}}(s_1,s_2)$$

$$\downarrow \quad s_1,\ s_2 \sim 0$$

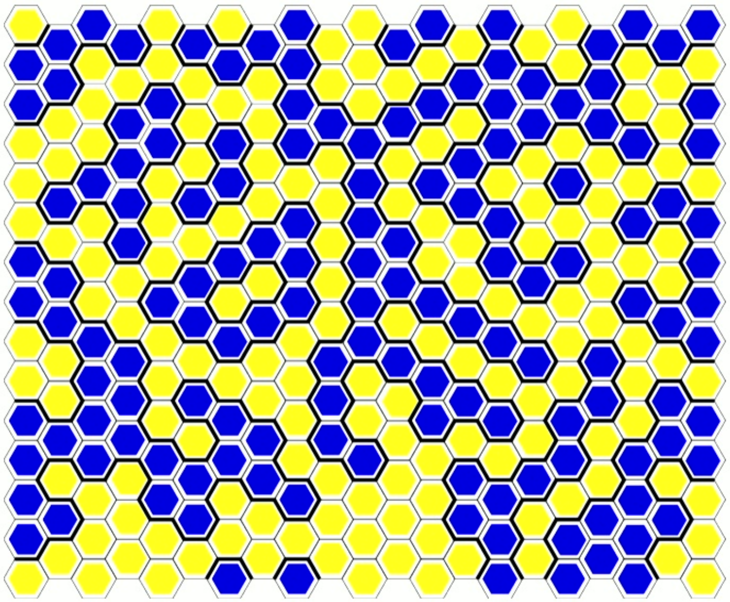
$$= -\frac{1}{s_1 s_2 s_3} + \frac{2}{(s_1+s_2)}\left[\sum_n \frac{(2\lambda+n)}{\lambda n^2 n!}\left(-\frac{n\lambda}{n+\lambda}\right)_n + \zeta(2)\right] + \mathcal{O}(1)$$

Ramanujan's π and physics

Modular equations and approximations to π

Quarterly Journal of Mathematics, XLV, 1914, 350 – 372

CFT: Conformal Field Theory link



- **Non-unitary critical theories called log-CFTs**
- **Describe physics of percolation for eg.**

- The starting point of the Ramanujan series involves squares of Gauss Hypergeometric functions
- These arise in the calculation of correlation functions in log-CFTs
- Interestingly in a class of theories, they are also related to the second Renyi entropy of two disjoint intervals (Renyi entropy: a generalization of entanglement entropy)

$$\frac{4}{\pi} \approx -\frac{56496149211311 + 674161005879900 \log^2(2) - 557725361939340 \log(2)}{3435973836800\sqrt{2}}.$$

- New kinds of formulas
- Arise on expanding using the natural quantum numbers that arise in the physics problem.

Work in progress with F. Bhat

The CFT calculation

- Let us imagine that there exists a correlator of the form (assume $h_1 = h_3, h_2 = h_4$) :
- $\langle \Phi_{h_1}(0)\Phi_{h_2}(z,\bar{z})\Phi_{h_3}(1)\Phi_{h_4}(\infty)\rangle \sim \underbrace{{}_2F_1(\sigma,1-\sigma,1,z)\,{}_2F_1(\sigma,1-\sigma,1,1-\bar{z})}_{L} + \underbrace{{}_2F_1(\sigma,1-\sigma,1,\bar{z})\,{}_2F_1(\sigma,1-\sigma,1,1-z)}_{R}$
- Then we have $\mathscr{L} := z(1-z)\partial_z - \bar{z}(1-\bar{z})\partial_{\bar{z}} = (\ell_0 - \ell_1) + (\bar{\ell}_0 - \bar{\ell}_1)$ obeying (the generalised Legendre relation)

$$\mathscr{L}\,{}_2F_1(\sigma,1-\sigma,1,z)\,{}_2F_1(\sigma,1-\sigma,1,1-\bar{z})\Big|_{\bar{z}=z} = \frac{\sin \pi\sigma}{\pi}$$

- This is the origin of the Ramanujan 1/pi series. To get the precise form, one proceeds by taking the diagonal limit and using the Clausen identity to convert the ${}_2F_1 \times {}_2F_1$ to a ${}_3F_2$ to get a single sum instead of a double sum (and faster convergence). This step is a bit subtle and involves the use of modular equations of various orders. Equivalently, one finds a differential operator *after* taking the diagonal limit and massages into the Legendre relation form at special points called singular moduli.

$${}_2F_1(1/2,1/2,1,z_0) = \sqrt{n}\,{}_2F_1(1/2,1/2,1/1-z_0) \qquad z_0 : \text{singular moduli}$$

- The modular equations need usage of theta function identities which Ramanujan cleverly leverages. However, from the CFT perspective, we would not proceed this way. Rather we would expand the correlator in terms of the natural quantum numbers of operators in the OPE. So the first question is to identify the external dimensions of the operators.

- Very naturally, we consider 2d generalised minimal models. We identify:

$$h_1 = -\frac{(\sigma-1)^2}{4\sigma} = h_3\,, \quad h_2 = \frac{3\sigma-2}{4} = h_4$$

$$c = 13 - 6(\sigma + \frac{1}{\sigma})$$

- The Ramanujan series correspond to $\sigma \in (1/2, 1/3, 1/4, 1/6) \implies c = (-2, -7, -25/2, -24)$ —logarithmic minimal models.

- These are all log-CFTs. The c=-2 is a famous one corresponding to the theory of symplectic fermions (Saleur '91, Gurarie '92) and contains the physics of critical dense polymers (and the nu=5/2 fractional quantum Hall effect). The c=-7, -25/2 are not studied while c=-24 appears in the discussion of percolation (Flohr et al).

- The logarithmic nature introduces 2 sets of partner operators, primaries and their log partners. It is precisely this nature that enables us to consider L and R separately in terms of just the log partners and the lambda-dispersion (and hence leverage the Legendre relation).

- The different rates of convergence of the Ramanujan series for the same c prompts us to examine the expansion of the correlator. Clearly there have to be different equivalent representations [not just s-channel OPE]. The lambda-dispersion relation can be re-purposed for this very purpose!

- Remarkably, in $\mathscr{L}\langle\Phi_{h1}\Phi_{h2}\Phi_{h1}\Phi_{h2}\rangle_L|_{\bar{z}=z}$ what we find is that in the large lambda limit, all the non-identity operators contribute like $O(1/\lambda)$ and only the log-identity operator survives —can verify that it indeed contributes 1/pi. Thus dialing lambda controls how much each operator contributes!

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- $$\int_1^\infty d\xi\, \frac{(z-1)z\left(6\xi^3-7\xi^2+\xi(2(z-1)z+3)+(z-1)z\right)}{\xi^{3/2}(z-\xi)^2(\xi+z-1)^2} = -2\,,$$

Only the log partner of the identity operator is enough to produce the Legendre identity in the large lambda limit

Conclusions

- …………let curiosity be the guide!

Thank you