Title: Graphs, curves, and their moduli spaces (Part 2 of 2)

Speakers: Michael Borinsky

Collection/Series: Mathematical Physics

Subject: Mathematical physics

Date: March 06, 2025 - 11:00 AM

URL: https://pirsa.org/25030130

Abstract:

I will give a gentle introduction to the moduli space of graphs and its fine moduli space cousin known as Outer Space. This moduli space of graphs has many applications to various branches of mathematical physics, algebraic geometry, and geometric group theory. It is a natural object to consider while studying Feynman amplitudes in parametric space, and it can be seen as the configuration space of one-dimensional quantum gravity. I will explain how this moduli space of graphs recently became the largest provider of information on the homology of the moduli space of curves of genus g and how associated graph complexes can be used to shed light on the 'dark-matter problems' of these moduli space's cohomology.

Graphs, Curves and their moduli spaces Part $\underline{\mathrm{II}}$

$$\mathbb{F}_{Lie}\langle \sigma_3, \dots \rangle \hookrightarrow \mathfrak{grt}_1\big|_g \hookrightarrow H_*(GC^{(g)}) \hookrightarrow H^{6g-6-*}(\mathcal{M}_g; \mathbb{Q})$$

Brown '12 Willwacher '15 Chan-Galatius-Payne '21

degree = 2g (top-weight iso)

Conjectured to be an iso

Lie bracket $\displaystyle [G, H] = \sum_{\substack{\text{insert} \\ G \to H}} G \circ_i H - \sum_{\substack{\text{insert} \\ H \to G}} H \circ_i G$

Q: How much of $H^*(\mathcal{M}_g)$ can CGP explain?

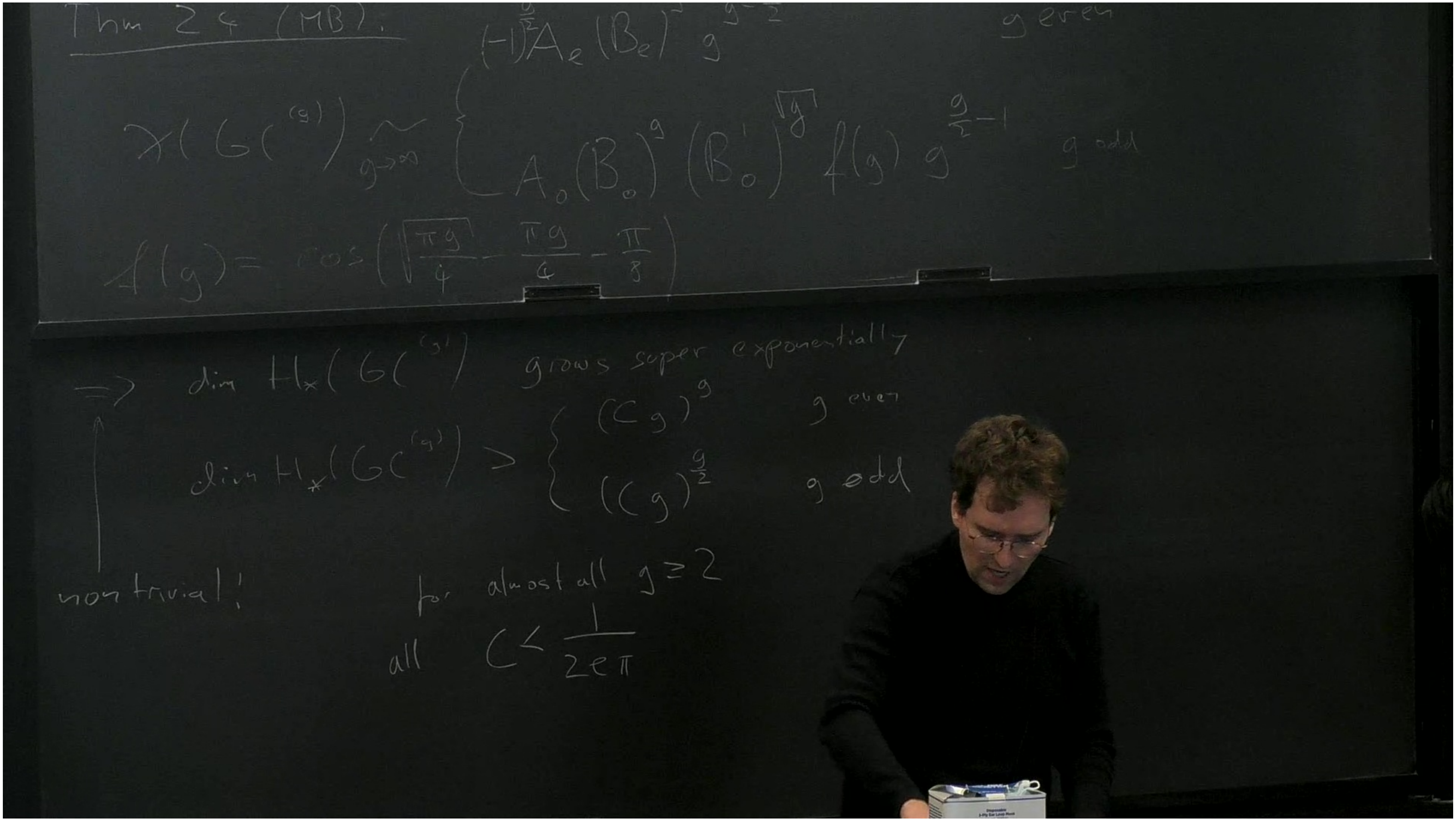
⇒ Euler characteristic of $GC^{(g)}$

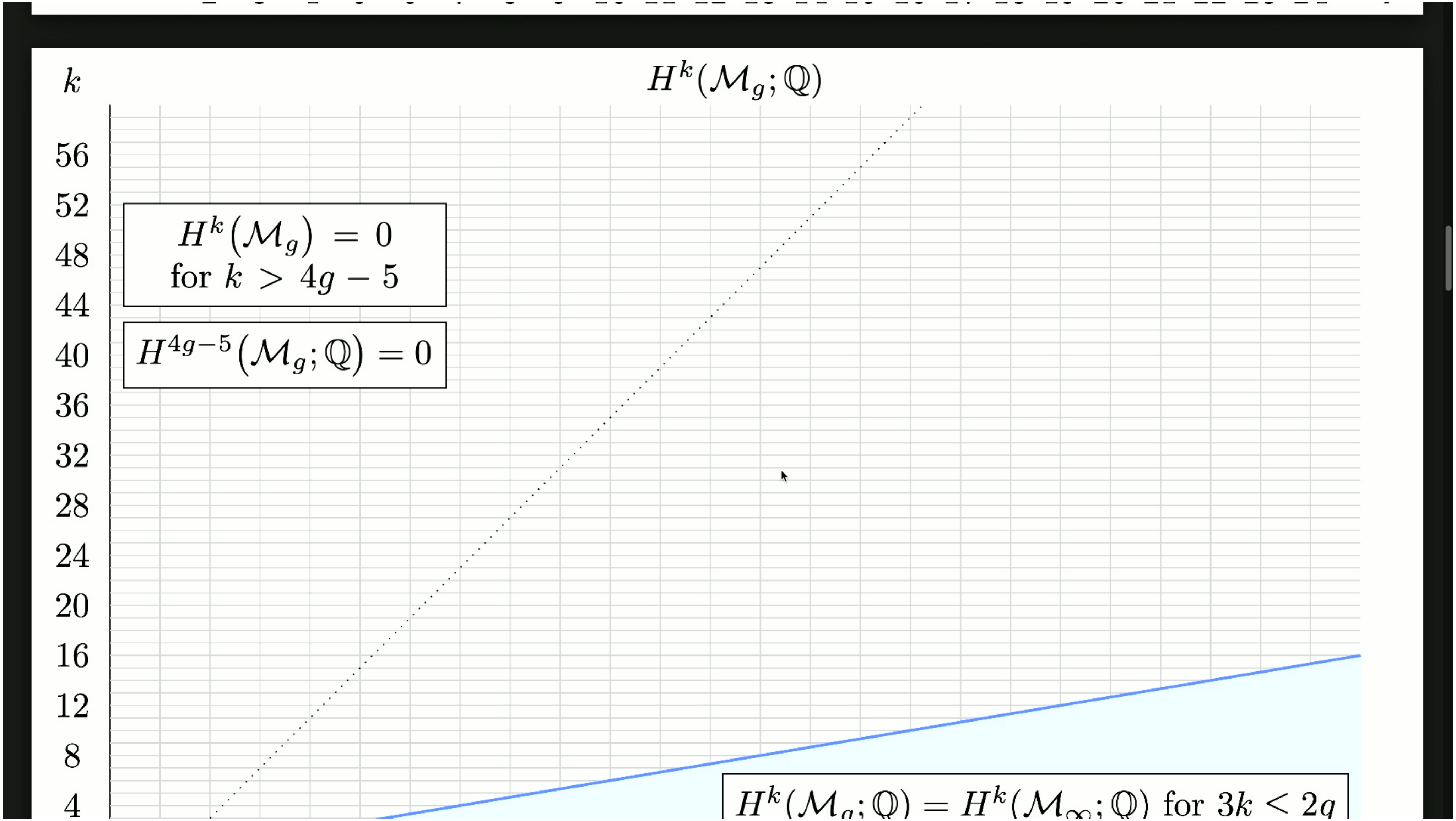
$$\chi(GC^{(g)}) = \sum_k (-1)^k \dim H_k(GC^{(g)})$$

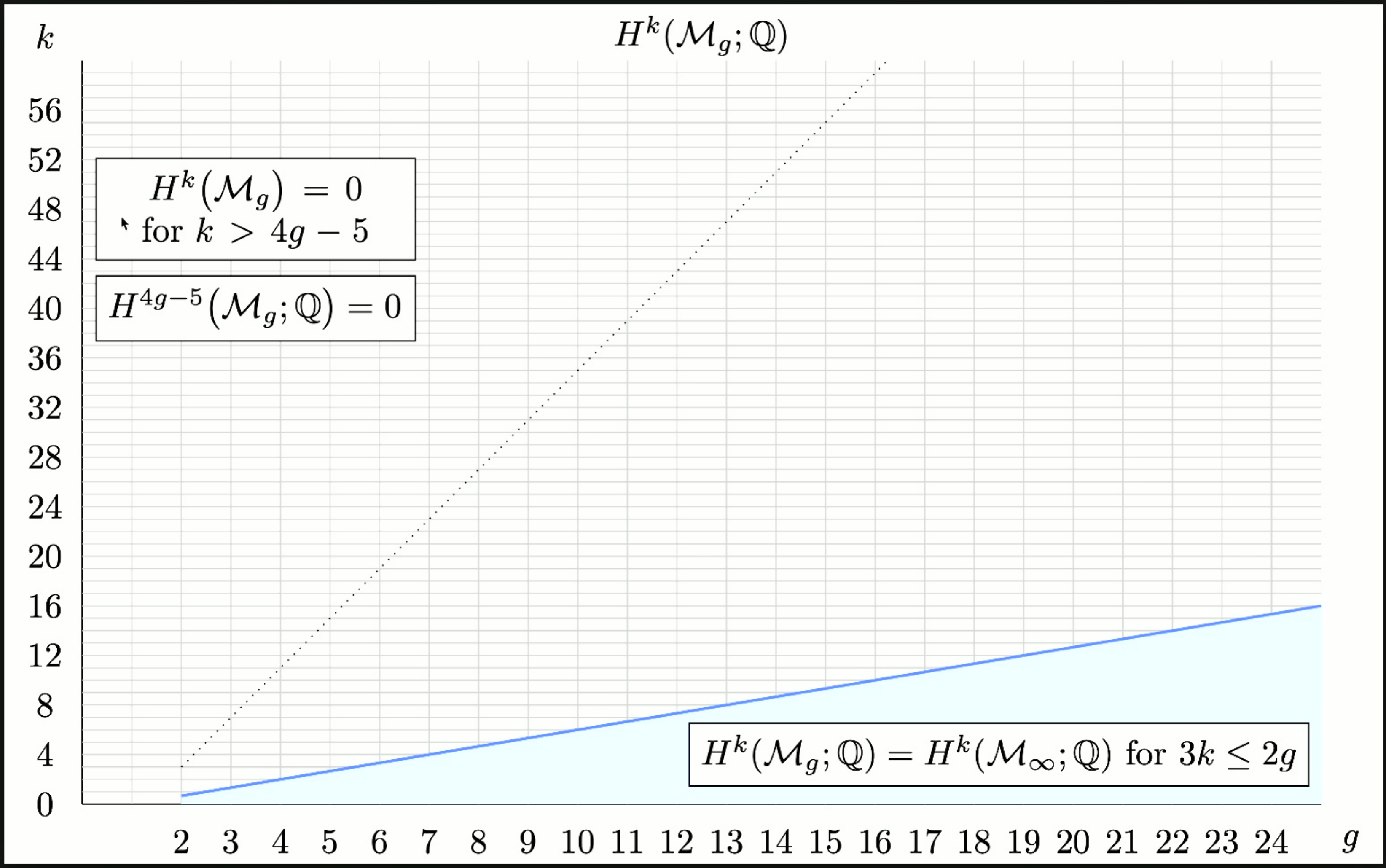
Thm 2.4 (MB):

$$\chi(GC^{(g)}) \underset{g\to\infty}{\sim} \begin{cases} (-1)^{\frac{g}{2}} A_e (B_e)^g \, g^{g-\frac{3}{2}} & g \text{ even} \\ A_o (B_o)^g (B_o')^{\sqrt{g}} f(g)\, g^{\frac{g}{2}-1} & g \text{ odd} \end{cases}$$

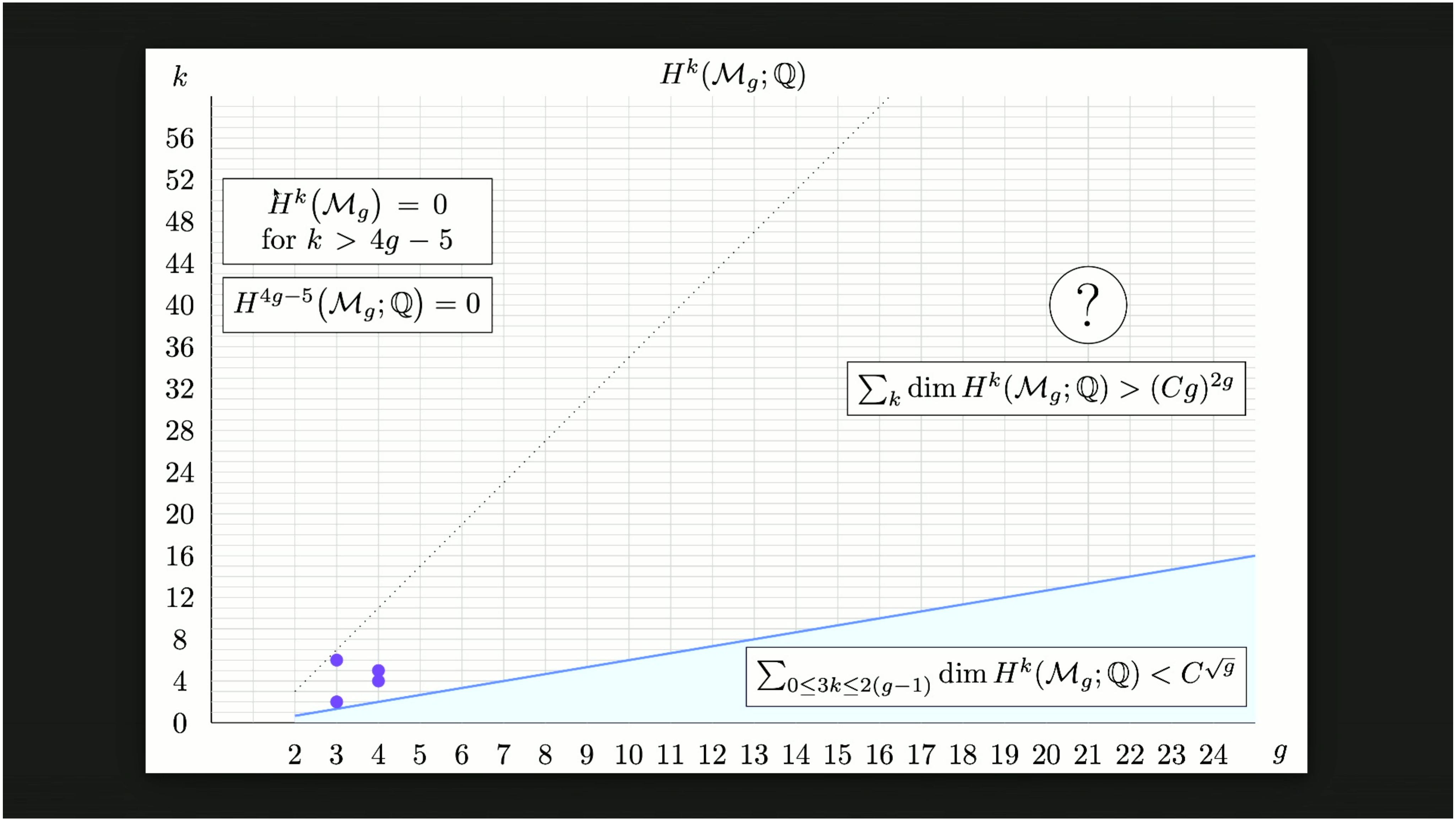
$$f(g) = \cos\left(\sqrt{\frac{\pi g}{4}} - \frac{\pi g}{4} - \frac{\pi}{8}\right)$$


Thm 2.4 (MB):
$\chi(G^{(g)}) \underset{g\to\infty}{\sim} \begin{cases} (-1)^{\frac{g}{2}} A_e (B_e)^g g^{g-\frac{1}{2}} & g \text{ even} \\ A_o (B_o)^g (B_o')^{\sqrt{g}} f(g)\, g^{\frac{g}{2}-1} & g \text{ odd} \end{cases}$
$f(g) = \cos\left(\sqrt{\frac{\pi g}{4}} - \frac{\pi g}{4} - \frac{\pi}{8}\right)$
$\Rightarrow$ dim $H_*(G^{(g)})$ grows super exponentially
dim $H_*(G^{(g)}) > \begin{cases} (Cg)^g & g \text{ even} \\ (Cg)^{\frac{g}{2}} & g \text{ odd} \end{cases}$
non trivial!
for almost all $g \geq 2$
all $C < \frac{1}{2e\pi}$

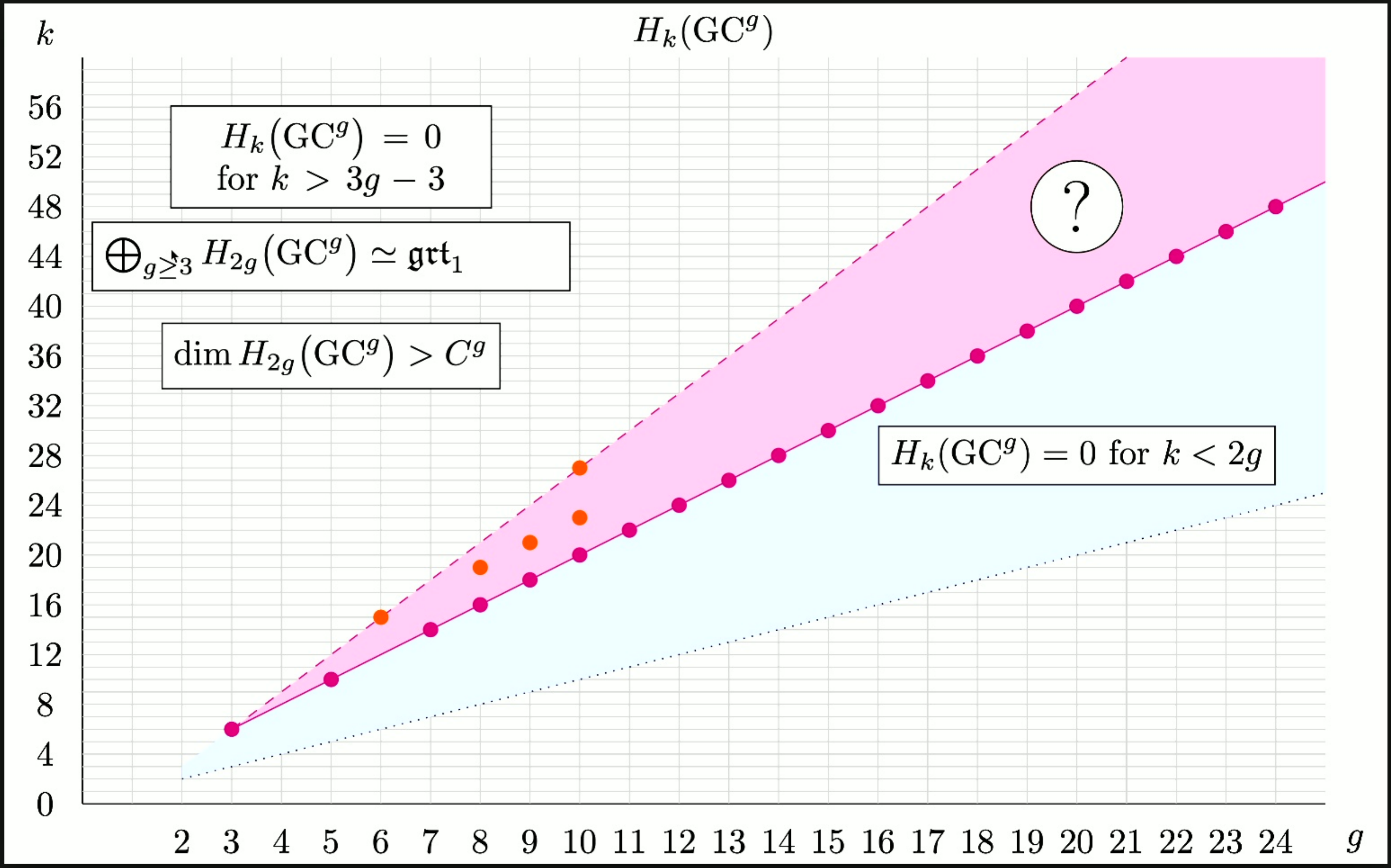

k
$H^k(\mathcal{M}_g;\mathbb{Q})$
56
52
48
44
40
36
32
28
24
20
16
12
8
4
$H^k(\mathcal{M}_g) = 0$
for $k > 4g-5$
$H^{4g-5}(\mathcal{M}_g;\mathbb{Q}) = 0$
$H^k(\mathcal{M}_g;\mathbb{Q}) = H^k(\mathcal{M}_\infty;\mathbb{Q})$ for $3k < 2g$

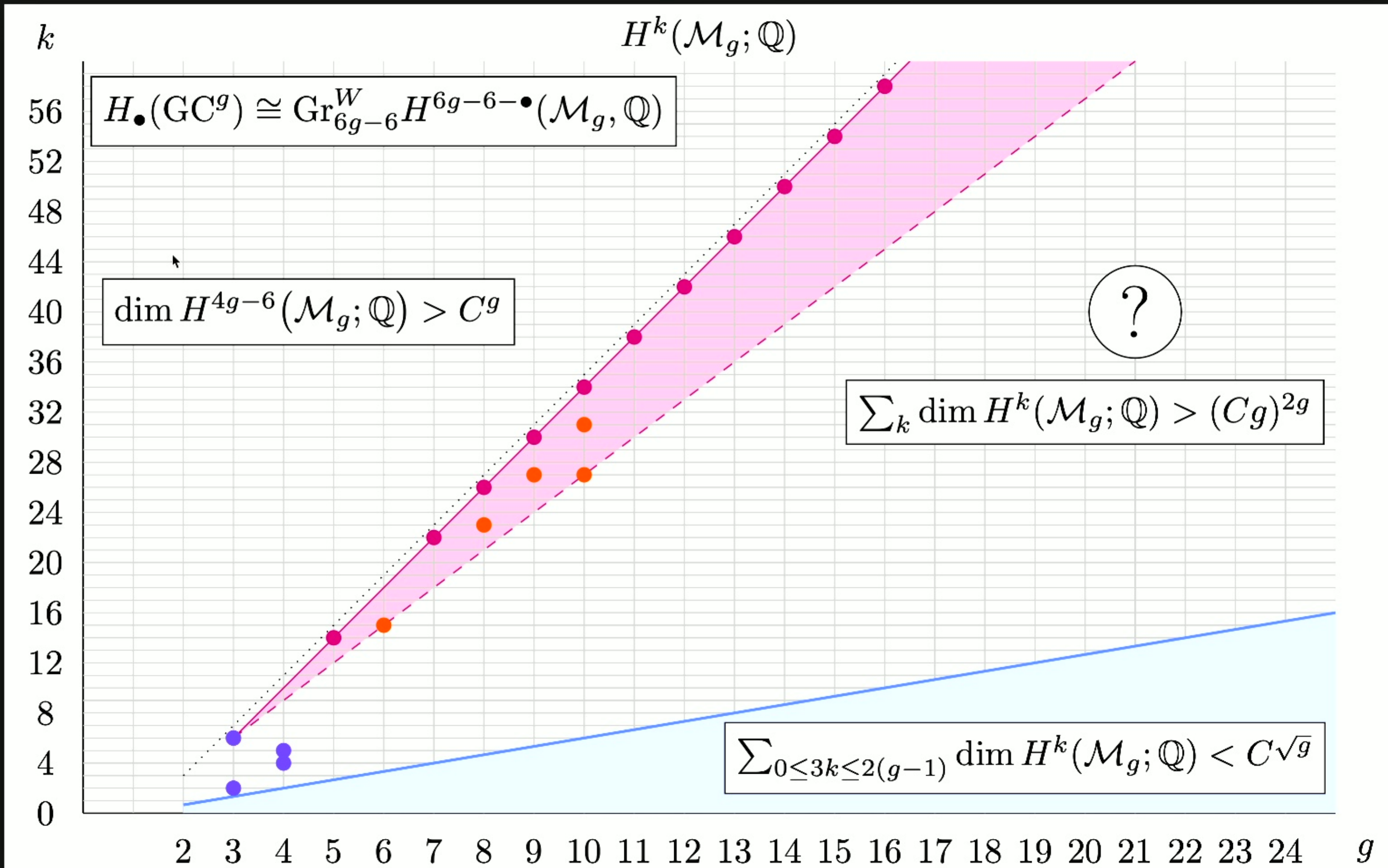


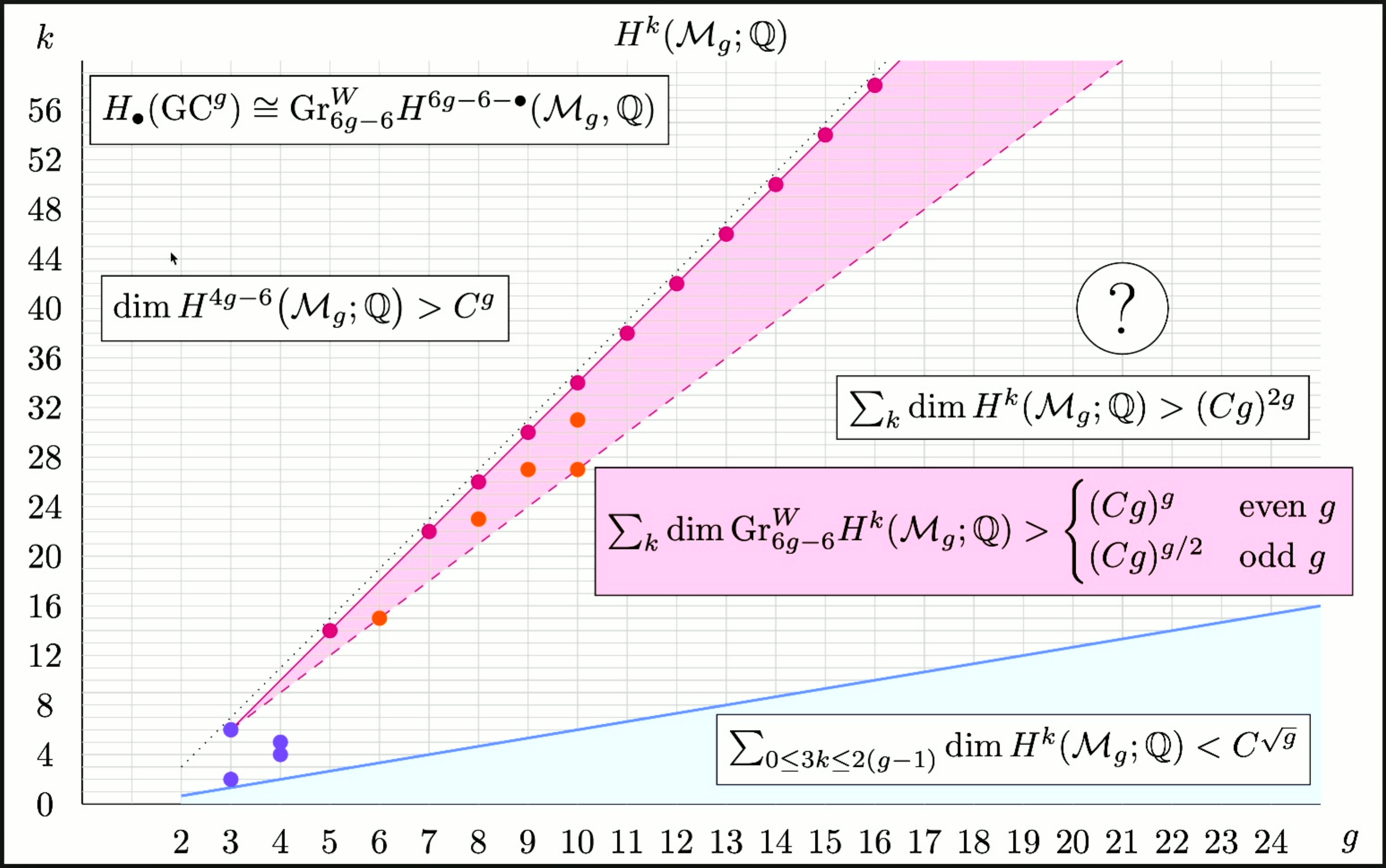
$H^k(\mathcal{M}_g;\mathbb{Q})$
k
$H^k(\mathcal{M}_g) = 0$ for $k > 4g-5$
$H^{4g-5}(\mathcal{M}_g;\mathbb{Q}) = 0$
$H^k(\mathcal{M}_g;\mathbb{Q}) = H^k(\mathcal{M}_\infty;\mathbb{Q})$ for $3k \le 2g$
0 4 8 12 16 20 24 28 32 36 40 44 48 52 56
2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
g



k
$H^k(\mathcal{M}_g;\mathbb{Q})$
$H^k(\mathcal{M}_g) = 0$ for $k > 4g-5$
$H^{4g-5}(\mathcal{M}_g;\mathbb{Q}) = 0$
?
$\sum_k \dim H^k(\mathcal{M}_g;\mathbb{Q}) > (Cg)^{2g}$
$\sum_{0\leq 3k\leq 2(g-1)} \dim H^k(\mathcal{M}_g;\mathbb{Q}) < C^{\sqrt{g}}$
56 52 48 44 40 36 32 28 24 20 16 12 8 4 0
2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
g


$H_k(\mathrm{GC}^g)$
k
g
$H_k(\mathrm{GC}^g) = 0$ for $k > 3g - 3$
$\bigoplus_{g\geq 3} H_{2g}(\mathrm{GC}^g) \simeq \mathfrak{grt}_1$
$\dim H_{2g}(\mathrm{GC}^g) > C^g$
?
$H_k(\mathrm{GC}^g) = 0$ for $k < 2g$
0 4 8 12 16 20 24 28 32 36 40 44 48 52 56
2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24


$H^k(\mathcal{M}_g;\mathbb{Q})$
k
$H_\bullet(\mathrm{GC}^g) \cong \mathrm{Gr}^W_{6g-6} H^{6g-6-\bullet}(\mathcal{M}_g,\mathbb{Q})$
$\dim H^{4g-6}(\mathcal{M}_g;\mathbb{Q}) > C^g$
?
$\sum_k \dim H^k(\mathcal{M}_g;\mathbb{Q}) > (Cg)^{2g}$
$\sum_{0\le 3k\le 2(g-1)} \dim H^k(\mathcal{M}_g;\mathbb{Q}) < C^{\sqrt{g}}$
56 52 48 44 40 36 32 28 24 20 16 12 8 4 0
2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
g



k
$H^k(\mathcal{M}_g;\mathbb{Q})$
$H_\bullet(\mathrm{GC}^g) \cong \mathrm{Gr}^W_{6g-6} H^{6g-6-\bullet}(\mathcal{M}_g, \mathbb{Q})$
$\dim H^{4g-6}(\mathcal{M}_g;\mathbb{Q}) > C^g$
?
$\sum_k \dim H^k(\mathcal{M}_g;\mathbb{Q}) > (Cg)^{2g}$
$\sum_k \dim \mathrm{Gr}^W_{6g-6} H^k(\mathcal{M}_g;\mathbb{Q}) > \begin{cases} (Cg)^g & \text{even } g \\ (Cg)^{g/2} & \text{odd } g \end{cases}$
$\sum_{0\le 3k\le 2(g-1)} \dim H^k(\mathcal{M}_g;\mathbb{Q}) < C^{\sqrt{g}}$
56 52 48 44 40 36 32 28 24 20 16 12 8 4 0
2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
g

Conjectured to be an iso.

Lie bracket: $[G, H] = \sum_{\substack{\text{insert} \\ G \to H}} G \circ_i H - \sum_{\substack{\text{insert} \\ H \to G}} H \circ_i G$

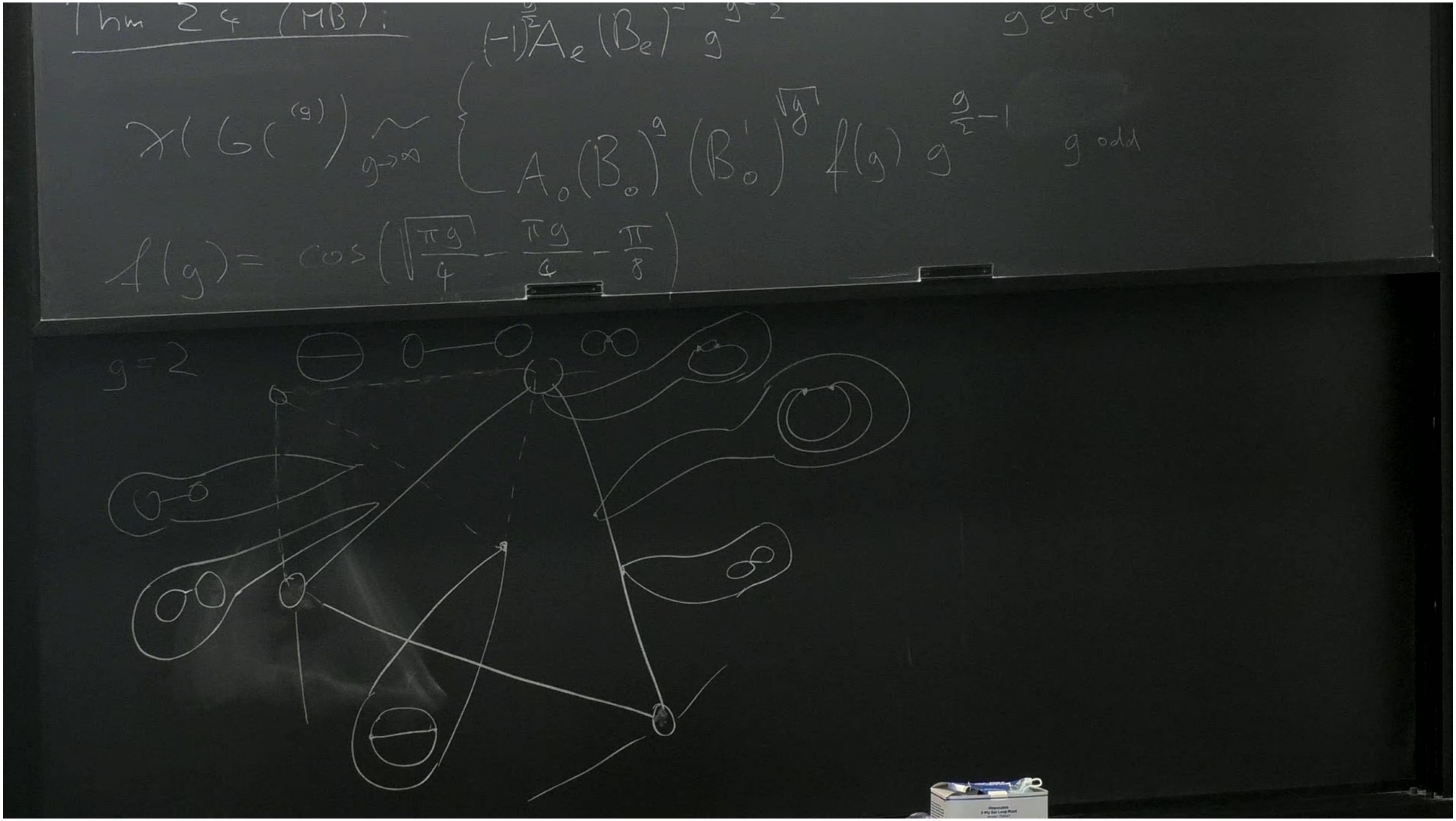
... geometry

Def. Each point in ... g

× G is a graph with g loops vertex deg ≥ 3

× Each edge has a length $m_e \in (0, \infty)$

s.t. $\sum_e m_e = 1$


Thm 2.4 (MB):
$\chi(G^{(g)}) \underset{g\to\infty}{\sim} \begin{cases} (-1)^{\frac{g}{2}} A_e (B_e)^g g^{g-2} & g \text{ even} \\ A_o (B_o)^g (B_o')^{\sqrt{g}} f(g)\, g^{\frac{g}{2}-1} & g \text{ odd} \end{cases}$
$f(g) = \cos\left(\sqrt{\frac{\pi g}{4}} - \frac{\pi g}{4} - \frac{\pi}{8}\right)$
$g=2$

Conjectured to be an iso. (so)

Lie bracket: $$[G,H] = \sum_{\substack{\text{insert}\\ G\to H}} G\circ_i H - \sum_{\substack{\text{insert}\\ H\to G}} H\circ_i G$$

Known about $\mathcal{M}G_g$:

* Culler-Vogtmann '86: $H^*(\mathcal{M}G_g;\mathbb{Q}) \cong H^*(Out(F_g);\mathbb{Q})$
* $H_{*-1}(\mathcal{M}G_g, \partial\mathcal{M}G_g) \cong H_*(GC^{(g)})$
* Natural space to study Feynman amplitudes (objects pQFT)

... MB-Vogtmann MB-Vermaseren

study $H_*(\mathcal{M}G_{g,n};\mathbb{Q})^{\Sigma_n}$

non trivial! for almost all $g \geq 2$

$$X_g = \left\{ \begin{array}{c} \text{symmetric } g \times g \text{ matrices} \\ \text{pos. definite} \end{array} \quad \text{real entries} \right\}$$

action $\boxed{Gl_g \mathbb{Z}}$ $\quad M \in X_g \mapsto G^T M G \quad$ for $G \in Gl_g \mathbb{Z}$

$$P_g = X_g / Gl_g \mathbb{Z}$$

stable cohomology of P_g is gen by $\omega_k = \mathrm{Tr}\left((M^{-1} dM)^k\right)$

$k = 5, 9, 13, \ldots$

Each edge has a length $m_e \in (0,\infty)$

s.t. $\sum_e m_e = 1$

Map $\eta : \mathcal{M}G_g \to P_g$

$\eta^* \omega_k \Big|_G$ = „Feynman forms" $= \sum_\ell \dfrac{Q_\ell}{\psi_G^\ell}\,\Omega$

polynomials

std vol form on simplex

Symanzik polynomial

QFT:

$A_g = \displaystyle\int_{\mathcal{M}G_g} \mu$ $\quad\Big|\quad$ $\mu\big|_G = \sum_\ell \dfrac{\tilde{Q}_\ell}{\psi_G^\ell}\,\Omega$

$\Omega_{g,n}, \psi \,|\, \int \Sigma_n$