Title: Combinatorial interpretation of the causal set d'Alembertian

Speakers: Karen Yeats

Collection/Series: Emmy Noether Workshop: Quantum Space Time

Subject: Quantum Gravity

Date: March 14, 2025 - 9:15 AM

URL: https://pirsa.org/25030067

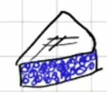
Abstract:

Causal set theory is an approach to quantum gravity where the underlying spacetime is a locally finite poset. Glaser gave a formula for the causal set theory analogue of the d'Alembertian in general dimension (growing out of previous work of Sorkin, Benincasa and Dowker, and Dowker and Glaser). The formula contains rational coefficients. Who can resist trying to find something that these coefficients count -- not me! -- so I will tell you about such a something.

Combinatorial interpretation of the causal set d'Alembertian

Karen Yeats

Department of Combinatorics and Optimization
University of Waterloo

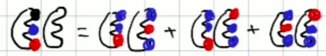
Mar 14, 2025 

Quantum Spacetime Workshop

arXiv: 2412.14036

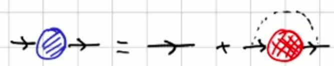
① What are we doing?

Not this stuff:



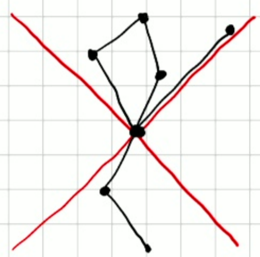
$$M_{\times} = M_{=} + M_{||} + M_{\times}$$

Also not this stuff



$$\phi B_+ = \Lambda \phi$$

But this stuff



What is the unifying theme for this and more?

Combinatorics / discrete math in fundamental physics.

Physics asks rich questions that a pure perspective wouldn't ask.

Math can solve physics problems.

Everyone wins!

Talk to your local mathematicians. Branch out beyond geometry.

② Causal set d'Alembertian

Earlier in the week we heard about causal set theory

- take the causal relationship as fundamental

  Spacetime is a poset

- make it discrete

  Spacetime is a locally finite poset.

- volume of a spacetime interval is replaced by number of elements in a poset interval.

Want the physics to emerge from clean minimalistic axioms, but get useful examples to study (Lorentz invariant on average) by

taking a Lorentzian manifold

choosing points on it at random by a Poisson distribution (Sprinkling)

using the induced causal order.

Spacetime is a poset

- make it discrete

Spacetime is a locally finite poset.

- volume of a spacetime interval is replaced by number of elements in a poset interval.

Want the physics to emerge from clean minimalistic axioms, but get useful examples to study (Lorentz invariant on average) by

taking a Lorentzian manifold

choosing points on it at random by a Poisson distribution

(Sprinkling)

using the induced causal order.

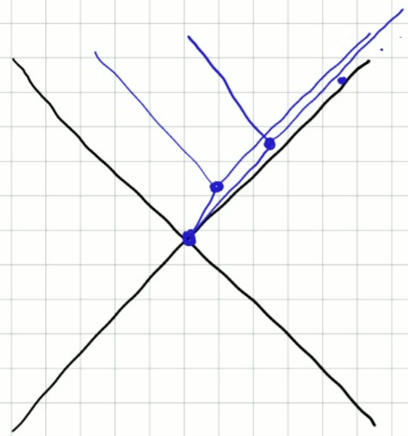
In this case the probability to find m poset elements in volume V is $\dfrac{(\rho V)^m e^{-\rho V}}{m!}$

ρ is density prameter. Also useful: discreteness scale ℓ, $\rho = \ell^{-d}$ ← dimension

Want analogues for everything. How about the d'Alembertian?

Differential operator should turn into difference operator

but:



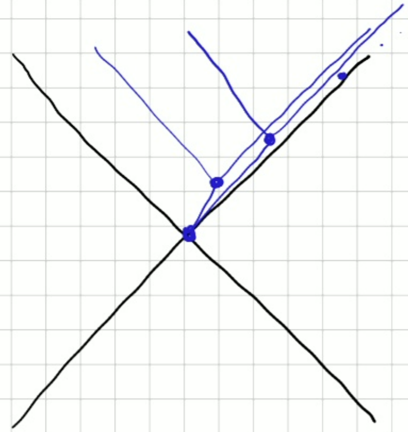
Sorkin ('09) showed a particular alternating linear combination will converge in 2d

$$\lim_{\rho\to\infty} \frac{1}{\sqrt{\rho}} \langle B\phi(x) \rangle = \Box\phi$$

expectation with respect to Poisson sprinkling

Generalized by Benincasa and Dowker ('10) to 4d

by Dowker and Glaser ('13) to general dimension but no explicit formula for but:



Sorkin ('09) showed a particular alternating linear combination will converge in 2d

$$\lim_{\rho \to \infty} \frac{1}{\sqrt{\rho}} \langle B\phi(x) \rangle = \Box\phi$$

expectation with respect to Poisson sprinkling

Generalized by Benincasa and Dowker ('10) to 4d

by Dowker and Glaser ('13) to general dimension but no explicit formula for $d > 7$

Explicit general formula by Glaser ('14)

③ So you're sitting in a physics talk and there's this table of integers...

	C_1	C_2	C_3	C_4	C_5
$d=1$	1	$-\frac{1}{2}$			
$d=2$	1	-2	1		
$d=3$	1	$-\frac{27}{8}$	$\frac{9}{4}$		
$d=4$	1	-9	16	-8	
$d=5$	1	$-\frac{215}{16}$	$\frac{225}{8}$	$-\frac{125}{8}$	
$d=6$	1	-34	141	-189	81

What do you do?

(Glaser)

$$B^{(d)}\phi(x) = \frac{1}{\ell^2}\left(\alpha_d\,\phi(x) + \beta_d \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor + 2} C_i^{(d)} \sum_{y \in L_i} \phi(y)\right) \quad \text{where} \quad C_i^{(d)} = \begin{cases} \displaystyle\sum_{k=0}^{i-1} \binom{i-1}{k} (-1)^k \frac{\Gamma\left(\frac{d}{2}(k+1)+2\right)}{\Gamma\left(\frac{d}{2}+2\right)\Gamma\left(1+\frac{dk}{2}\right)} & \text{even } d \\[2ex] \displaystyle\sum_{k=0}^{i-1} \binom{i-1}{k} (-1)^k \frac{\Gamma\left(\frac{d}{2}(k+1)+\frac{3}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)\Gamma\left(1+\frac{dk}{2}\right)} & \text{odd } d \end{cases}$$

$$L_i(x) = \{ y \in C : |[y,x]| = i+1 \}$$

$d=1$	1	$-\frac{1}{2}$			
$d=2$	1	-2	1		
$d=3$	1	$-\frac{27}{8}$	$\frac{9}{4}$		
$d=4$	1	-9	16	-8	
$d=5$	1	$-\frac{215}{16}$	$\frac{225}{8}$	$-\frac{125}{8}$	
$d=6$	1	-34	141	-189	81

What do you do?

(Glaser)

$$B^{(d)}\phi(x) = \frac{1}{\ell^2}\left(\alpha_d\,\phi(x) + \beta_d \sum_{i=1}^{\lfloor \frac{d}{2}\rfloor+2} C_i^{(d)} \sum_{y\in L_i}\phi(y)\right) \quad \text{where} \quad C_i^{(d)} = \begin{cases} \displaystyle\sum_{k=0}^{i-1}\binom{i-1}{k}(-1)^k \frac{\Gamma\left(\frac{d}{2}(k+1)+2\right)}{\Gamma\left(\frac{d}{2}+2\right)\Gamma\left(1+\frac{dk}{2}\right)} & \text{even } d \\ \displaystyle\sum_{k=0}^{i-1}\binom{i-1}{k}(-1)^k \frac{\Gamma\left(\frac{d}{2}(k+1)+\frac{3}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)\Gamma\left(1+\frac{dk}{2}\right)} & \text{odd } d \end{cases}$$

$$L_i(x) = \{y\in C : |[y,x]| = i+1\}$$

α_d, β_d also explicit

"aesthetically unsatisfying"
Glaser

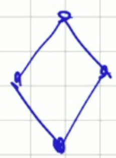
$d=2$	1	-2	1		
$d=3$	1	$-\frac{27}{8}$	$\frac{9}{4}$		
$d=4$	1	-9	16	-8	
$d=5$	1	$-\frac{215}{16}$	$\frac{225}{8}$	$-\frac{125}{8}$	
$d=6$	1	-34	141	-189	81

What do you do?

(Glaser)

$$B^{(d)}\phi(x) = \frac{1}{\ell^2}\left(\alpha_d\,\phi(x) + \beta_d \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor + 2} C_i^{(d)} \sum_{y \in L_i} \phi(y)\right) \quad \text{where} \quad C_i^{(d)} = \begin{cases} \displaystyle\sum_{k=0}^{i-1} \binom{i-1}{k} (-1)^k \frac{\Gamma\left(\frac{d}{2}(k+1)+2\right)}{\Gamma\left(\frac{d}{2}+2\right)\Gamma\left(1+\frac{dk}{2}\right)} & \text{even } d \\ \displaystyle\sum_{k=0}^{i-1} \binom{i-1}{k} (-1)^k \frac{\Gamma\left(\frac{d}{2}(k+1)+\frac{3}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)\Gamma\left(1+\frac{dk}{2}\right)} & \text{odd } d \end{cases}$$

$$L_i(x) = \{ y \in C : |[y,x]| = i+1 \}$$



"aesthetically unsatisfying"

Glaser

α_d, β_d also explicit

④ Interpret in two phases

ⓐ Phase 1

eg:

$\mathcal{B}_{n,m}$ set of partial rooted chord diagrams with n chords on m points

and
- each chord red, blue, or black
- each blue or red chord has a designated first end

with
- inside of every blue or red chord has every point in a black chord and no black chord not inside red or blue
- noncrossing

$$\mathcal{B} = \bigcup_{\substack{m \geq 2n \\ n \geq 0}} \mathcal{B}_{n,m}$$

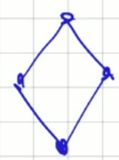
Note:

		0	1		
$d=4$	1	-9	16	-8	
$d=5$	1	$-\frac{215}{16}$	$\frac{225}{8}$	$-\frac{125}{8}$	
$d=6$	1	-34	141	-189	81

(Glaser)

$$B^{(d)}\phi(x) = \frac{1}{\ell^2}\left(\alpha_d\,\phi(x) + \beta_d \sum_{i=1}^{\lfloor \frac{d}{2}\rfloor + 2} C_i^{(d)} \sum_{y\in L_i} \phi(y)\right) \quad \text{where} \quad C_i^{(d)} = \begin{cases} \displaystyle\sum_{k=0}^{i-1} \binom{i-1}{k}(-1)^k \frac{\Gamma\left(\frac{d}{2}(k+1)+2\right)}{\Gamma\left(\frac{d}{2}+2\right)\Gamma\left(1+\frac{dk}{2}\right)} & \text{even } d \\ \displaystyle\sum_{k=0}^{i-1} \binom{i-1}{k}(-1)^k \frac{\Gamma\left(\frac{d}{2}(k+1)+\frac{3}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)\Gamma\left(1+\frac{dk}{2}\right)} & \text{odd } d \end{cases}$$

$$L_i(x) = \{y \in C : |[y,x]| = i+1\}$$



α_d, β_d also explicit

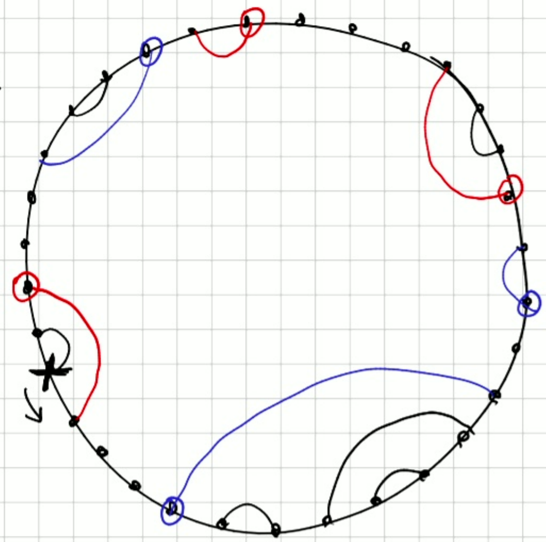
"aesthetically unsatisfying"

Glaser

(4) Interpret in two phases

(a) Phase 1

eg:



$\mathcal{B}_{n,m}$ set of partial rooted chord diagrams with n chords on m points

and
- each chord red, blue, or black
- each blue or red chord has a designated first end

with
- inside of every blue or red chord has every point in a black chord and no black chord not inside red or blue
- noncrossing

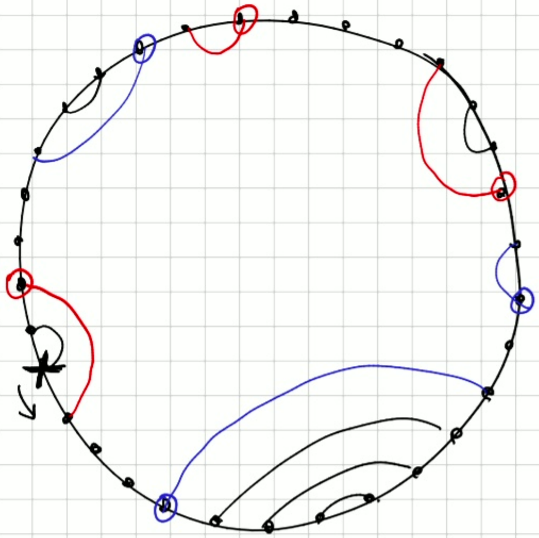
$$\mathcal{B} = \bigcup_{\substack{m \geq 2n \\ n \geq 0}} \mathcal{B}_{n,m}$$

Note: 

(4) Interpret in two phases

(a) Phase 1

eg:



$\mathcal{B}_{n,m}$ set of partial rooted chord diagrams with n chords on m points

and
- each chord red, blue, or black
- each blue or red chord has a designated first end

with
- inside of every blue or red chord has every point in a black chord and no black chord not inside red or blue
- noncrossing

$$\mathcal{B} = \bigcup_{\substack{m \geq 2n \\ n \geq 0}} \mathcal{B}_{n,m}$$

Note:

elementary)

Get $\left|\mathcal{B}_{\lfloor \frac{d}{2}\rfloor+1,\ 2\lfloor\frac{d}{2}\rfloor+2+dk}\right| = \dfrac{(2dk+4\lfloor\frac{d}{2}\rfloor+4)(2dk+4\lfloor\frac{d}{2}\rfloor)\cdots(2dk+4)}{(\lfloor\frac{d}{2}\rfloor+1)!}$

$$= \begin{cases} 2^{d+2}\dfrac{\Gamma\left(\frac{d}{2}(k+1)+2\right)}{\Gamma\left(\frac{d}{2}+2\right)\Gamma\left(1+\frac{dk}{2}\right)} & \text{even } d \\[2ex] 2^{d+1}\dfrac{\Gamma\left(\frac{d}{2}(k+1)+\frac{3}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)\Gamma\left(1+\frac{dk}{2}\right)} & \text{odd } d \end{cases}$$

(b) Phase 2

idea – the binomial coefficient is the number of ways to insert blocks of bare chords before first ends of red and blue chords.

eg

detail – need number of insertion places corresponding to width

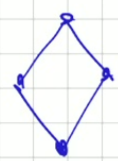
$d=1$	1	$-\frac{1}{2}$			
$d=2$	1	-2	1		
$d=3$	1	$-\frac{27}{8}$	$\frac{9}{4}$		
$d=4$	1	-9	16	-8	
$d=5$	1	$-\frac{215}{16}$	$\frac{225}{8}$	$-\frac{125}{8}$	
$d=6$	1	-34	141	-189	81

What do you do?

(Glaser)

$$B^{(d)}\phi(x) = \frac{1}{\ell^2}\left(\alpha_d\,\phi(x) + \beta_d \sum_{i=1}^{\lfloor \frac{d}{2}\rfloor + 2} C_i^{(d)} \sum_{y\in L_i} \phi(y)\right) \quad \text{where} \quad C_i^{(d)} = \begin{cases} \displaystyle\sum_{k=0}^{i-1} \binom{i-1}{k}(-1)^k \frac{\Gamma\left(\frac{d}{2}(k+1)+2\right)}{\Gamma\left(\frac{d}{2}+2\right)\Gamma\left(1+\frac{dk}{2}\right)} & \text{even } d \\[2ex] \displaystyle\sum_{k=0}^{i-1} \binom{i-1}{k}(-1)^k \frac{\Gamma\left(\frac{d}{2}(k+1)+\frac{3}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)\Gamma\left(1+\frac{dk}{2}\right)} & \text{odd } d \end{cases}$$

$$L_i(x) = \{ y \in C : |[y,x]| = i+1 \}$$



α_d, β_d also explicit

"aesthetically unsatisfying"
Glaser

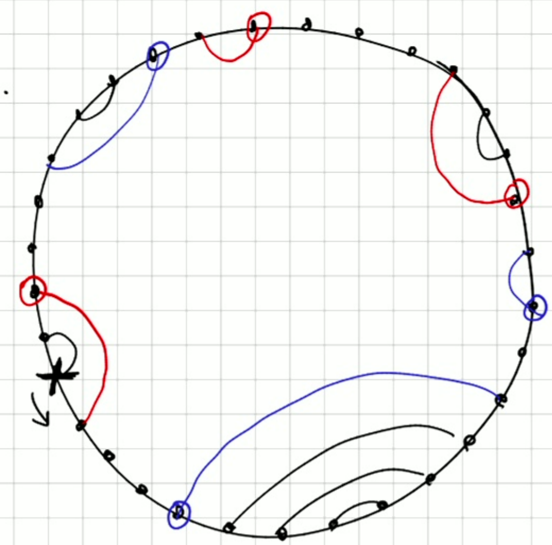
$$= \begin{cases} 2 \quad \dfrac{\cdots}{\Gamma\left(\frac{d}{2}+2\right)\Gamma\left(1+\frac{dk}{2}\right)} & \text{even } d \\[2ex] 2^{d+1} \dfrac{\Gamma\left(\frac{d}{2}(k+1)+\frac{3}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)\Gamma\left(1+\frac{dk}{2}\right)} & \text{odd } d \end{cases}$$

(b) Phase 2

idea – the binomial coefficient is the number of ways to insert blocks of bare chords before first ends of red and blue chords.

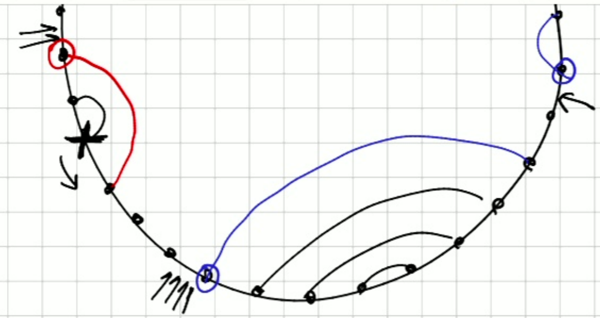
detail – need number of insertion places corresponding to width

eg



ends of red and blue chords.

detail — need number of insertion places corresponding to width



$\mathcal{B}_{n,m,c,e}$ set of elements of $\mathcal{B}_{n,m}$ with fewer than c consecutive bare points before a first and belonging to any of the first e insertion places.

Theorem $$2^{2\lfloor \frac{d}{2} \rfloor + 2} C_i^{(d)} = (-1)^{i-1} \left| \mathcal{B}_{\lfloor \frac{d}{2} \rfloor + 1,\ 2\lfloor \frac{d}{2} \rfloor + 2 + d(i-1),\ d,\ i-1} \right|$$

Simplifications to get rid of one power of 2 always and all of them for d even

$\mathcal{B}_{n,m,c,e}$ set of elements of $\mathcal{B}_{n,m}$ with fewer than c consecutive bare points before a first and belonging to any of the first e insertion places.

$$\text{Theorem} \qquad 2^{2\lfloor \frac{d}{2}\rfloor+2}\, C_i^{(d)} = (-1)^{i-1} \left|\mathcal{B}_{\lfloor \frac{d}{2}\rfloor+1,\ 2\lfloor \frac{d}{2}\rfloor+2+d(i-1),\ d,\ i-1}\right|$$

Simplifications to get rid of one power of 2 always and all of them for d even

5 So what?

Theorem $2^{2\lfloor \frac{d}{2} \rfloor + 2} C_i^{(d)} = (-1)^{i-1} \left| \mathcal{B}_{\lfloor \frac{d}{2} \rfloor + 1,\ 2\lfloor \frac{d}{2} \rfloor + 2 + d(i-1),\ d,\ i-1} \right|$

Simplifications to get rid of one power of 2 always and all of them for d even

(5) So what?

$$\frac{\alpha_d}{\beta_d} = \begin{cases} -\frac{1}{2} \mathrm{Cat}_{\frac{d}{2}} & \text{even } d \\ -\frac{2^{d-1}}{d+1} & \text{odd } d \end{cases}$$

the "C_0" term.

Thanks!