Title: Quantum Dynamics of Causal Sets: Results and Challenges

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Subject: Quantum Gravity

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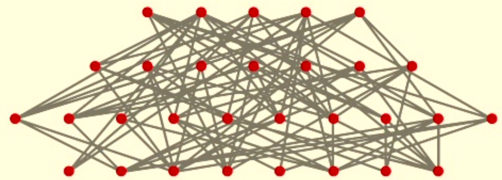
Abstract:

I will discuss aspects of the path sum of causal set theory, in which the continuum is replaced by a sample space of locally finite partially ordered sets. This sample space not only contains continuumlike causals of different dimensions, but an entire zoo of non-continuumlike ones. An open question is whether this path sum can be treated as a UV regularisation of the continuum path integral, as the number of elements increases. I will discuss some results as well as challenges in answering this question.

Quantum Dynamics of Causal Sets: Results and Challenges



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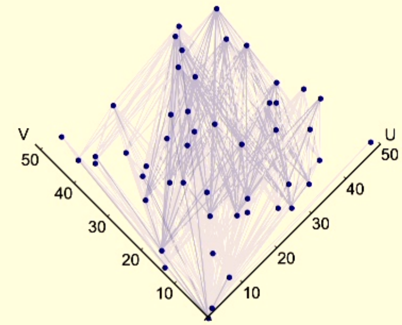
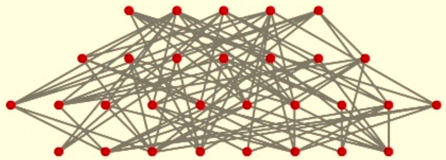
Outline

- Lorentzian geometry, Posets and Causal Sets
- Understanding the Lorentzian Path Sum
- Open Questions

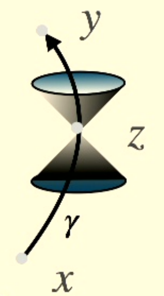




Posets and Lorentzian Geometry

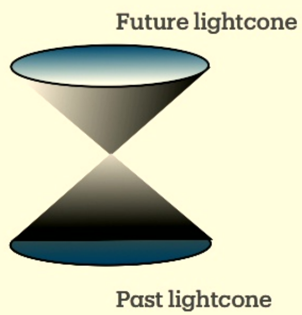
—Robb, 1914, "A theory of time and space"

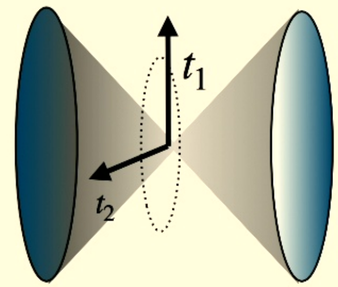
- **Causality on (M, g) defines an order relation on space of events $M : x \prec y$**
- **Principle of Causality: (M, g) is a causal spacetime $\Rightarrow \prec$ is acyclic : $x \prec y \Rightarrow y \nprec x$**



Acyclic: $x \prec y \Rightarrow y \nprec x$
Transitive: $x \prec y, y \prec z \Rightarrow x \prec z$

$(M, \prec)$ is a partially ordered set





Poset associated with $(-, +, +, +, \ldots)$

$(-, -, +, +)$ has no associated poset

Causal Structure as the ``Essence” of Lorentzian Geometry

How much information does $(M, \prec)$ contain?

- Causal Bijection: $f : (M_1, \prec_1) \to (M_2, \prec_2), \quad f(x) \prec_2 f(y) \Leftrightarrow x \prec_1 y, \forall\, x, y \in M_1$
- Conformal Isometry : $F : (M_1, g_1) \to (M_2, g_2), \quad g_2 = \Omega^2 g_1$

Theorem: ***The group of causal automorphisms is isomorphic to the group of conformal transformations on*** $\mathbb{M}^{\mathrm{d}}$**.**

— Alexandrov and Ovchinnikova, 1953, Zeeman, 1964

Theorem: ***If a chronological bijection exists between two future and past distinguishing spacetimes then they are conformally isometric.***

— Kronheimer and Penrose, 1967

— Hawking, King, McCarthy, 1976, Malament, 1977

HKMMKP theorem: (M, g) ``=“ $(M, \prec)$ + ϵ

Distinguishing spacetimes: $I^{\pm}(x) \neq I^{\pm}(y)$

Suggests a non-Riemannian order-theoretic route to quantising spacetime

Discretising the Causal Structure

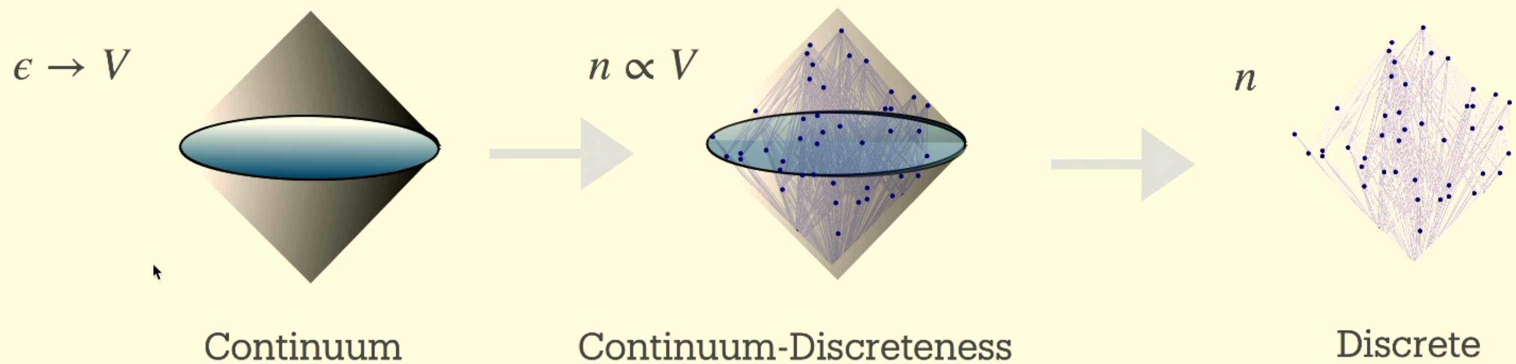
$$(M, g) = (M, \prec) + \epsilon$$

Acyclic: $x \prec y \Rightarrow y \nprec x$

Transitive: $x \prec y, y \prec z \Rightarrow x \prec z$

+

Local Finiteness: $|\mathrm{Fut}(x) \cap \mathrm{Past}(y)| < \infty$



Causal Set: A locally finite poset

Causal Structure as the ``Essence” of Lorentzian Geometry

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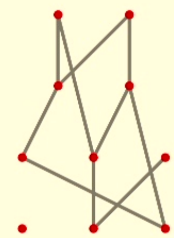
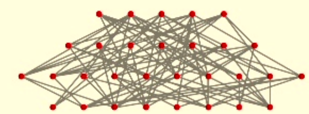
The Causal Set Hypothesis

-- Myrheim, 1978
-- Bombelli, Lee, Meyer and Sorkin,1987

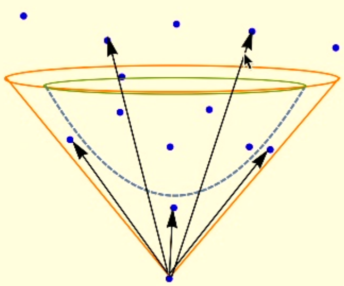
1. Causal Sets are the fine grained structure of spacetime
2. Continuum Spacetime is an approximation of underlying causal sets



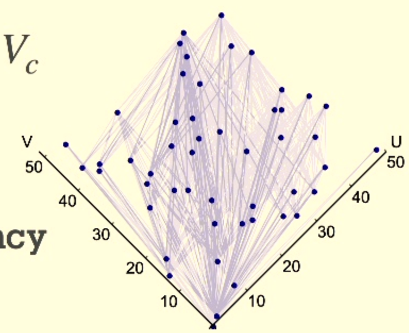


Order ↔ Causal Order
Number ↔ Spacetime Volume

$\langle n \rangle \propto V$, via a Poisson point process: $P_V(n) = \frac{(\rho V)^n}{n!} e^{-\rho V}, \quad \rho^{-1} = V_c$



- Lorentz Invariance is preserved for $C \approx \mathbb{M}^d$
- Non-locality: resulting graph does not have a fixed/finite valency

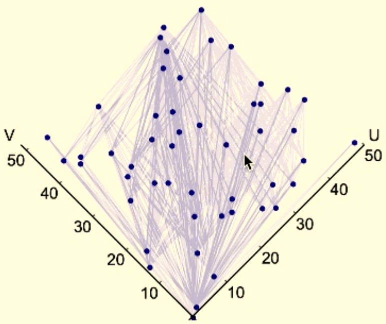


Continuumlike causal set ~ "random lattice"

Geometric Reconstruction: spacetime from causal sets

Order + Number ~ Spacetime

The ensemble of continuum like causal sets contains a lot of topological and geometric information about spacetime

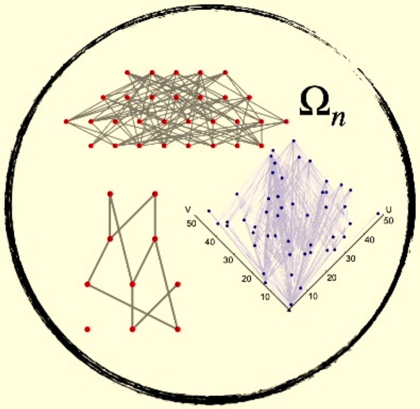


- Timelike distance — *Brightwell & Gregory, Roy, Sinha & Surya*
- Dimension estimators — *Myrheim, Myer, Sorkin, Glaser & Surya, ..*
- Spatial homology —*Major, Rideout & Surya*
- D'Alembertian — *Sorkin, Henson, Benincasa & Dowker & Glaser*
- Scalar curvature —*Benincasa, Dowker, Glaser, Roy, Sinha & Surya*
- Einstein-Hilbert Action —*Benincasa, Dowker, Glaser*
- GHY boundary terms — *Buck, Dowker, Jubb & Surya*
- Spatial and Spacelike Distance — *Rideout & Wallden, Eichhorn, Mizera & Surya, Eichhorn, Surya, Versteegen*
- Horizon area/entropy —*Dou & Sorkin, Barton, Counsell, Dowker, Gould & Jubb, ..*
- Null Geodesics — *Bhattacharya, Mathur, Surya*
- Scalar Field theory —*Johnston, Sorkin, Yazdi, Surya, Nomaan X,*
  — *Jubb, Rejzner, Dable-Heath, Fewster, Wood, Dowker, Albertini, Nasiri, Zalel*
- Entanglement Entropy —*Sorkin, Yazdi, Surya, Nomaan X*
- ......

Covariant Observables → Order Invariants

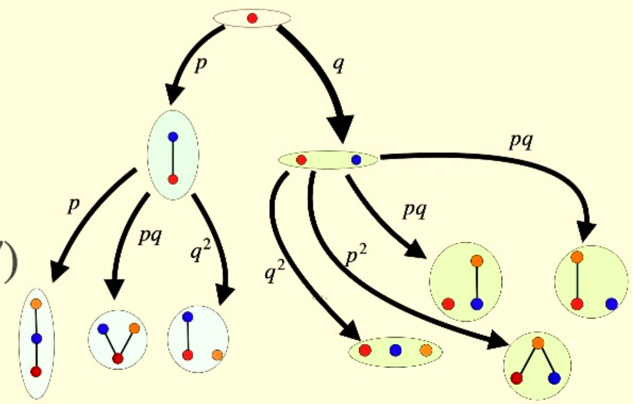
Quantum Dynamics for Causal Sets

$$Z_{cont} = \int \mathscr{D}[g] e^{\frac{i}{\hbar} S_{EH}(g)} \rightarrow Z_n = \sum_{c \in \Omega_n} e^{\frac{i}{\hbar} S^d_{BDG}(c)}$$



Sample Space of all n element causal sets

- **Lorentzian Path Sum:** $Z_n = \sum_{c \in \Omega_n} e^{\frac{i}{\hbar} S^d_{BDG}(c)}$
- **Lorentzian State Sum:** $Z_{n,\beta} = \sum_{c \in \Omega_n} e^{-\beta S^d_{BDG}(c)}$
- **Sequential Growth Dynamics :** $D(\alpha, \alpha') = \sum_{c \in \alpha} \sum_{c' \in \alpha'} D(c, c')$



The Lorentzian Path Sum

$$Z_n = \sum_{c \in \Omega_n} e^{\frac{i}{\hbar} S^d_{BDG}(c)}$$

Can Z_n be thought of as a UV regulated continuum path integral in the large n limit?

Quantum Dynamics for Causal Sets

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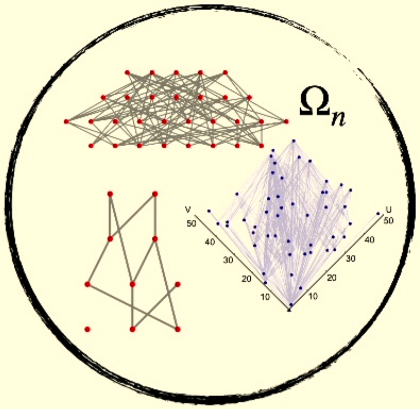

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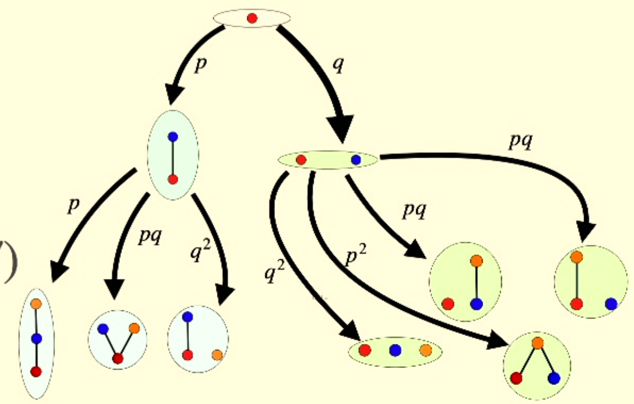
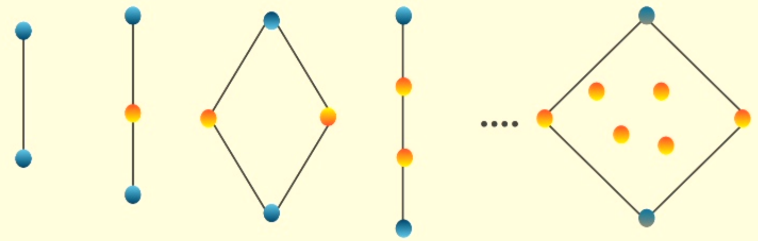


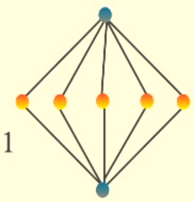
Discrete Einstein-Hilbert Action: The Benincasa-Dowker-Glaser Action(s)

— Benincasa & Dowker, 2010,
— Dowker & Glaser, 2012,
— Glaser, 2014
— Yeats, 2024

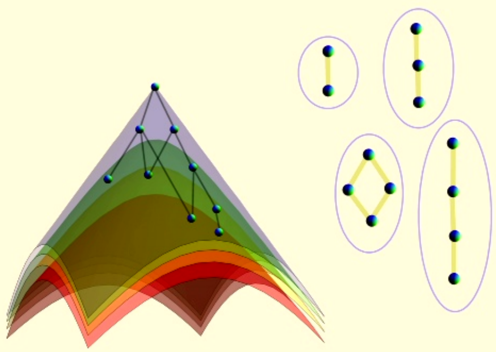


N_J = # of J-element intervals

example:
$N_0 = 10,\ N_5 = 1$



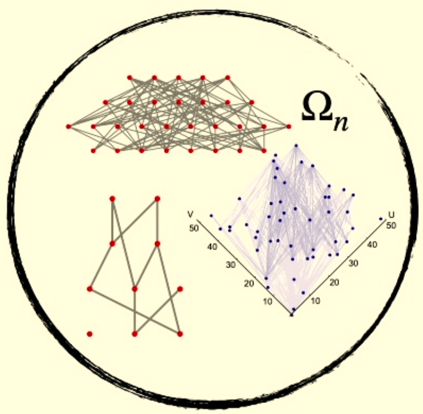
$J^-(p)$ is "foliated" by the the J -element intervals to the past of $p \in \mathbb{M}^d$



- $\frac{1}{\hbar} S^{(d)}_{BDG}(C) = -\alpha_d \left(\frac{\ell}{\ell_p}\right)^{d-2} \left(n + \frac{\beta_d}{\alpha_d} \sum_{J=0}^{J_{max}} C^d_J N_J\right)$
- ℓ_p: Planck Length, ℓ: discreteness length,
- α_d, β_d, C^d_J: known consts.
- $\frac{1}{\hbar} S^{(4)}_{BDG} = -\frac{4}{\sqrt{6}} \left(\frac{\ell}{\ell_p}\right)^2 \left(n - N_0 + 9N_1 - 16N_2 + 8N_3\right)$

$$\lim_{\rho \to \infty} \langle S_{BDG} \rangle = S_{EH} + \text{bdry terms}$$

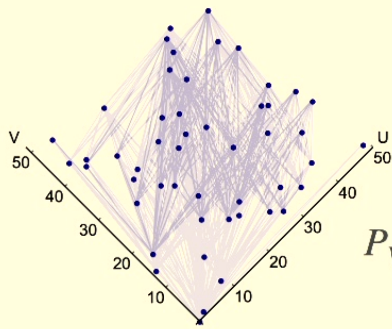
The Sample Space Ω_n



- Causal Spacetimes of *all* dimensions
- Topology changing spacetimes (without causality violation)
- Most are non-continuumlike causal sets

$$|\Omega_n| \sim 2^{\frac{n^2}{4}+\frac{3n}{2}+o(n)}$$

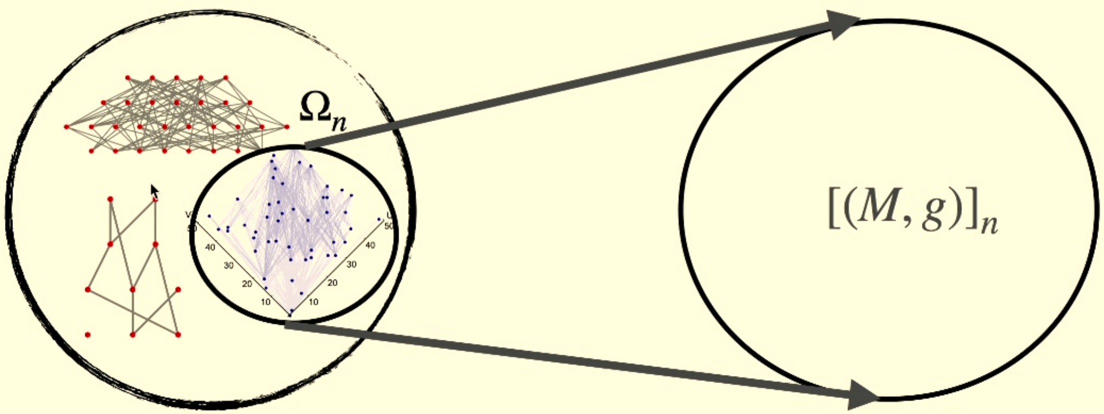
Continuumlike causal sets



- $(M, g) \sim_n (M', g')$ if $\mathscr{C}_n(M, g) = \mathscr{C}_n(M', g')$

$$P_V(n) = \frac{(\rho V)^n}{n!} e^{-\rho V}, \quad \rho^{-1} = V_c$$

- Only equivalence classes of spacetimes $[(M, g)]_n$ are relevant to causal sets



- Causal Spacetimes of *all* dimensions
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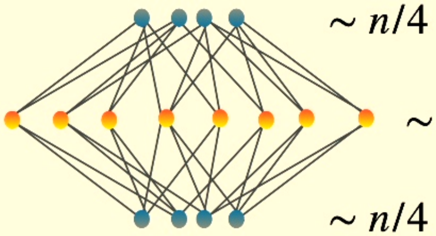
Entropic Contributions



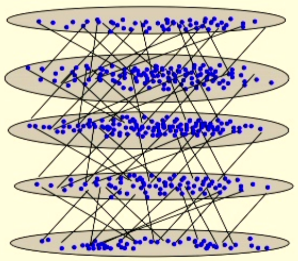
Almost surely, a causal set is a KR poset

- **Kleitmann-Rothschild (KR):** $|\Omega_{KR}| \sim 2^{\frac{n^2}{4}+\frac{3n}{2}+o(n)}$

- Kleitman and Rothschild, Trans AMS, 1975

  - **3 layers:** $\mathbb{L}_k, \quad k=1,2,3, \quad |\mathbb{L}_{1,3}| \sim \frac{n}{4}, |\mathbb{L}_2| \sim \frac{n}{2}$

- **K-layered poset:** $C = \mathbb{L}_1 \sqcup \mathbb{L}_2 \ldots \mathbb{L}_K : e \prec e', e \in \mathbb{L}_k, e' \in \mathbb{L}_{k'} \Rightarrow k < k'$
- $|\Omega_n^{(K)}| \sim 2^{c(K)n^2+o(n^2)}, \quad c(K) \leq 1/4,,$
- **Dominance hierarchy:** $|\Omega_n^{(3)}| > |\Omega_n^{(2)}| > |\Omega_n^{(4)}| > |\Omega_n^{(5)}| \ldots$
- **For $K \ll n$ none of these are like continuumlike**

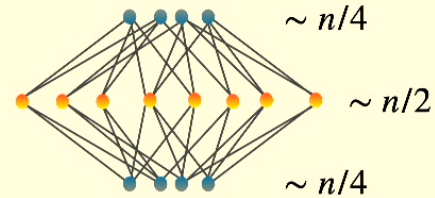


-- D. Dhar, JMP, 1978
-- Promel, Steger, Taraz 2001

Onset of asymptotic regime $n \sim 100$

—J. Henson, D. Rideout, R. Sorkin and S.Surya, JEM, 2015

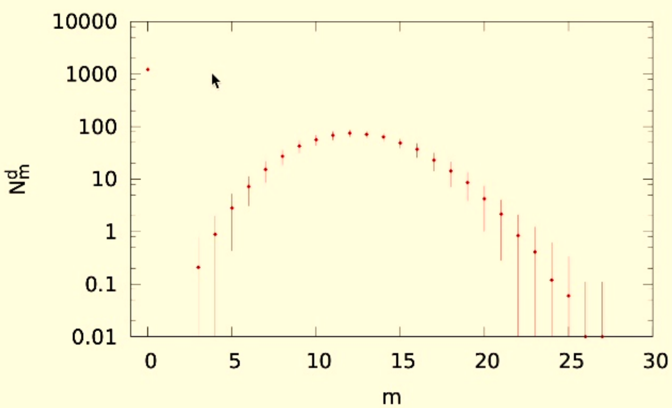
A KR poset is not continuum-like

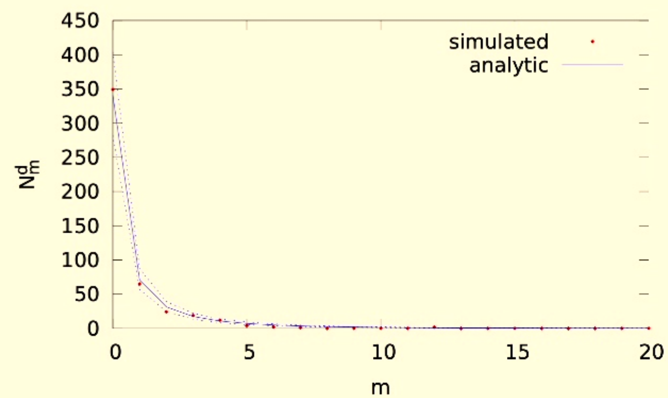


- Does not arise from Poisson Point process into any continuum (M, g)
  - Myrheim-Myer Continuum Dimension is fractional:

$$\frac{\langle R \rangle}{n^2} = \frac{\Gamma(d+1)\Gamma(d/2)}{4\Gamma(3d/2)} \Rightarrow \frac{\Gamma(d_{KR}+1)\Gamma(d_{KR}/2)}{4\Gamma(3d_{KR}/2)} = \frac{3}{16} \Rightarrow d_{KR} \sim 2.5$$

  - Maximal time-like distance: $H_{KR} = 3$
  - Interval Abundances are not like the continuum:

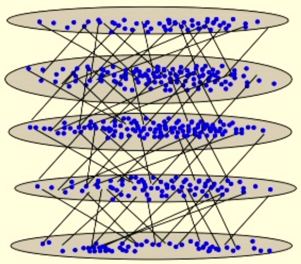




Contribution from K-layer orders

— Loomis and Carlip, 2017
-- A.Anand Singh, A.Mathur and Surya, 2021
-- P. Carlip, S. Carlip and S. Surya, 2023
-- P. Carlip, S. Carlip and S. Surya, 2024

$$Z_K = \sum_{c \in \cup_K \Omega_K} e^{i\mu(n+\lambda_0 N_0)}$$



- **The link action is enough..**
- **Filling fraction:** $\vec{\gamma} = \{\gamma_1, \gamma_2, \ldots \gamma_K\}$
- $N_{max} = \alpha(\vec{\gamma})n^2, \quad |P_{\vec{\gamma},p,n}| = \binom{\alpha(\vec{\gamma})n^2}{pn^2}, \quad \alpha(\vec{\gamma}) = \sum_{i=1}^{K-1} \gamma_i \gamma_{i+1}$
- $\alpha_{max}(\vec{\gamma}) = 1/4, \quad \gamma_x = (1/4 - x, 1/2, 1/4 + x), \quad x \in [-1/4, 1/2, 1/4]$
- $|P_{\vec{\gamma}_x,p,n}| \leq |Q^K_{p,n}| \leq K^n |P_{\vec{\gamma}_x,p,n}|$

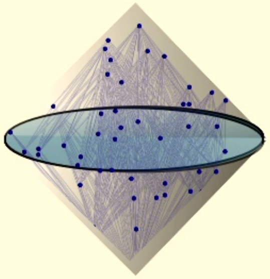
Suppression: $\tan\left[\frac{1}{2}\beta_d\left(\frac{\ell}{\ell_p}\right)\right] > \sqrt{\left(\frac{27}{4}e^{-1/2} - 1\right)} \Rightarrow \ell > \ell_0, \quad 1.136 \leq (\ell_0/\ell_p) < 2.33,$ for all $d > 2$

Contribution from Continuumlike Causal Sets

.. work in progress with Steve Carlip

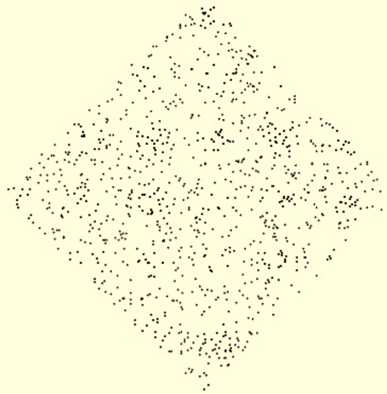
- $\mathscr{C}_n(M,g)$: Ensemble of causal sets generated from a Poisson point process into (M,g)
- Contribution from $[(M,g)]_n$ is:

$$Z_n(M,g) = \sum_{C \in \mathscr{C}_n(M,g)} e^{\frac{i}{\hbar} S_{BDG}(C)}$$

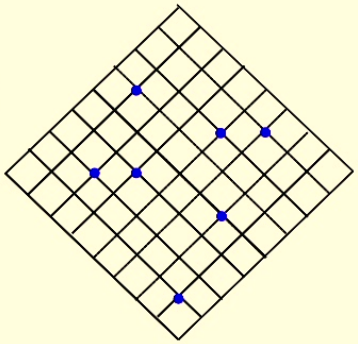


- $\mathscr{C}_n(M,g)$ overlaps with $\mathscr{C}_n(M',g')$, but as n increases, the overlap decreases.
- How many causal sets in $\mathscr{C}_n(M,g)$ are there?

$d = 2$



- Consider the causal diamond $\mathbb{D}^2 \subset \mathbb{M}^2$
- d=2 orders $U \cap V$ on n elements
  - $u_i, v_j \in (1, \ldots, n)$
  - $U = (u_1, u_2, \ldots u_n), V = (v_1, v_2, \ldots v_n),$
  - $C = U \cap V: \quad e_i = (u_i, v_i) \prec e_j = (u_j, v_j) \Leftrightarrow u_i < u_j, v_i < v_j$
- Order theoretic dimension and continuum dimension are the same for $d = 2$

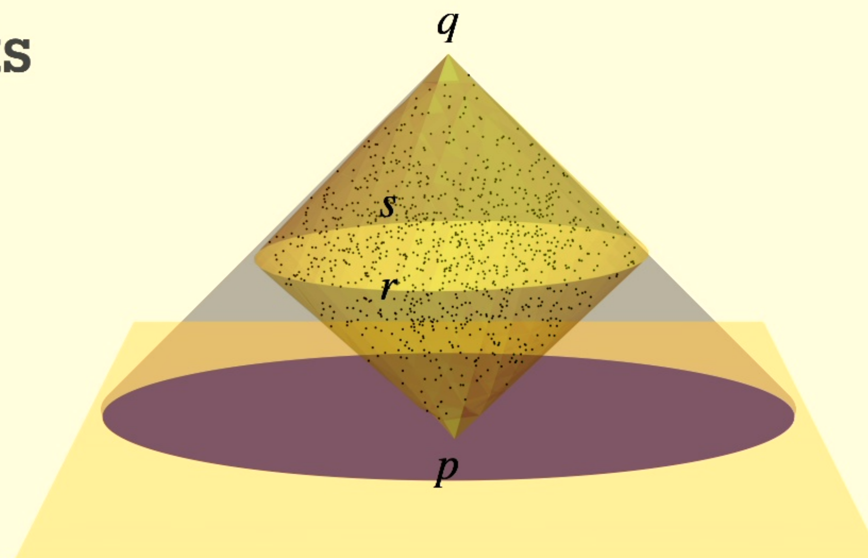


- How many 2d orders are there?
- $|\Omega_{2d}| \sim 2^{n \ln n} \ll |\Omega_{KR}|$

—Winkler, 1985
—El-Zahar & Sauer, 1988

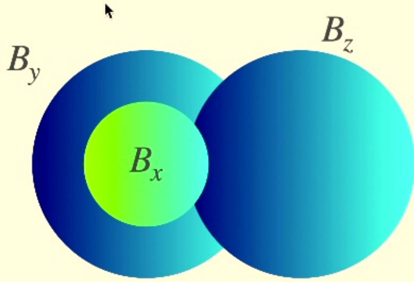
- Most of them are random orders, or $|\mathscr{C}_n(\mathbb{D}^2)| \sim |\Omega_{2d}|$

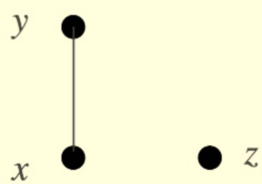
Counting Continuumlike Causal Sets

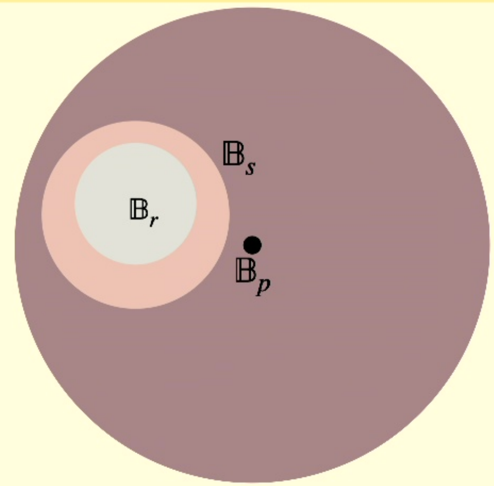


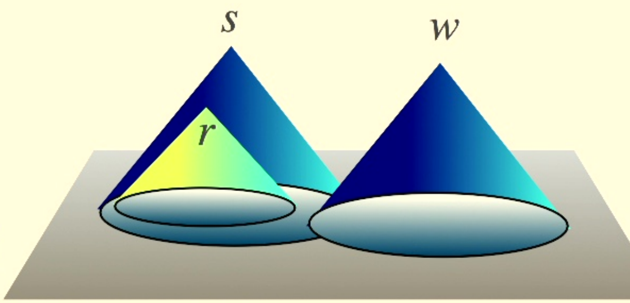
- Causal diamond $I[p,q] \subset (M,g)$
- Let $p \in \Sigma_p \sim \mathbb{R}^{d-1}, \quad \mathbb{B}^d_q \equiv I^-(q) \cap \Sigma_p$
- For any $r, s \in I[p,q],\ r \prec s \Leftrightarrow \mathbb{B}^d_r \subset \mathbb{B}^d_s$
- $I[p,q]$-like causal sets are Sphere Orders

-Meyer, 1989
-Brightwell and Winkler, 1989





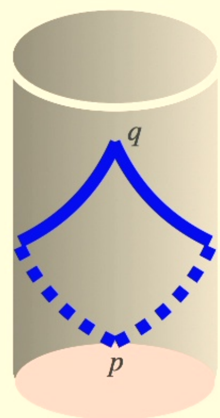




- **Example :** $\mathbb{M}^d$
  - $P_{rs}(x,y,t) \equiv (x_r - x_s)^2 + (y_r - y_s)^2 - (t_r - t_s)^2 < 0, \quad t_r - t_s < 0$
  - $|\Omega_{\mathscr{B}_n^{d-1}}| \sim |\mathscr{C}_n(I[p,q])|$
  - $|\Omega_{\mathscr{B}_n^{d-1}}| \sim 2^{d\,n \ln n}$

— ***Alon and Schienerman, 1988***
— ***Sauermann, 2021***

- **For generic spacetimes:**
  - $P_{rs}(x,y,t) < 0$



- **However, if Σ_p is compact, then the causal sets are not sphere orders.**

Contribution from Continuum-like Posets

- $$Z_n(M,g) = \sum_{C \in \mathscr{C}_n(M,g)} e^{\frac{i}{\hbar} S^{(d)}_{BDG}(C)}$$

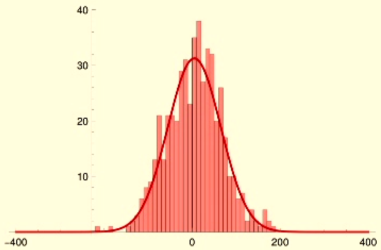
- **If** $d' = d$**, then** $S^{(d)}_{BDG}(C) = \langle S^{(d)}_{BDG} \rangle + \Delta S$**,**

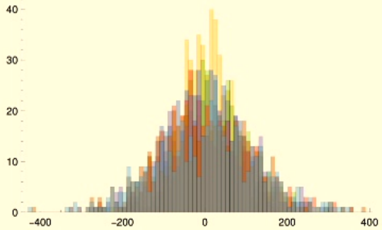
- $$\lim_{n\to\infty} \langle S^{(d)}_{BDG} \rangle = S_{EH}(M,g) + boundary\ terms$$

- $$Z_n(M,g) = \sum_{C \in \mathscr{C}_n(M,g)} e^{\frac{i}{\hbar}(\langle S^{(d)}_{BDG} \rangle + \Delta S)} \sim e^{\frac{i}{\hbar}\langle S^{(d)}_{BDG} \rangle} \sum_{\Delta S} F(\Delta S) e^{\frac{i}{\hbar}\Delta S}$$

- $$F(\Delta S) = |\mathscr{C}_n(M,g)| \times \mathscr{F}(\Delta S)$$

- A first guess :
  - Gaussian : $F(\Delta S) = |\mathscr{C}_n(M,g)| \times \frac{1}{\pi}\sqrt{\frac{\alpha}{2}}\, e^{-\alpha(\Delta S)^2}$
  - $\sum_{\Delta S} \rightarrow \int d(\Delta S)$
  - $Z([M,g]) = e^{\frac{i}{\hbar}\langle S^{(d)}_{BDG}(C)\rangle} |\mathscr{C}_n(M,g)| \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{4\alpha}}$



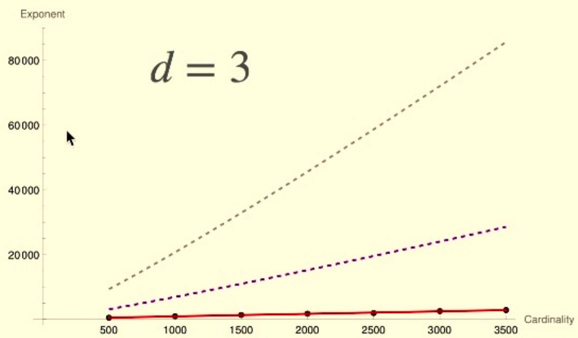


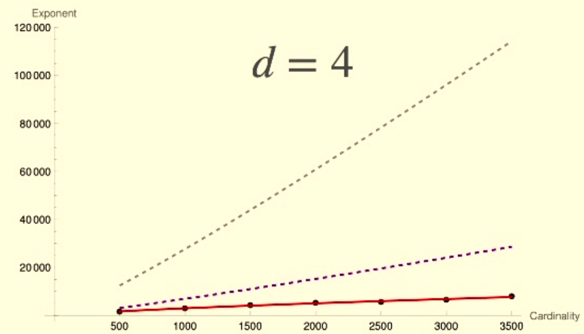
- A competition between $|\mathscr{C}_n(M,g)|$ and $e^{-\frac{1}{4\alpha}}$

Estimating $\alpha(n)$

- **If the volume of the causal diamond is fixed:**
  - **Numerical Simulations in** $\mathbb{M}^d$, $d = 3{,}4$
  - $\dfrac{1}{\alpha(n)} < n \ln n$

- **Fine balance of exponents:** $> dn \ln n - n \ln n = (d-1) n \ln n$ **!**

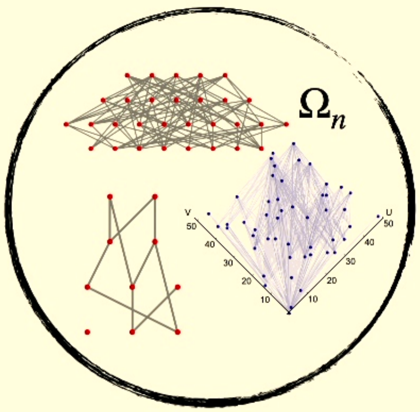




Continuum contributions $Z_n(M, g) \sim 2^{(d-1) n \ln n} e^{\frac{i}{\hbar} \langle S^{(d)}_{BGG} \rangle}$ **grow with** n

Other Contributions..

- In $\mathbb{M}^d$, $d \neq d'$:
  - $S^{(d')}_{BDG}(C) \neq S_{EH}(M, g) + \Delta S$
  - $d > 2: \langle N_m \rangle = n^{2-2/d} \dfrac{\Gamma(d)\Gamma(2/d + m)}{m! g(d)} + \ldots$
  - $d = 2 : \langle N_m \rangle = n \ln n + \ldots$
  - $d > 2 : Z_n(M, g) \sim \sum_{C \in \mathscr{C}(M,g)} e^{\frac{i}{\hbar}\beta(d) n^{2-2/d}}$

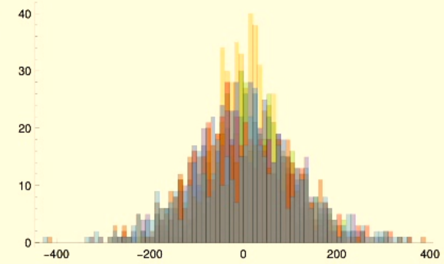


- Similar behaviour for RNNs: $\beta(d) \to \beta(d, R)$ — do these fluctuations cancel out?
- What are other non-continuumlike contributions?
  - k-d orders also grow as $2^{nk \ln n}$
  - What else?

— Alon and Schienerman, 1988
— Sauermann, 2021

Open Questions

- **For** $\mathrm{vol}(I[p,q]) \neq const$
  - **Numerical simulations suggest:** $1/\alpha(n)$ **grows faster than** $dn \ln n$
  - **Is the Gaussian assumption for** $F(\Delta S)$ **correct?**
  - **Is it possible to numerically sum** $\sum_{C \in \mathscr{C}_n(M,g)} e^{\frac{i}{\hbar}\Delta S(C)}$ **?**



- **Is** $S^{(d)}_{BDG}$ **the only choice of action?**
  - **Higher order curvature terms ?** — *Brito, Eichhorn & Pfeiffer, 2023*
  - **The chain action — smaller fluctuations?** — *Roy, Sinha & Surya, 2013* — *Bombelli and Pilgrim, 2020* —*Kambor and Nomaan, 2020*

Can Z_n **be thought of as a UV regulated continuum path integral in the large** n **limit?**

Thank you!