Title: Lecture - Strong Gravity, PHYS 777

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Subject: Strong Gravity

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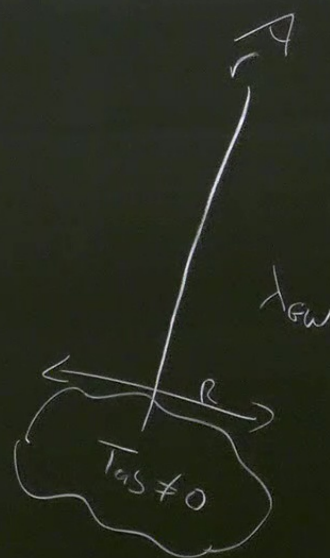
Gravitational waves

$$\bar{h}_{ij} = \frac{2}{r}\ddot{I}_{ij}(t-r)$$

$$I_{ij} = \int T^{00} x_i x_j \, d^3x$$

Projection tensor:

$$P_{ij} = \delta_{ij} - \frac{x^i}{r}\frac{x^j}{r}$$



$r >> R, \lambda_{GW}$

$\lambda_{GW} >> R$

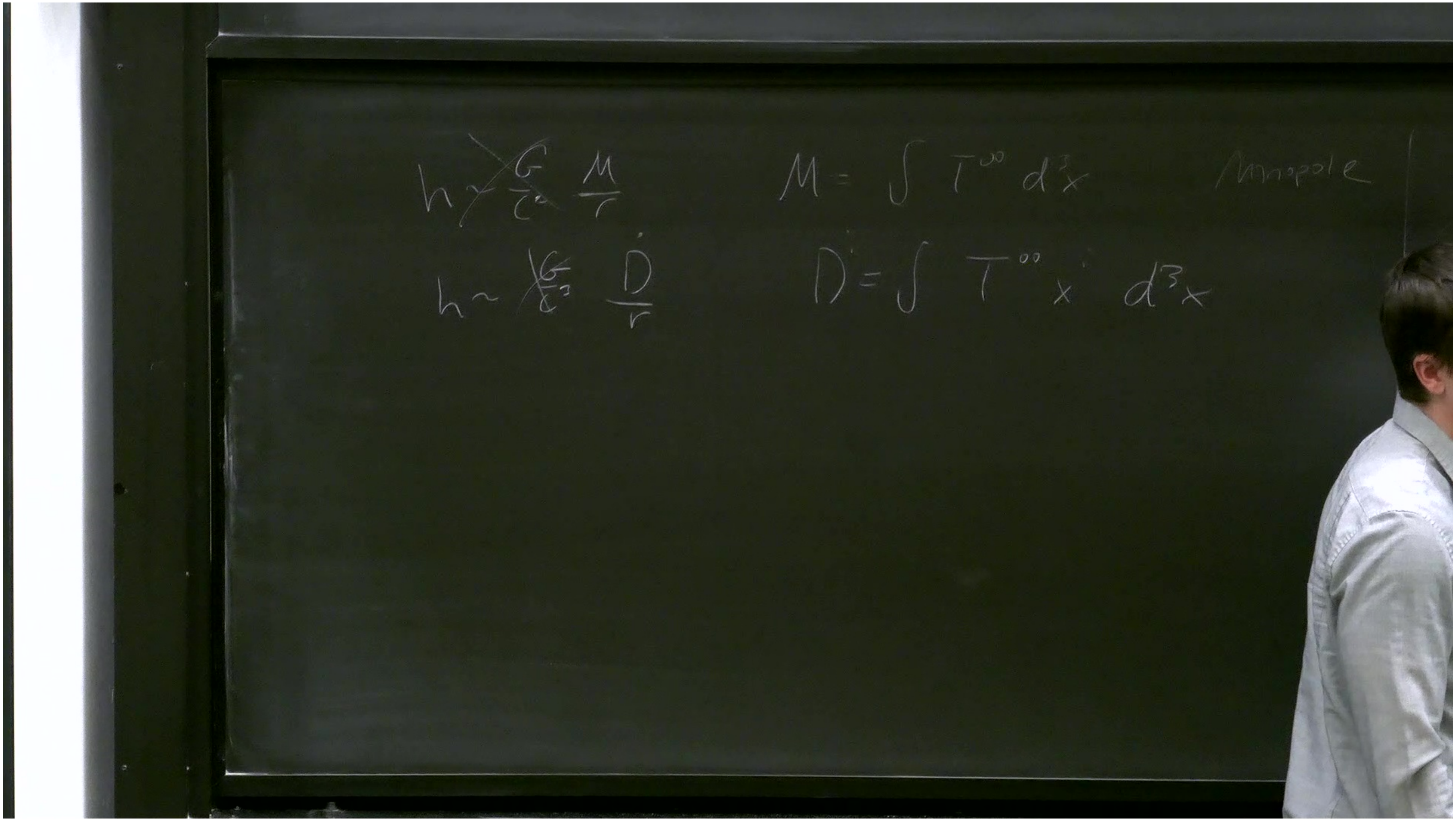
Transverse component

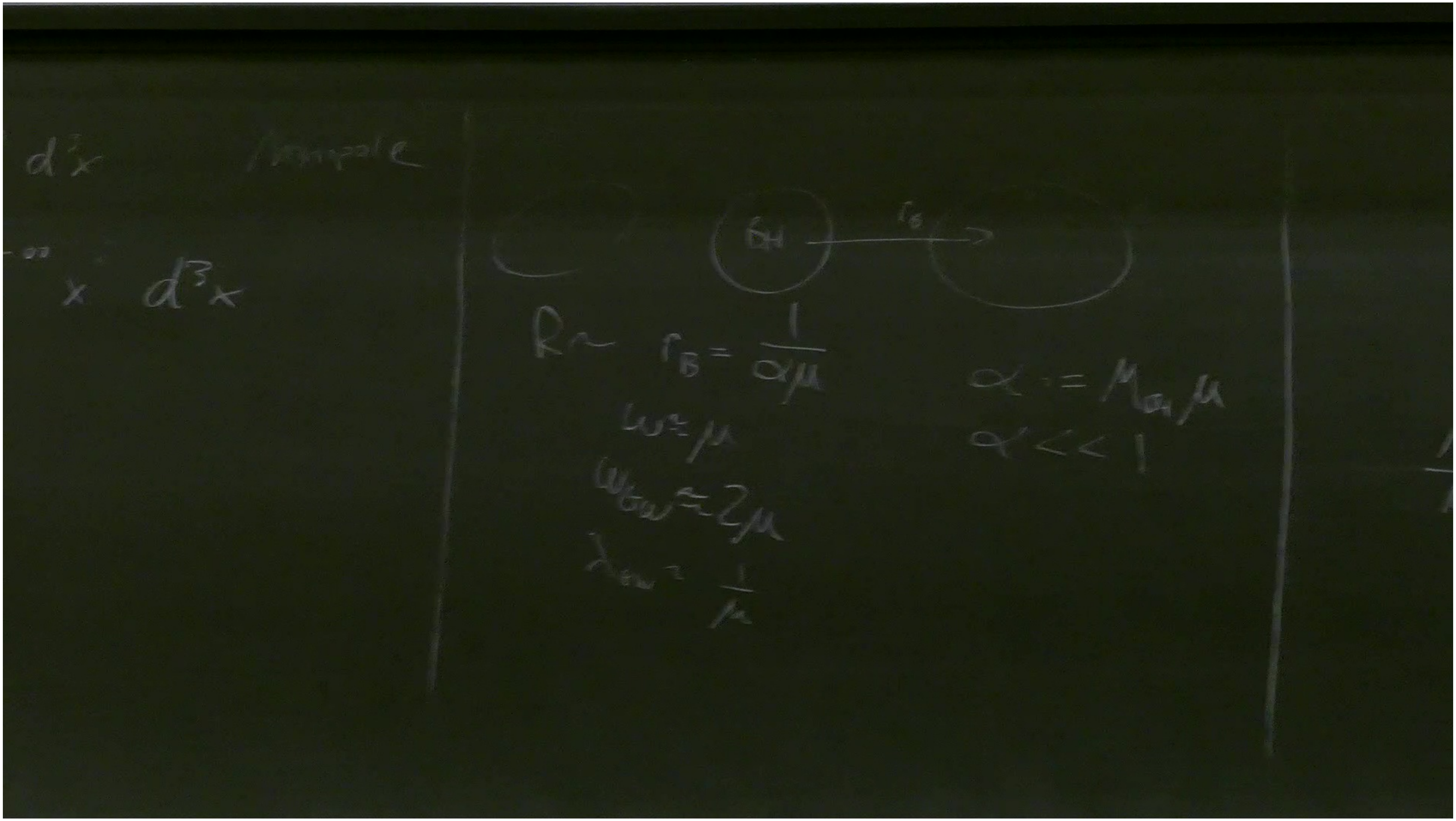
$$\bar{h}^{T}_{ij} = \bar{h}^{k\ell} P_{ik} P_{j\ell}$$

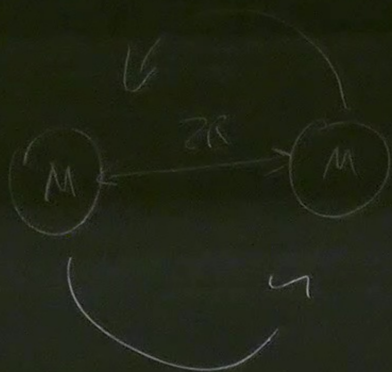
$$h^{TT}_{ij} = \bar{h}^{k\ell} P_{ik} P_{j\ell} - \frac{1}{2} P_{ij} P_{k\ell} \bar{h}^{k\ell}$$

$$h^{TT}_{ij} = \frac{2}{r} \ddot{\mathcal{I}}^{k\ell} \left[P_{ik} P_{j\ell} - \frac{1}{2} P_{k\ell} P_{ij} \right]$$

$$\mathcal{I}_{ij} = I_{ij} - \frac{1}{3} \delta_{ij} I$$


$h \sim \cancel{\frac{G}{c^2}} \frac{M}{r}$
$M = \int T^{00} d^3x$
Monopole
$h \sim \cancel{\frac{G}{c^3}} \frac{\dot{D}}{r}$
$D^i = \int T^{00} x^i d^3x$


d³x
d³x
BH
r_B
R ~ r_B = 1/(αμ)
ω ≈ μ
ω_GW ≈ 2μ
λ_GW ≈ 1/μ
α := M_BH μ
α << 1



$$\omega_{GW} \approx 2\omega_0 = \frac{4\pi}{T} = \sqrt{\frac{M}{R^3}}$$

$$= \frac{1}{R}\left(\frac{M}{R}\right)^{1/2}$$

$$\frac{M}{R_0} << 1$$

$$\lambda_{GW} = R\left(\frac{R}{M}\right)^{1/2}$$

Energy and angular momentum in GWs

$$g_{ab} = \hat{g}_{ab} + h^{(1)}_{ab} + h^{(2)}_{ab}$$ second order

$$R_{ab} = \cancel{R^{(0)}_{ab}}^{\to 0} + R^{(1)}_{ab} + R^{(2)}_{ab}$$

1st order $\quad G^{(1)}_{ab}[h^{(1)}_{cd}] = 0$

2nd order : $\quad G^{(1)}_{ab}[h^{(2)}_{cd}] + G^{(2)}_{ab}[h^{(1)}_{ab}] = 0 \iff$

$$G^{(1)}_{ab}[h^{(2)}_{cd}] = 8\pi t_{ab} = -G^{(2)}_{ab}[h^{(1)}_{ab}]$$

Caution: not gauge invariant

Get around by averaging over several wavelengths

$$t_{ab} = \frac{1}{32\pi} \left\langle \left(\partial_a h^{TT}_{cd}\right)\left(\partial_b h^{cd}_{TT}\right) \right\rangle$$

in general

$$= \frac{1}{32\pi} \left\langle \partial_a \bar{h}_{cd} \partial_b \bar{h}^{cd} - \frac{1}{2}\partial_a \bar{h} \partial_b \bar{h} - 2\left[\partial_d \bar{h}_{(b|c} \partial_{a)} \bar{h}^{cd}\right] \right\rangle$$

Lorenz traceless $\partial_a \to \nabla_a$

$\langle \partial_a X \rangle = 0$

$\langle X \partial_a Y \rangle = -\langle Y \partial_a X \rangle$

$\Gamma^{d}_{ab} = 0$

$$G^{(1)}_{ab}\left[h^{(2)}_{cd}\right] = 8\pi t_{ab} = -G^{(2)}_{ab}\left[h_{ab}\right]$$

$$E_{GW} = \int_{\Sigma_t} t_{00}\, d^3x$$

For plane wave . $E_{GW} = \frac{\omega^2}{32\pi}\left\langle h_+^2 + h_\times^2 \right\rangle$

$$\frac{dE_{GW}}{dt} = \int t_{0i}\, \hat{n}^i\, dS$$

unit normal

$$\frac{dJ^i_{GW}}{dt} = \int \varepsilon^{ijk}\left(\hat{n}_j r\right) t_{0k}\, dS$$

$$\frac{dE_{GW}}{dt} = \frac{1}{5}\left\langle \dddot{I}_{ij}\,\dddot{I}^{ij} \right\rangle \Big|_{t=t-r}$$

$$\frac{dJ^{i}_{GW}}{dt} = \frac{2}{5}\,\varepsilon^{ijk}\left\langle \ddot{I}_{lj}\,\dddot{I}^{l}{}_{k} \right\rangle$$

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Scalar (test) field

$$g^{ab} \nabla_a \nabla_b \varphi = 0$$

Kerr
(B-L coördinates)

Ansatz: $\varphi = R(r)\, \Theta(\theta)\, e^{im\phi}\, e^{-i\omega t}$

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$$g^{ab}\nabla_a\nabla_b\varphi = 0$$

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(*

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$$(*) \quad \frac{d}{dr}\left(\Delta \frac{dR}{dr}\right) + \left[\frac{\omega^2(r^2+a^2) - 4Ma\omega r + m^2a^2}{\Delta} - (\omega^2 a^2 + \Lambda) \right] R = 0$$

$$(**) \quad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left[a^2\omega^2 \cos^2\theta - \frac{m^2}{\sin^2\theta} + \Lambda \right] \Theta = 0$$

separation constant

$a = 0$

$$\Theta \to P_{\ell m}(\cos\theta), \quad \Lambda \to \ell(\ell+1), \quad \Theta e^{im\phi} = Y_{\ell m}(\theta, \phi)$$

$a \neq 0$

$c = a\omega$

$$\Theta = S_{\ell m}(\cos\theta; c)$$

$$a=0, \quad (*) \Rightarrow \quad \frac{d^2R}{dr_*^2} + (\omega^2 - V) R = 0$$

$$r_* = r + 2M \log\left(\frac{r}{2M} - 1\right) \quad \text{tortise coördinates}$$

$$V = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2M\sigma}{r^3}\right] \quad (\Diamond) \qquad \sigma = 1$$

General Solution

$$\psi = \sum_{\ell,m} \int dt \, e^{-i\omega t} R_{\ell m\omega}(r) \, S_{\ell m}(\theta, c=a\omega) \, e^{im\phi}$$