Title: Lecture - Strong Gravity, PHYS 777

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Subject: Strong Gravity

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Linear perturbations of the metric

$$g_{ab} = \hat{g}_{ab} + h_{ab} \qquad |h_{ab}| << |\hat{g}_{ab}|, \quad \text{Assume} \quad \hat{R}_{ab} = 0$$

$$\delta\Gamma^a{}_{bc} = \frac{1}{2}\hat{g}^{ad}\left(\hat{\nabla}_c h_{bd} + \hat{\nabla}_b h_{dc} - \hat{\nabla}_d h_{bc}\right)$$

$$\underline{\delta R^a{}_{bcd} = \hat{\nabla}_c \delta\Gamma^a{}_{bd}} - \hat{\nabla}_d \delta\Gamma^a{}_{bc}$$

$$\Rightarrow \delta R^a{}_{bcd} = \frac{1}{2}\left[\hat{\nabla}_c\hat{\nabla}_b h^a{}_d + \hat{\nabla}_c\hat{\nabla}_d h^a{}_b - \hat{\nabla}_c\hat{\nabla}^a h_{bd} - \hat{\nabla}_d\hat{\nabla}_b h^a{}_c - \hat{\nabla}_d\hat{\nabla}_c h^a{}_b + \hat{\nabla}_d\hat{\nabla}^a h_{bc}\right]$$

$$\Rightarrow \delta R_{bd} = \frac{1}{2}\left(\nabla_c \nabla_b h^c{}_d + \nabla_c \nabla_d h^c{}_b - \nabla_c \nabla^c h_{bd} - \nabla_d \nabla_b h^c{}_c - \nabla_d \nabla_c h^c{}_b + \nabla_d \nabla^c h_{bc}\right)$$

$$8\pi T_{bd} = \delta G_{bd} = -\frac{1}{2}\hat{\Box}\bar{h}_{bd} + \hat{R}_{adbc}\bar{h}^{ac} - \frac{1}{2}\hat{g}_{bd}\hat{\nabla}_a\hat{\nabla}_c\bar{h}^{ac} + \frac{1}{2}\hat{\nabla}_b\hat{\nabla}_a\bar{h}^a{}_d + \frac{1}{2}\hat{\nabla}_d \ldots$$

$$\hat{\nabla}_a \bar{h}^{ab} = 0 \quad (\text{Lorenz Gauge})$$

$$\hat{\Box}\bar{h}_{bd} - 2\hat{R}_{adbc}\bar{h}^{ac} = -16\pi T_{bd} \quad (*)$$

$$x^a \to x^{a'} = x^a + \xi^a$$

$$h_{ab} \to h'_{ab} = h_{ab} - 2\nabla_{(a}\xi_{b)}$$

$$\hat{\nabla}^a \bar{h}'_{ab} = \hat{\nabla}^a \bar{h}_{ab} - \hat{\Box}\xi_b$$

Original $\xi^a(t, x^i)$

In Lorenz gauge $\xi^a(t=0, x^i)$, $\partial_t \xi^a(t=0 \ldots$

$\bar{h}^{ac} + \frac{1}{2}\hat{\nabla}_b \hat{\nabla}_a \bar{h}^a{}_d + \frac{1}{2}\hat{\nabla}_a \hat{\nabla}_a \bar{h}^a{}_b$

$$\bar{h}_{ab} = h_{ab} - \frac{1}{2} h \hat{g}_{ab}$$

$$\bar{h} = -h$$

e)

$$\hat{\Box} \bar{h} = -16\pi T$$

$\xi^a(t, x^i)$

uge $\xi^a(t=0, x^i)$, $\partial_t \xi^a(t=0, x^i)$

Particular cases: (i) Vaccum ($T_{ab}=0$)

Choose 2 free functions from, $\xi_a, \partial_t \xi_a$ at $t=0$

to have $\bar{h}=0$, $h_{ab}=\bar{h}_{ab}$

(ia) Vacum, flat background $\hat{g}_{ab}=\eta_{ab}$

$h_{ti}=\partial_t h_{ti}=0$ at $t=0$

(ib) $\hat{g}_{ab}=\hat{g}^{Kerr}_{ab}$

(ii) Non-vacuum ($T_{ab}\neq 0$), $\hat{g}_{ab}=\eta_{ab}$

Propagation of GWs

Use 4 gauge DOF to set $h_{ti} = 0$, $\partial^a h_{ab} = 0$

$$0 = \partial_a h^{a0}_{TT} = \partial_0 h^{00}_{TT}$$

Choose $h^{00} = 0$, $\Rightarrow h^{00} = 0$ for all time

$$\Box h^{TT}_{ij} = 0$$

$$h^{TT}_{ij} = \underbrace{A_{ij}}_{\text{constant}} \exp(i k_a x^a)$$

$0 = \Box h_{ij}$

$(ab \neq 0)$

$= 0 \quad , \eta^{ab} h_{ab} = 0$

all time

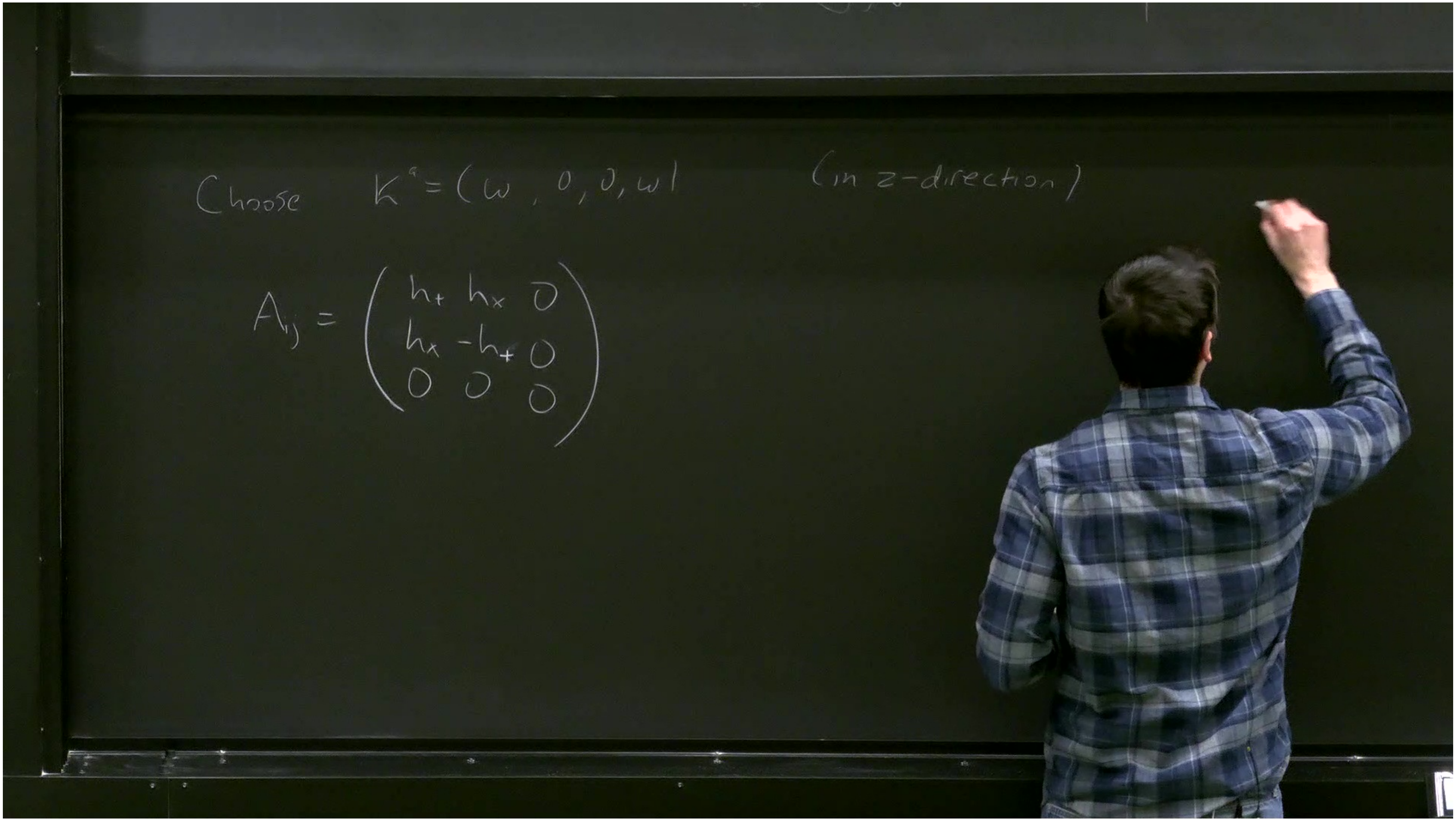
$0 = \Box h_{ij}^{TT}$

$= \eta^{ab} (i k_a i k_b) h_{ij}^{TT}$

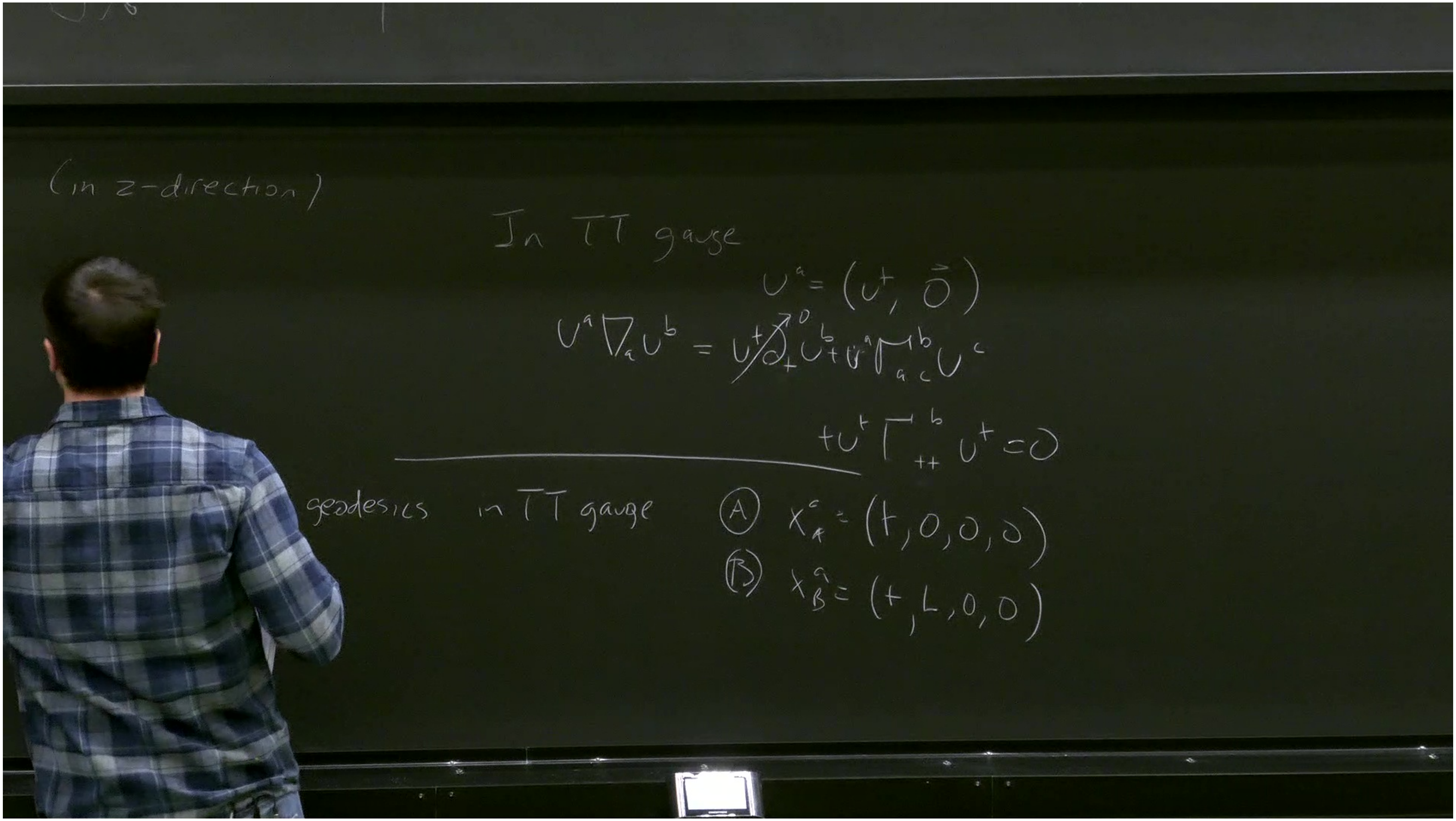
$= - k_a k^a h_{ij}^{TT}$

$k_a k^a = 0 \Leftrightarrow k^a$ is null

$0 = \partial_a h^{ab}_{TT} = k_a h^{ab} = 0$



Choose $k^a = (\omega, 0, 0, \omega)$ (in z-direction)
$A_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$


(in z-direction)
In TT gauge
$U^a = (U^t, \vec{0})$
$U^a \nabla_a U^b = \cancel{U^t \partial_t U^b} + U^a \Gamma^b{}_{ac} U^c$
$+ U^t \Gamma^b{}_{tt} U^t = 0$
geodesics in TT gauge
A $x^a_A = (t, 0, 0, 0)$
B $x^a_B = (t, L, 0, 0)$

gauge

$$U^a = (U^t, \vec{0})$$

$$U^b = \cancel{U^t \partial_t U^b} + U^a \Gamma^b{}_{ac} U^c$$

$$+ U^t \Gamma^b{}_{tt} U^t = 0$$

e

(A) $x^a_A = (t, 0, 0, 0)$

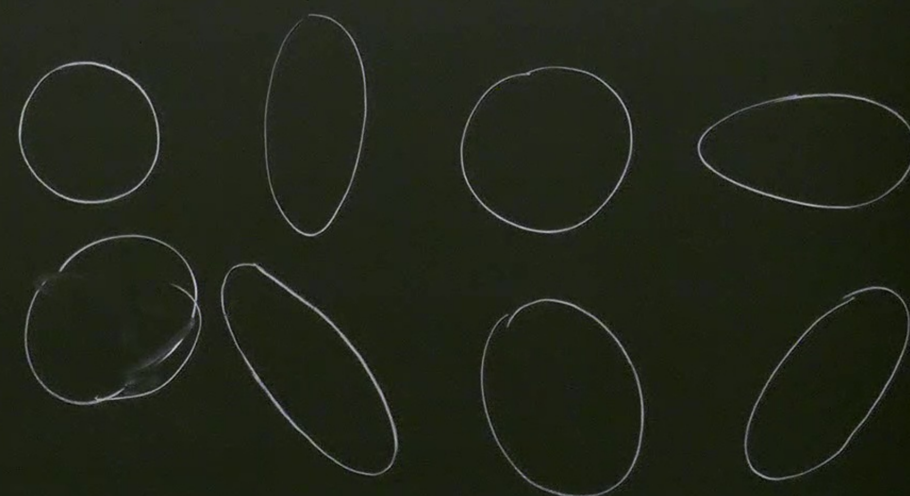
(B) $x^a_B = (t, L, 0, 0)$

For h_+ wave

$$S_{AB}^2 = [L(1 + h_+ \cos(\omega t))]^2$$

h_+ :

$h_\times$



Constant

Rotate by ϕ about z-axis

$$h_\times \pm i h_+ \to (h_\times \pm i h_+) e^{\mp 2 i \phi}$$

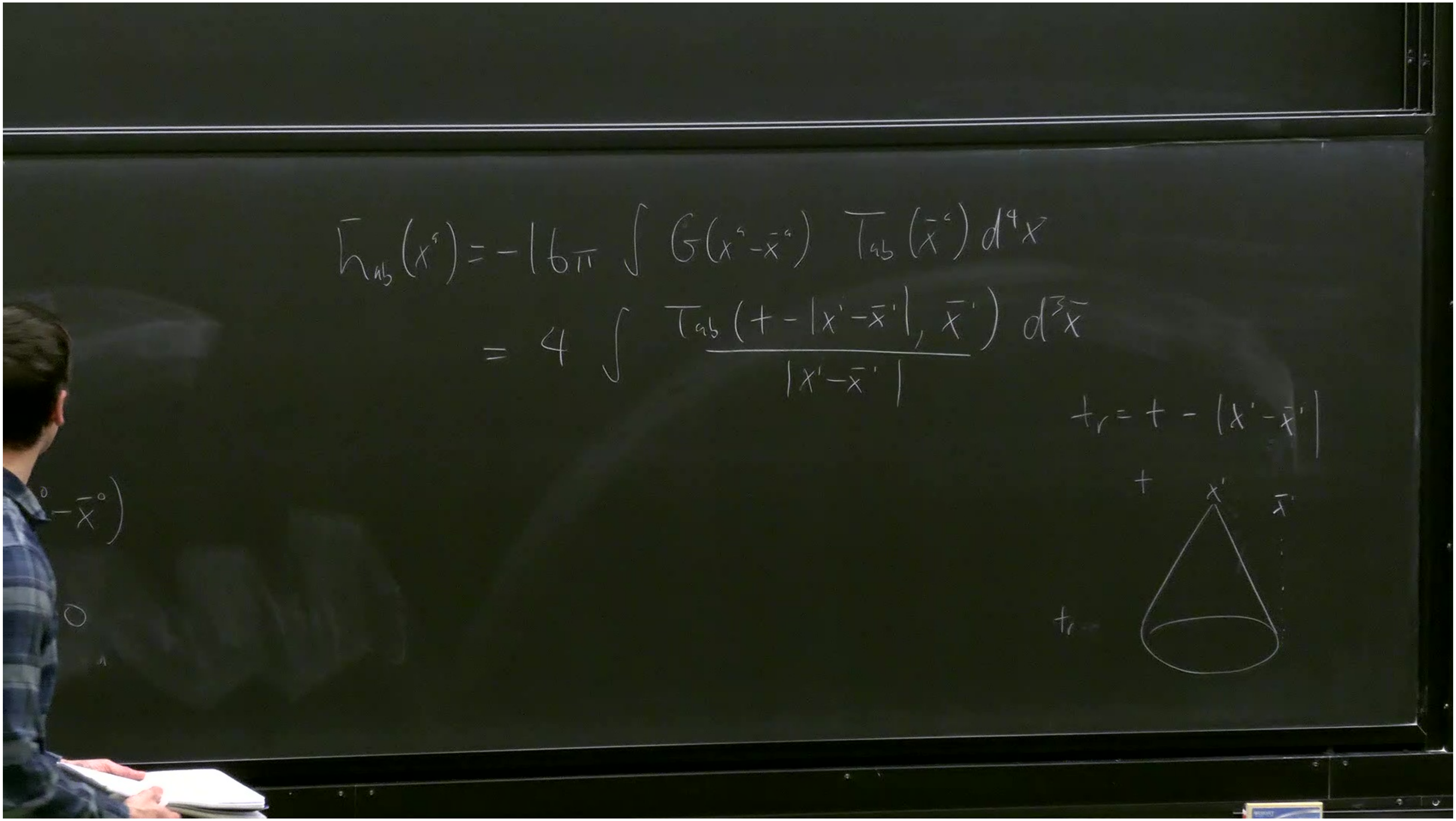
spin-2

$$\Box \bar{h}_{ab} = -16\pi T_{ab}$$

Green function $\quad \Box G(x^a - \bar{x}^a) = \delta^4(x^a - \bar{x}^a)$

$$G(x^a - \bar{x}^a) = \frac{-1}{4\pi |x^i - \bar{x}^i|}\,\delta\left[\,|x^i - \bar{x}^i| - (x^0 - \bar{x}^0)\right]\Theta(x^0 - \bar{x}^0)$$

$$\Theta(x^0 - \bar{x}^0) = \begin{cases} 1 & \text{when } x^0 - \bar{x}^0 > 0 \\ 0 & \text{otherwise} \end{cases}$$


$\bar{h}_{ab}(x^a) = -16\pi \int G(x^a - \bar{x}^a)\, T_{ab}(\bar{x}^a)\, d^4\bar{x}$
$= 4 \int \frac{T_{ab}(t - |x^i - \bar{x}^i|, \bar{x}^i)}{|x^i - \bar{x}^i|}\, d^3\bar{x}$
$t_r = t - |x^i - \bar{x}^i|$
t
x^i
$\bar{x}^i$
t_r

Transform to Fourier Space

$$\tilde{\bar{h}}_{ab} = \int dt\, e^{i\omega t}\, \bar{h}_{ab}$$

$$= 4 \int dt_r \int d^3\bar{x}\, \frac{T_{ab}(t_r, x')}{|x' - \bar{x}'|}\, e^{i\omega(t_r + |x' - \bar{x}'|)}$$

$$= 4 \int d^3\bar{x}\, \frac{\tilde{T}_{ab}}{|x' - \bar{x}'|}\, e^{i\omega |x' - \bar{x}'|}$$

$$\tilde{h}^{0a} = \frac{1}{\omega}\, \partial_i\, \tilde{h}$$

Assume

$\bar{h}^{\mu\nu}$ (Lorenz)

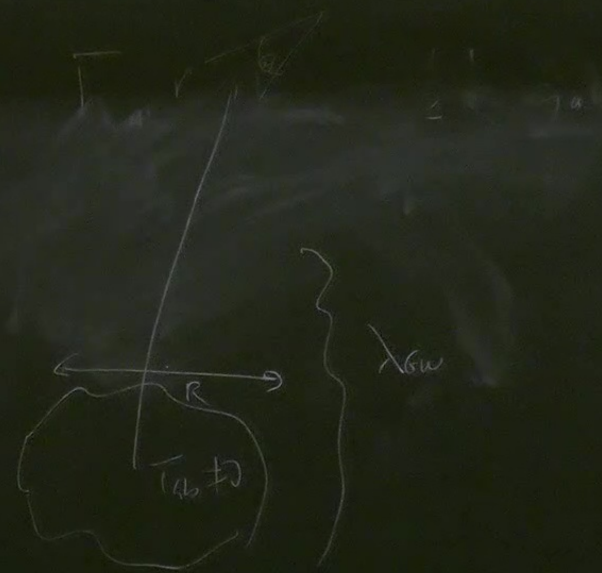
$r \gg \frac{1}{\omega}$ (wave zone)

$R_{source} \ll \lambda_{GW} = \frac{1}{\omega}$

$|x^i - x'^i|$

$\simeq \frac{e^{i\omega r}}{r}$

$|\vec{x} - \vec{x}'|$



Transform to Fourier Space

$$\tilde{\bar{h}}_{ab} = \int dt\, e^{i\omega t}\, \bar{h}_{ab}$$

$$= 4\int dt_r \int d^3\bar{x}\, \frac{T_{ab}(t_r, x')}{|x'-\bar{x}'|}\, e^{i\omega(t_r + |x'-\bar{x}'|)}$$

$$= 4\int d^3\bar{x}\, \frac{\tilde{T}_{ab}}{|x'-\bar{x}'|}\, e^{i\omega|x'-\bar{x}'|} \approx \frac{4e^{i\omega r}}{r} \int d^3\bar{x}\, \tilde{T}_{ab}$$

$\tilde{h}^{0a} = \frac{1}{\omega}\partial_i \tilde{h}$

Assume

$\frac{e^{i\omega|x'-}}{|x'-\bar{x}}$

Constant

$$\int d^3\bar{x}\, \tilde{T}_{ij} = \int \Big[\underbrace{\partial_k (\bar{x}^i \tilde{T}^{kj})}_{0} - \underbrace{\partial_k (\tilde{T}^{kj})}_{= i\omega \tilde{T}^{0j}} \bar{x}^i \Big] d^3\bar{x}$$

$$\nabla_a T^{ab} = 0$$

$$i\omega \tilde{T}^{0b} + \partial_i \tilde{T}^{ib} = 0$$

$$= \frac{i\omega}{2} \int \left(\tilde{T}^{0j} x^i + \tilde{T}^{i0} \bar{x}^j \right) d^3\bar{x}$$

$$= \frac{i\omega}{2} \int \Big[\partial_\ell (\bar{x}^i \bar{x}^j \tilde{T}^{0\ell}) - \bar{x}^i \bar{x}^j (\partial_\ell \tilde{T}^{0\ell}) \Big] d^3\bar{x}$$

$$= \frac{-\omega^2}{2} \underbrace{\int \bar{x}^i \bar{x}^j \tilde{T}^{00} d^3\bar{x}}_{\tilde{I}^{ij}}$$

0

$$\tilde{\bar{h}}_{ij} = -2\omega^2 \frac{e^{i\omega r}}{r} \tilde{I}_{ij}$$

$$\bar{h}_{ij} = \frac{2}{r} \ddot{I}_{ij}(t-r)$$