Title: Lecture - Strong Gravity, PHYS 777

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Subject: Strong Gravity

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Recap of CTT constraint Eqns

$$\gamma_{ij} = \psi^{4}\,\tilde{\gamma}_{ij}$$

$$K_{ij} = \frac{1}{3} K \gamma_{ij} + \psi^{-10}\left(\hat{A}^{ij}_{TT} + \mathcal{L}X^{ij}\right)$$

$$E = \psi^{-8}\tilde{E}$$

$$p^{i} = \psi^{-10}\tilde{p}^{i}$$

Hamiltonian

Momentum:

Free data:

Hamiltonian $\quad \tilde{D}^i \tilde{D}_i \psi - \frac{1}{8} \tilde{R} \psi + \frac{1}{8} \left[(\tilde{\mathcal{L}} X)_{ij} + \hat{A}^{TT}_{ij} \right]^2 \psi^{-7} - \frac{1}{12} K^2$

Momentum: $\quad (\tilde{\Delta}_L X)^{ij} - \frac{2}{3} \psi^6 \tilde{D}^i K = 8\pi \tilde{p}^i$

Free data: $\tilde{\gamma}_{ij}, \; K, \; \hat{A}^{ij}_{TT}, \; (\tilde{E}, \tilde{p}^i)$

$5, \; 1, \; 2$

Constrained $\quad \psi, \; X^i$

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$$\tilde{R}\psi + \frac{1}{8}\left[(\tilde{\mathcal{L}}X)_{;} + \hat{A}_{;}^{TT}\right]^2\psi^{-7} - \frac{1}{12}K^2\psi^5 + 2\pi\tilde{E}\psi^{-3} = 0$$

$$\frac{2}{3}\psi^6\tilde{D}^i K = 8\pi\tilde{p}^i$$

$(\tilde{E}, \tilde{p}^i)$ Constrained ψ, X^i

ψ

Simple Example

$\partial_i \partial$

Vacuum: $\tilde{E} = \tilde{p}^i = 0$

Maximal & "Waveless": $K = \hat{A}^{ij}_{TT} = 0$

Conformally flat: $\tilde{\gamma}_{ij} = f_{ij}$

BCs: $\psi \to 1$, $X^i = 0$ as $r \to \infty$

$$\partial_i \partial^i \psi + \frac{1}{8}\left[(\mathcal{L}X)^{ij} (\mathcal{L}X)_{ij} \right] \psi^{-7} = 0$$

$$\Delta_L X^i = 0$$

$= \hat{A}^{ij}_{TT} = 0$

$\Sigma_0 = \mathbb{R}^3$, Soln: $X^i = 0$, $\psi = 1$

$= f_{ij}$

as $r \to \infty$

Example 2: $\Sigma_0 = \mathbb{R}^3 / B_R$

BC: $r = R$ $\quad D_i s^i = 0 = \tilde{D}_i (\psi^4 \tilde{s}^i) = \frac{1}{r^2} \partial_r (\psi^4 r^2)$

$' = 0$

$$\partial_r \psi + \frac{\psi}{2r} = 0 \quad \text{at } r = R$$

$$\psi \to 1 \qquad r \to \infty$$

$$\partial_i \partial^i \psi = 0$$

$= 1$

Unique Soln: $\psi = 1 + \frac{R}{r}$

$$\gamma_{ij} = \left(1 + \frac{M}{2r}\right)^4 \left(dr^2 + r^2 d\Omega^2\right) \qquad M = 2R$$

Schwarzschild in isotropic coordinates

$\partial_r(\psi^4 r^2)$

	Free data	Constrained Data
Conformal Thin sandwich (CTS)	$\tilde{\gamma}_{ij}, K, \tilde{\alpha}, \partial_t \tilde{\gamma}_{ij}$	Ψ, β^i
Extended CTS	$\tilde{\gamma}_{ij}, K, \partial_t \tilde{\gamma}_{ij}, \partial_t K$	Ψ, α, β^i

Strategies for choosing free data:

$$\tilde{\gamma}_{ij} = \gamma^{(1)}_{ij} + \gamma^{(2)}_{ij} - f_{ij}$$

Quasi-Circular approximate Killing Vector

$$\mathcal{L}_\xi g_{ab} \stackrel{?}{\approx} 0$$

$$\xi^a = t^a + \Omega_{orb} \hat{\phi}$$

Bad time slicings, an example

Gaussian normal coordinates

$$\alpha = 1 \;, \quad \beta^i = 0 \quad \Longleftrightarrow \quad g_{tt} = -1 \;, \quad g_{ti} = 0$$

$$n^a = (1, 0)$$

$$n^a \nabla_a n^b = -\Gamma^b_{tt} = 0$$

$$K = g^{ab} K_{ab} = -g^{ab}\left(g^c_a + n_a n^c\right)\nabla_c n_b$$

$$= -\nabla_a n^a = -\Theta$$

↖ expansion of congruence of geo

Raychaudhari's Eqn

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 + \sigma_{ab}\sigma^{ab} - \omega_{ab}\omega^{ab} = -R_{ab}u^a u^b$$

σ_{ab} is shear

$\omega_{ab} = \frac{1}{2}(\nabla_a u_b - \nabla_b u_a)$ rotation tensor

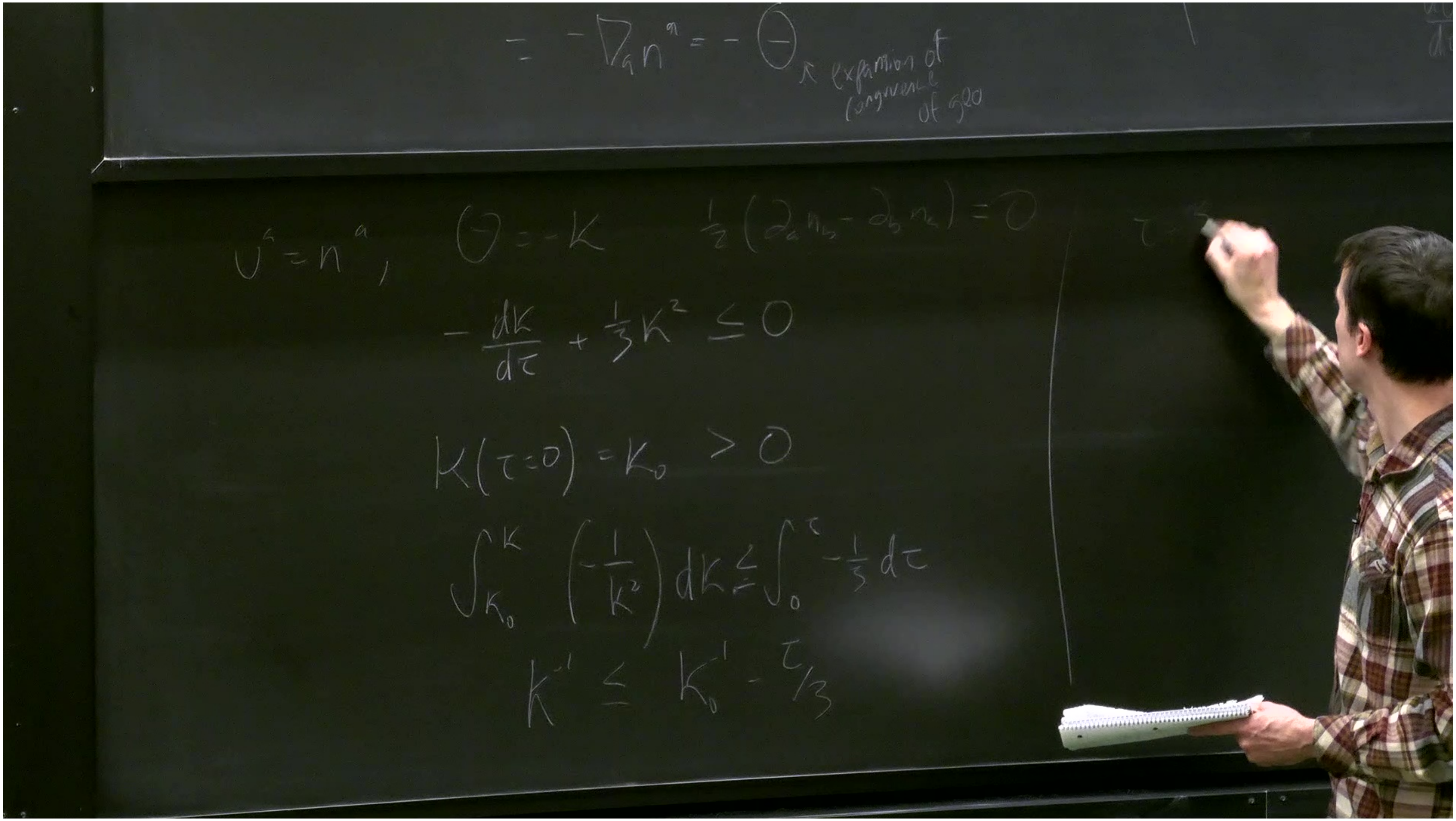
$\omega_{ab}\omega^{ab} \geq 0$ (spatial tensor)

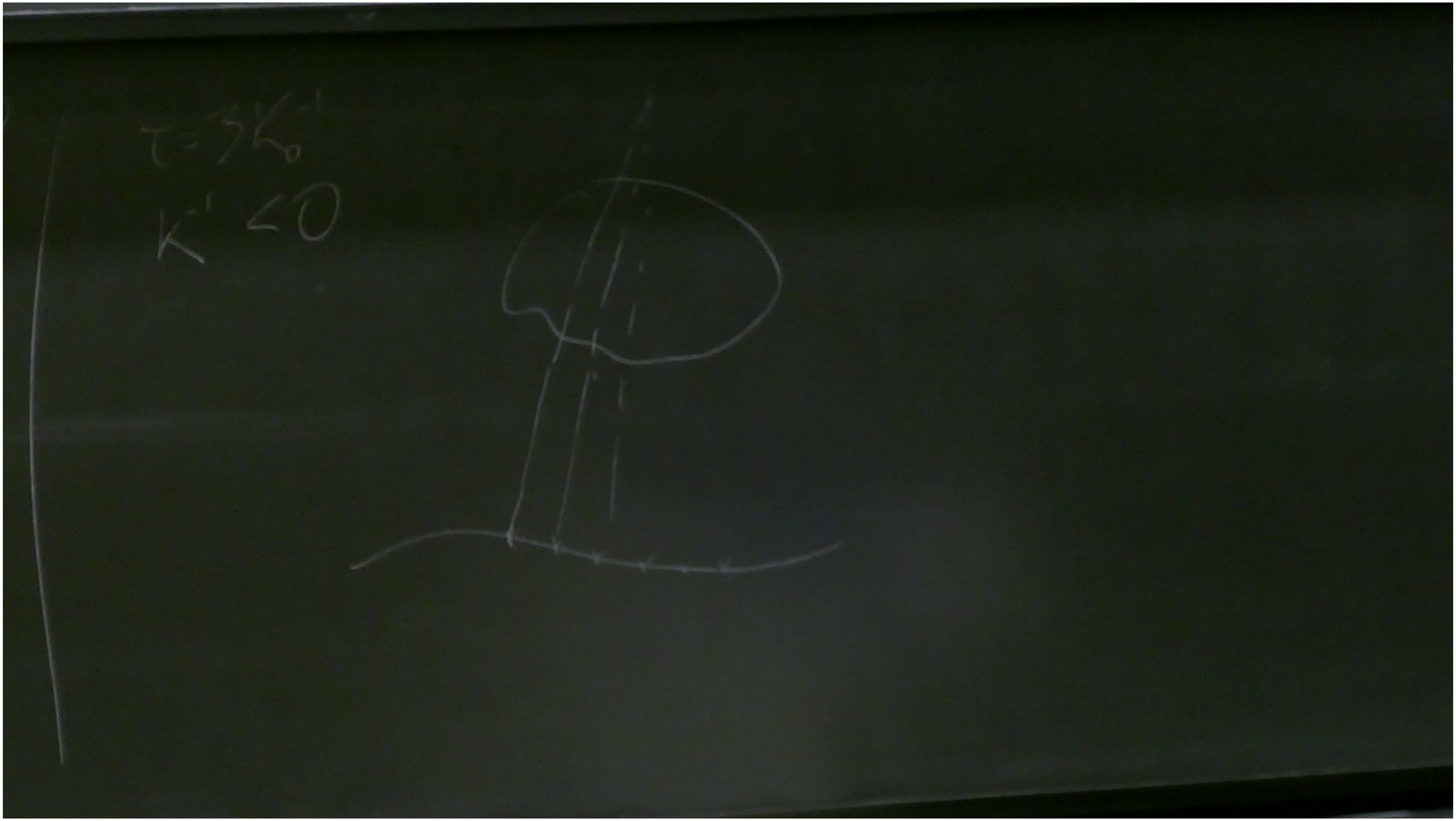
$$R_{ab}u^a u^b = 8\pi\left(T_{ab} - \frac{1}{2}T g_{ab}\right)u^a u^b \geq 0$$

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 - \omega_{ab}\omega^{ab} \leq 0$$

SEC

$$T_{ab}t^a t^b \geq \frac{1}{2}T t^a t_a$$


$= -\nabla_a n^a = -\Theta$
expansion of congruence of geo
$u^a = n^a$, $\Theta = -K$ $\frac{1}{2}(\partial_a n_b - \partial_b n_a) = 0$
$-\frac{dK}{d\tau} + \frac{1}{3}K^2 \le 0$
$K(\tau=0) = K_0 > 0$
$\int_{K_0}^{K} \left(-\frac{1}{K^2}\right) dK \le \int_0^{\tau} -\frac{1}{3}\, d\tau$
$K^{-1} \le K_0^{-1} - \tau/3$


τ = ζ K_0^{-1}
K' < 0

Generalized harmonic formulation of the Einstein Eqns

Fix gauge DoF:

$$\Box x^a = H^a$$ ← source functions (not vector)

$\nabla_a \nabla^a$

$$R_{ab} = 8\pi \left(T_{ab} - \frac{1}{2} T g_{ab} \right)$$

$$= -\frac{1}{2} g^{cd} \partial_c \partial_d g_{ab} - \partial_c g_{d(a} \partial_{b)} g^{cd} + \nabla_{(a} \Gamma_{b)} - \Gamma_c \Gamma^c{}_{ab} - \Gamma^c{}_{da} \Gamma^d{}_{cb}$$

Recap of CTT constraint Eqns

$$\Gamma_a = g^{bc}\Gamma_{abc} = -g_{ab}\nabla_c\nabla^c x^b$$

Evolve $\{g_{ab}, \partial_t g_{ab}, H_a, \partial_t H_a\}$

Rewrite use $H_a = -\Gamma_a$

$$-8\pi\left(2T_{ab} - g_{ab}T\right) = g^{cd}\partial_c\partial_d g_{ab} + F\left(g_{ab}, \partial_c g_{ab}, H_a, \partial_a H_b\right)$$

$$L(H_a) = 0$$

some evolution operator

$H_a = 0$ harmonic gauge

$H_a = F(g_{ab})$

$$(\partial_t - \mathcal{L}_\beta)\alpha = -\alpha^2 K$$

$$\Rightarrow \partial_t\beta^i = \beta^j\partial_j\beta^i + \alpha^2\,{}^3\Gamma^i_{jk}\gamma^{jk} - \alpha D^i\alpha$$

$$C^a = H^a - \Box x^a$$

$$R_{ab} - 4\pi\left(2T_{ab} - g_{ab}T\right) = \nabla_{(a}C_{b)}$$

$n^a n^b \nabla_{(a}C_{b)} = 0 \iff$ Hamiltonian constraint

$n^a \gamma^b_i \iff$ Momentum constraint

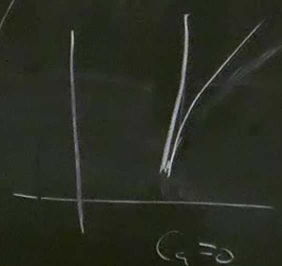
$\partial_t C_a = C_a = 0$ at $t=0$

operator $\quad (g_{ab})$

$$\left(R_{ab} - \frac{1}{2} R g_{ab}\right) - 8\pi T_{ab} = \nabla_{(a} C_{b)} - \frac{1}{2} g_{ab} g^{cd} \nabla_{(c} C_{d)}$$

Evolution of constraints

$$\nabla_a \nabla^a C_b = - R^a{}_b C_a$$



$$\kappa \left(n_{(a} C_{b)} - \frac{1}{2} g_{ab} n^c C_d \right)$$

$$\Rightarrow \quad = \dots + \kappa \nabla_b [n^{(b} C^{a)}]$$

$$= -\frac{1}{2} g^{cd} \partial_c \partial_d g_{ab} - \partial_c g_{d(a} \partial_{b)} g^{cd} + \nabla_{(a} \Gamma_{b)} - \Gamma_c \Gamma^c{}_{ab} - \Gamma_{da} \Gamma \ldots$$

$$C^a = H^a - \Box x^a$$

$$R_{ab} - 4\pi (2T_{ab} - g_{ab} T) = \nabla_{(a} C_{b)}$$

$n^a n^b \nabla_{(a} C_{b)} = 0 \iff$ Hamiltonian constraint

$n^a \gamma^b_i \qquad \iff$ Momentum constraint

$\partial_t C_a = C_a = 0$ at $t=0$

$g_{ab},\ \partial_t g_{ij},\ \partial_t g_{ta}$

$$(R_{ab} - \tfrac{1}{2} R g_{ab}) - 8 \ldots$$

$$\nabla_a \nabla^a C_b \ldots$$