Title: Lecture - Strong Gravity, PHYS 777

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3+1 Decomposition of EFEs

$$(\partial_t - \mathcal{L}_\beta)\gamma_{ij} = -2\alpha K_{ij}$$

$$(\partial_t - \mathcal{L}_\beta)K_{ij} = -D_i D_j \alpha + \alpha\left\{ {}^3R_{ij} + K K_{ij} - 2K_{ik}K^k{}_j + 4\pi\left[(S-E)\gamma_{ij} - 2S_{ij}\right]\right\}$$

$${}^3R + K^2 - K_{ij}K^{ij} = 16\pi E$$

$$D_j K^j{}_i - D_i K = 8\pi p_i$$

Types of Partial Differential Eqns

Wave Eqn	$(-\partial_t^2 + \partial_x^2) u = 0$	Hyperbolic PDE
Laplace Eqn	$(\partial_x^2 + \partial_y^2) u = 0$	Elliptic PDE
Heat Eqn	$(\partial_t - \partial_x^2) u = 0$	Parabolic PDE

General linear, second order PDE

$$a\,\partial_x^2 u + b\,\partial_x\partial_y u + c\,\partial_y^2 u + d\,\partial_x u + e\,\partial_y u + f u = 0$$

$b^2 - 4ac > 0$ hyperbolic can reduce to $(-\partial_x^2 + \partial_y^2)u + \text{l.o.t.} = 0$

$= 0$ parabolic $\partial_x^2 u = \text{l.o.t.}$

< 0 elliptic $(\partial_x^2 + \partial_y^2)u = \text{l.o.t.}$

$$\partial_t \vec{u} + \underline{A}\,\partial_x \vec{u} + B\vec{u} = 0 \quad (*)$$

$\underline{A}$ has all real eigenvalues $\Rightarrow$ (*) weakly hyperbolic

Also has complete set of eigenvectors $\Rightarrow$ (*) strongly hyperbolic

$\underline{A}$ has

Well-posedness: $\| \vec{u}(t) \| \leq K e^{\alpha t} \| \vec{u}(t=0) \|$

K, α independent of $\vec{u}(t=0)$

$\Longleftrightarrow$ strongly hyperbolic

Example. Wave Eqn

$$\vec{u} = \begin{pmatrix} u \\ \partial_t u \\ \partial_x u \end{pmatrix}$$

$$\partial_t \vec{u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \partial_x \vec{u} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{u}$$

$\underline{A}$ has all real eigenvalues $\Rightarrow$ (*) weakly hyperbolic

Also has complete set of eigenvectors $\Rightarrow$ (*) strongly hyperbolic

Eigenvalues $0, \pm 1$

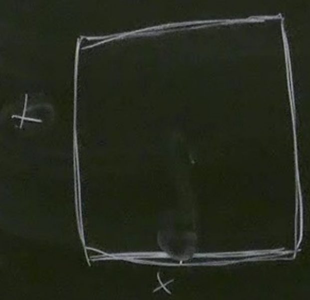
Eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$$+ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{U}$$

$\partial_t u \mp \partial_x u = \text{constant}$

$$\underline{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad : \quad \text{Eigenvalues} : \; 0, \pm i$$

ω $\qquad \sim e^{\omega t + i \omega x}$



$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \partial_x \begin{pmatrix} U \\ V \end{pmatrix}$$

Eigenvalue : 1 (multiplicity 2)

one eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

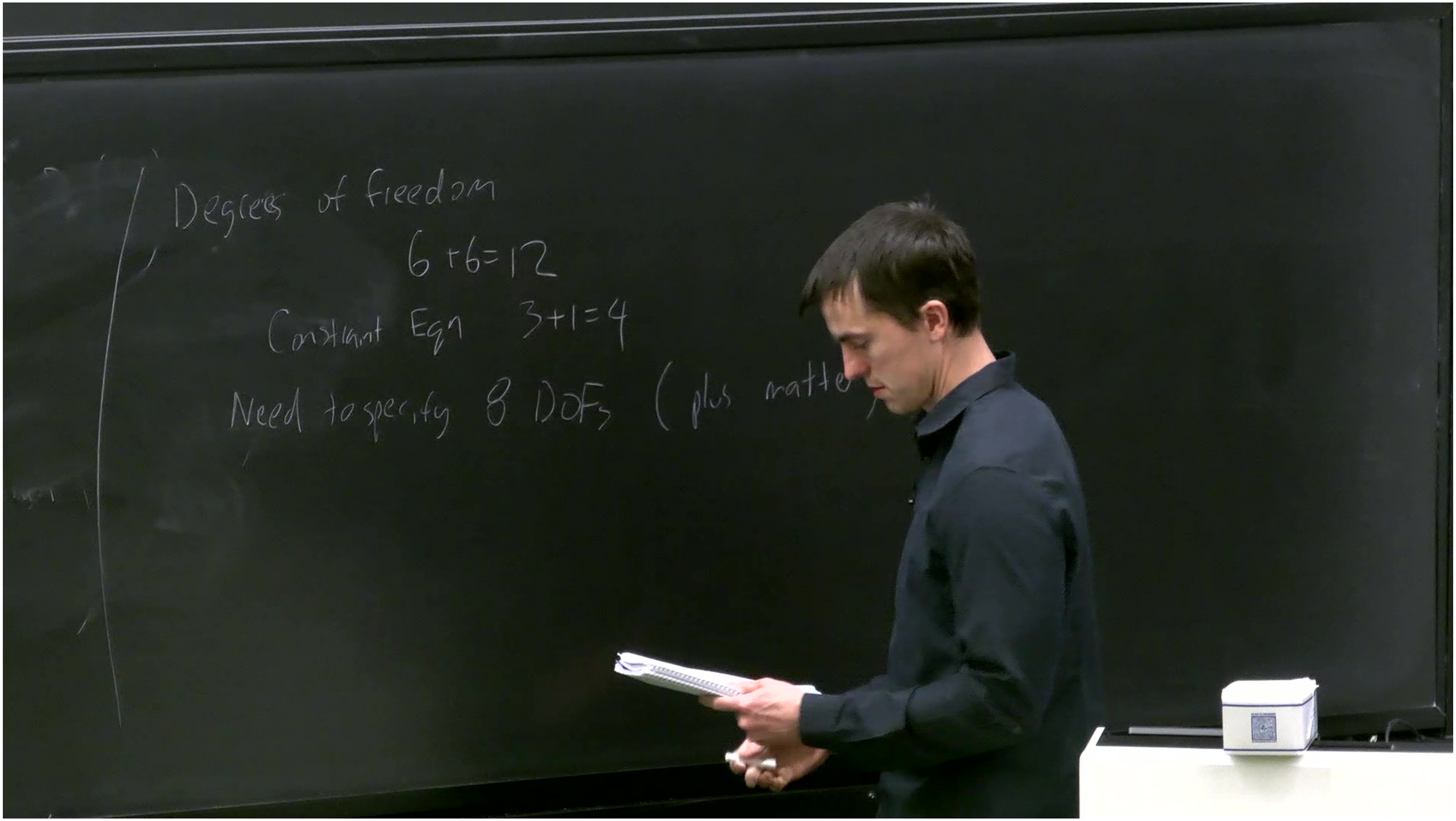
$$\begin{pmatrix} U \\ V \end{pmatrix} \sim \begin{pmatrix} \frac{1}{2} i\omega t \\ 1 \end{pmatrix} e^{i\omega(t+x)}$$

Initial Data in GR (Chap. 8 in arXiv: gr-qc/0703035)

Two goals: 1) Choose γ_{ij}, K_{ij} satisfy Ham. + Mom. constraints

2) Choose " " to represent physical system of interest

Divide DOFs into free + constrained data


Degrees of freedom
6+6=12
Constraint Eqn 3+1=4
Need to specify 8 DOFs (plus matte

(see also Ch. 6, Appendix G Carroll)

Conformal transformation

$$C^{c}_{ab} = \Gamma^{c}_{ab} - \tilde{\Gamma}^{c}_{ab} = \omega^{-1}\left[\delta^{c}_{a}\tilde{\nabla}_{b}\omega + \delta^{c}_{b}\tilde{\nabla}\right.$$

$$g_{ab} = \omega^{2}\tilde{g}_{ab}$$

n-dim

Scalar: $\nabla_a \phi = \tilde{\nabla}_b \phi = \partial_a \phi$

vector: $\nabla_a V_b = \tilde{\nabla}_a V_b - \left(\delta^{c}_{a}\delta^{d}_{b} + \delta^{d}_{a}\delta^{c}_{b} - \tilde{g}_{ab}\tilde{g}^{cd}\right) \times \omega^{-1}($

Ricci curvature: $R = \omega^{-2}\tilde{R} - 2(n-1)\tilde{g}_{ab}\omega^{-3}(\nabla_a\nabla_b\omega) - (n-1)(n-$

Conformal Transverse Traceless Decomposition

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$$

$$^3R = {}^3\tilde{R}\,\psi^{-4} - 8\psi^{-5}\tilde{D}_i \tilde{D}^i \psi$$

$$K^{ij} = A^{ij} + \frac{1}{3} K \gamma^{ij}$$

trace-free

$$A^{ij} = \psi^{-10} \hat{A}^{ij}$$

Ham. Co

osition

Ham. constraint $\tilde{D}_i \tilde{D}^i \psi - \frac{1}{8}\tilde{R}\psi + \frac{1}{8}\hat{A}^{ij}\hat{A}_{ij}\psi^{-7} - \frac{1}{12}K^2\psi^5 + 2\pi E\psi^5 = 0$

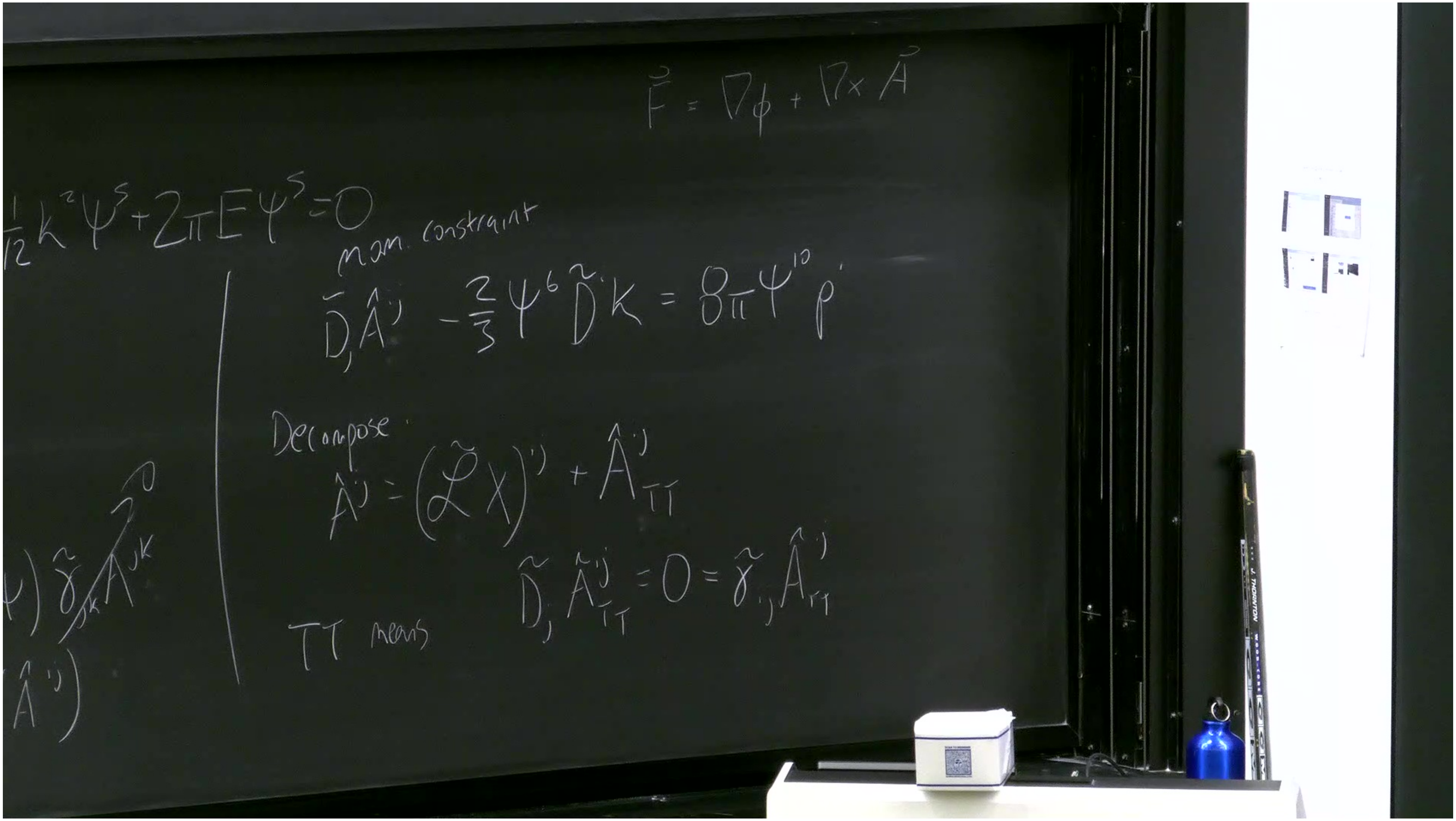
mom. $\tilde{D}_j \hat{A}^{ij}$

$$D_j K^{ij} = D_j A^{ij} + \frac{1}{3} D^i K$$

$$D_j A^{ij} = \tilde{D}_j A^{ij} + C^i_{jk} A^{kj} + C^j_{jk} A^{ik}$$

$$= \tilde{D}_j A^{ij} + 10 A^{ij} \tilde{D}_j \ln\psi - 2(\tilde{D}^i \ln\psi)\underbrace{\cancel{\tilde{\gamma}_{jk} A^{jk}}}_{0}$$

$$= \psi^{-10} \tilde{D}_j (\psi^{10} A^{ij}) = \psi^{-10} \tilde{D}_j (\hat{A}^{ij})$$

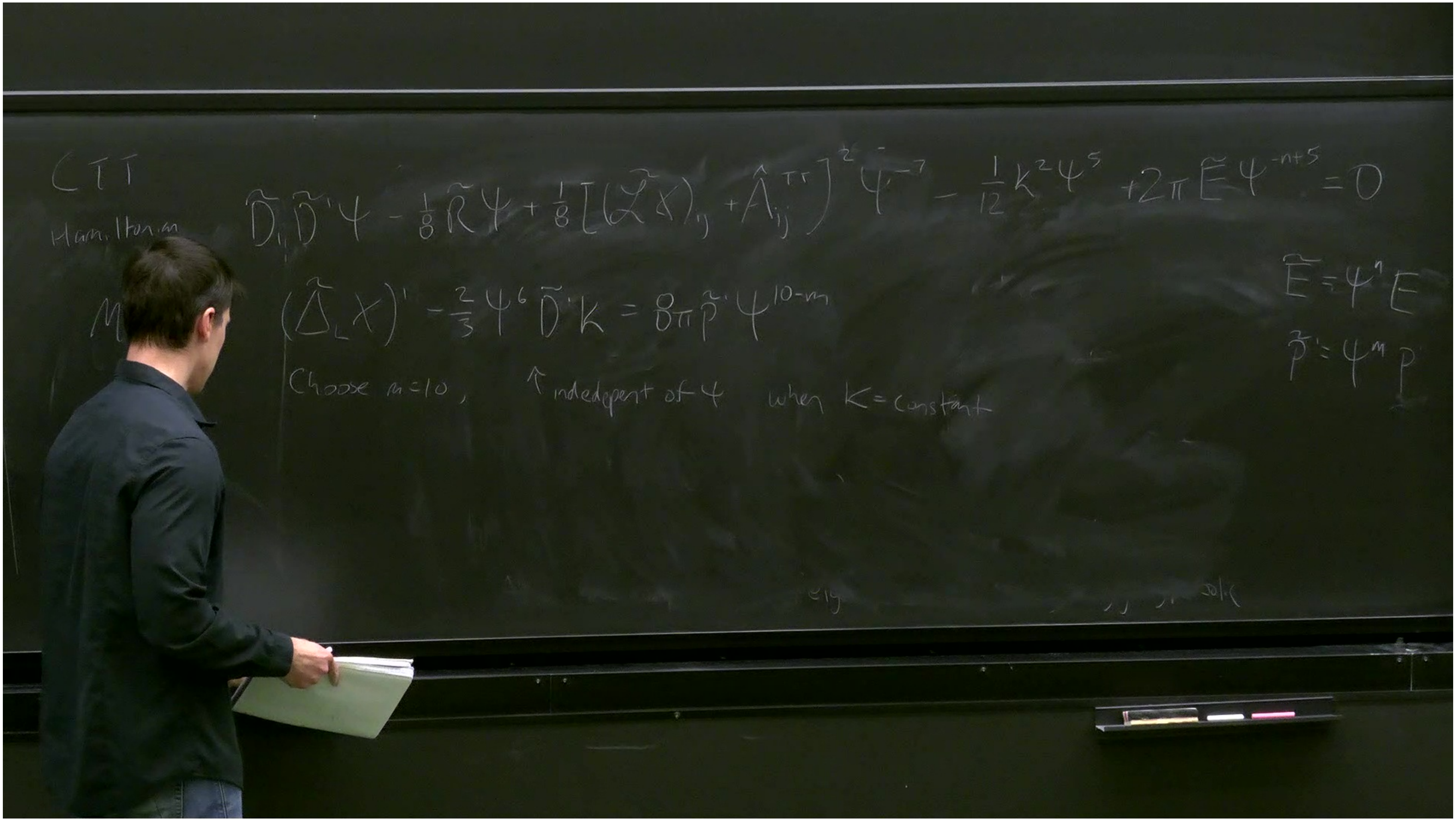


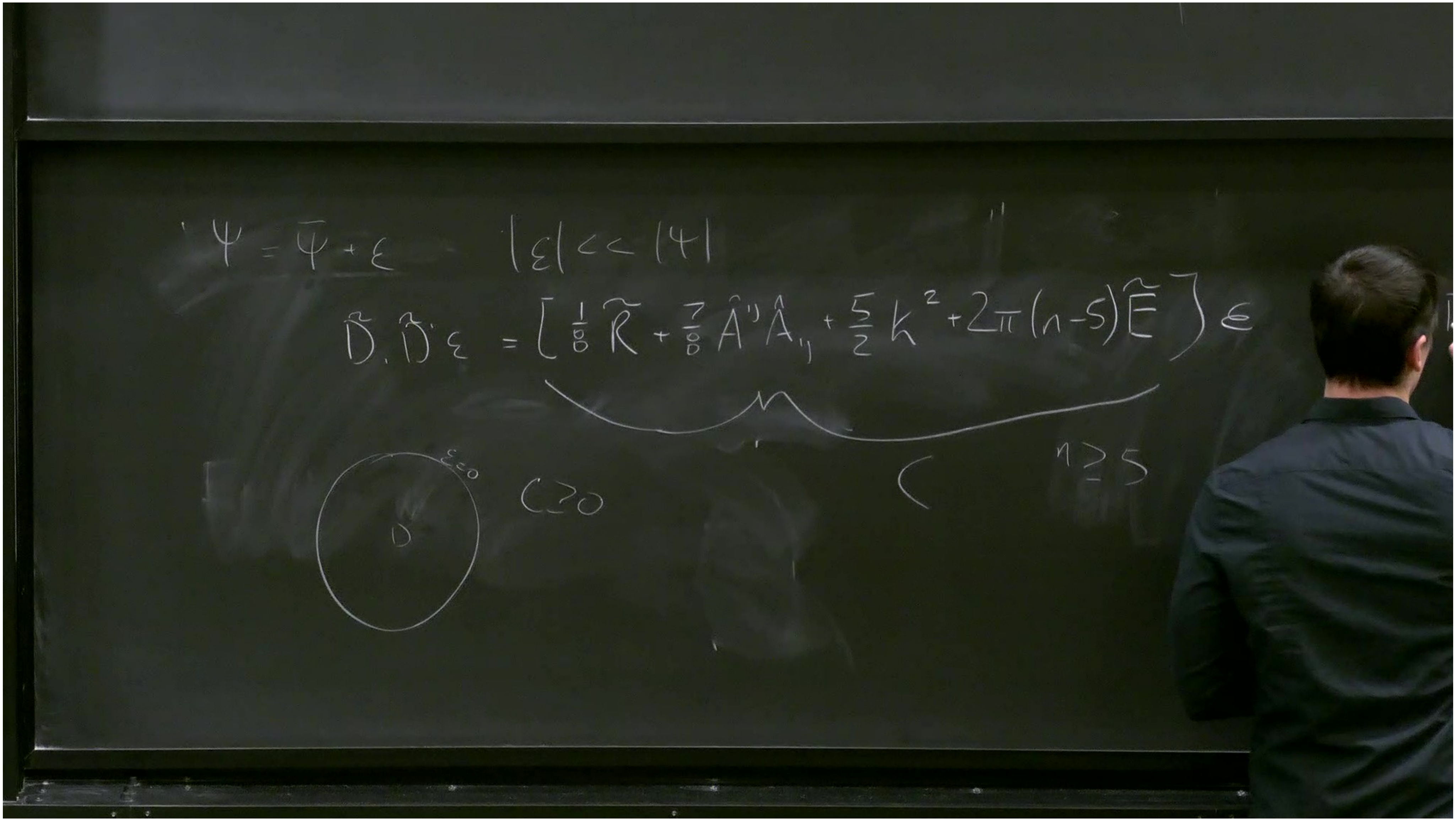
$\vec{F} = \nabla\phi + \nabla\times\vec{A}$
$\frac{1}{2}k^2\psi^5 + 2\pi E\psi^5 = 0$
mom. constraint
$\tilde{D}_j\hat{A}^{ij} - \frac{2}{3}\psi^6\tilde{D}^i K = 8\pi\psi^{10}p^i$
Decompose:
$\hat{A}^{ij} = (\tilde{\mathbb{L}}X)^{ij} + \hat{A}^{ij}_{TT}$
TT means
$\tilde{D}_j\hat{A}^{ij}_{TT} = 0 = \tilde{\gamma}_{ij}\hat{A}^{ij}_{TT}$

Conformal Killing Operator

$$(\tilde{\mathcal{L}}X)^{ij} = \tilde{D}^i X^j + \tilde{D}^j X^i - \frac{2}{3}\left(\tilde{D}_k X^k\right)\tilde{\gamma}^{ij}$$

$$(\tilde{\Delta}_L X)^i = \tilde{D}_j(\tilde{\mathcal{L}}X)^{ij} = \tilde{D}_j \hat{A}^{ij}$$


CTT
Hamiltonian
$\tilde{D}_i \tilde{D}^i \psi - \frac{1}{8} \tilde{R} \psi + \frac{1}{8} \left[(\tilde{\mathbb{L}} X)_{ij} + \hat{A}_{ij}^{TT} \right]^2 \psi^{-7} - \frac{1}{12} K^2 \psi^5 + 2\pi \tilde{E} \psi^{-n+5} = 0$
M
$(\hat{\Delta}_L X)^i - \frac{2}{3} \psi^6 \tilde{D}^i K = 8\pi \tilde{p}^i \psi^{10-m}$
Choose $m=10$, ↑ indedepent of ψ when K = constant
$\tilde{E} = \psi^n E$
$\tilde{p}^i = \psi^m p^i$


$\Psi = \bar{\Psi} + \varepsilon \qquad |\varepsilon| << |\Psi|$
$\hat{D}\cdot\hat{D}\,\varepsilon = \left[\frac{1}{8}\tilde{R} + \frac{7}{8}\hat{A}^{ij}\hat{A}_{ij} + \frac{5}{2}k^2 + 2\pi(n-5)\tilde{E}\right]\epsilon$
$\varepsilon=0$
D
$C\geq 0$
$n \geq 5$

$n = \beta$

$n-5)\tilde{E}\,]\;\epsilon$

Dominant EC: $-T^a_b\, n^a \le 0$

$$-E^2 + p_i p^i \le 0$$

$n \ge 5$

$$\tilde{E}^2 \ge \tilde{p}^i \tilde{p}_i$$

$$\iff E^2 \ge p^i p_i$$