Title: Lecture - Strong Gravity, PHYS 777

Speakers: William East

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Subject: Strong Gravity

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Extrinsic Curvature

$$K_{ab} = -\nabla_a n_b - \underbrace{(n^c \nabla_c n_a)}_{= a_a} n_b$$

Foliation

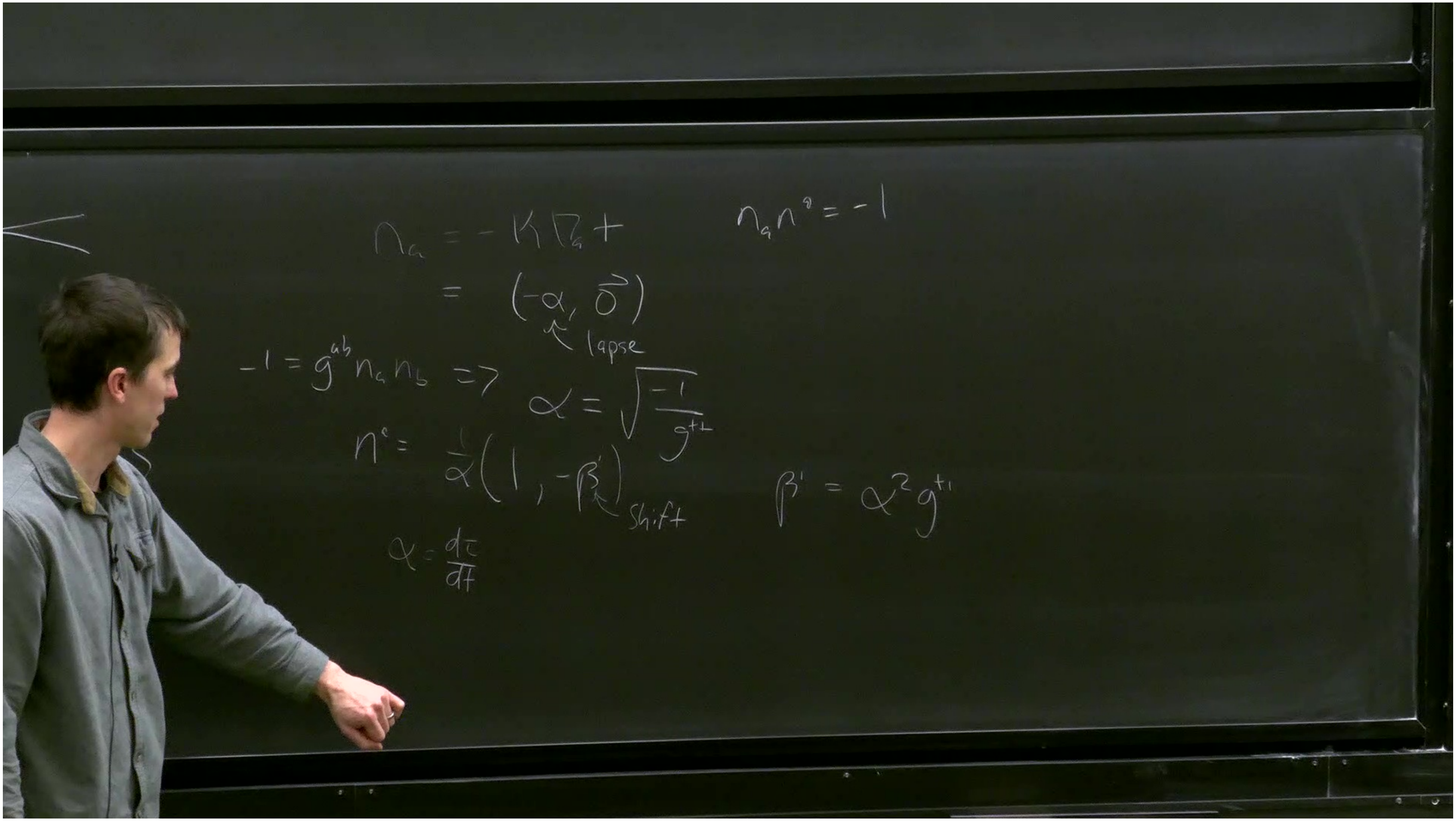
$$\mathcal{M} = \{\Sigma_t\}_{t \in \mathbb{R}}$$

$$\Sigma_t = \{x^a \in \mathcal{M} : \hat{t}(a) = t\}$$

slice or leaf

$\hat{t}$ smooth, $\nabla_a \hat{t} \neq 0$

$\mathcal{M}$ is globally hyperbolic



$n_a = -K \nabla_a t$
$n_a n^a = -1$
$= (-\alpha, \vec{0})$
lapse
$-1 = g^{ab} n_a n_b \Rightarrow \alpha = \sqrt{\frac{-1}{g^{tt}}}$
$n^a = \frac{1}{\alpha}(1, -\beta^i)$
shift
$\beta^i = \alpha^2 g^{ti}$
$\alpha = \frac{d\tau}{dt}$

$$n_a = -K \nabla_a t \qquad n_a n^a = -1$$

$$= (-\alpha, \vec{0})$$

lapse

$$-1 = g^{ab} n_a n_b \Rightarrow \alpha = \sqrt{\frac{-1}{g^{tt}}}$$

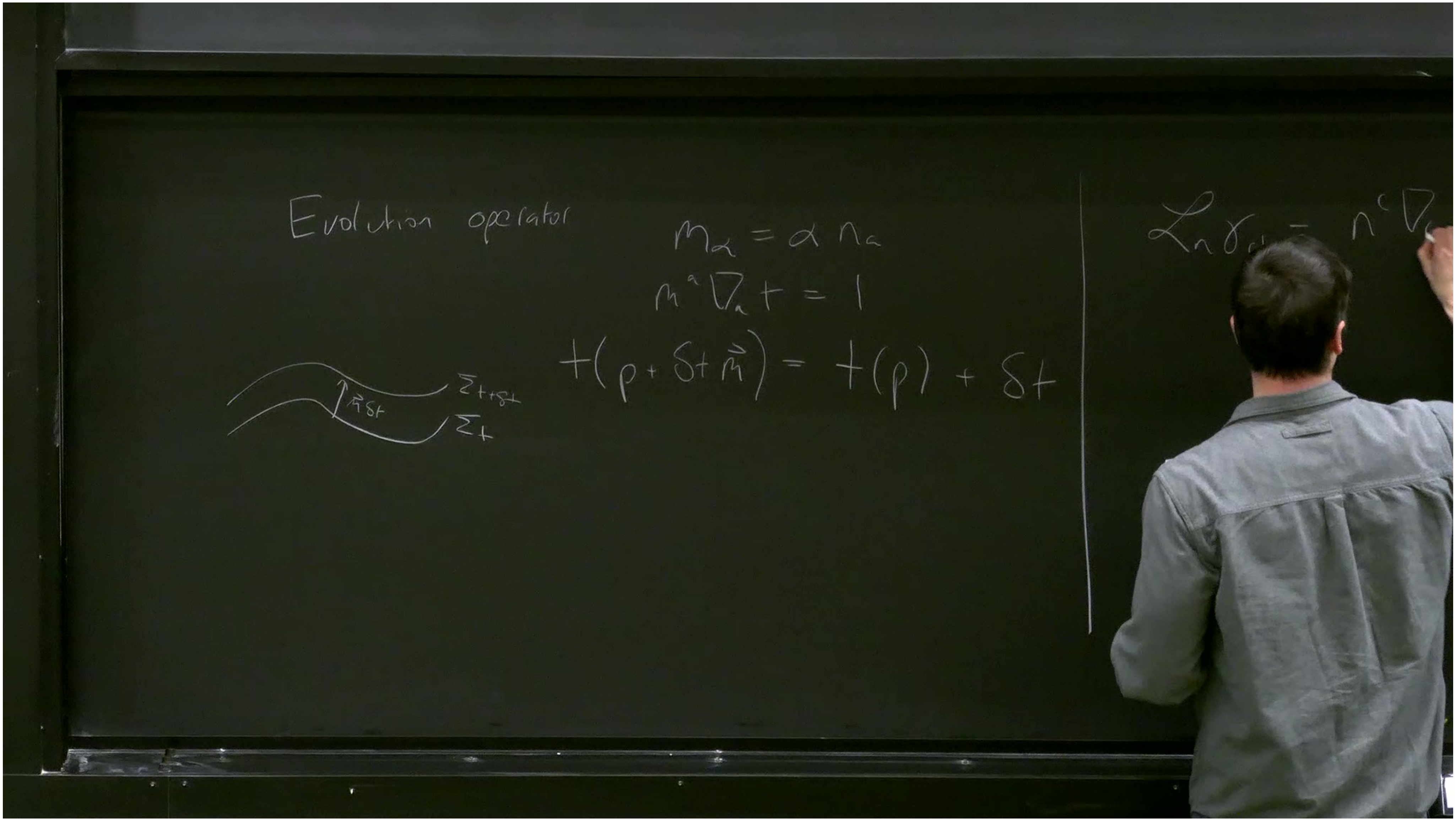
$$n^a = \frac{1}{\alpha}(1, -\beta^i)$$

shift

$$\beta^i = \alpha^2 g^{ti}$$

$$\alpha = \frac{d\tau}{dt} \quad , \quad -\beta^i = \frac{dx^i}{dt}$$

$$a) = t \}$$


Evolution operator
$m_\alpha = \alpha n_a$
$m^a \nabla_a t = 1$
$t(p + \delta t \vec{m}) = t(p) + \delta t$
$\vec{n}\delta t$
$\Sigma_{t+\delta t}$
Σ_t

M is globally hyperbolic

$) + \delta t$

$$\mathcal{L}_n \gamma_{ab} = n^c \nabla_c \gamma_{ab} + \gamma_{cb} \nabla_c n^c + \gamma_{ac} \nabla_b n^c$$

$$\nabla_c \gamma_{ab} = \nabla_c (n_a n_b)$$

$$= -2K_{ab}$$

$$\mathcal{L}_m \gamma_{ab} = -2\alpha K_{ab}$$

$$ {}^{(3)}R_{abc}{}^{d} u_d = (D_a D_b - D_b D_a) u_c $$

$$ R^c{}_{dab} n^d = (\nabla_a \nabla_b $$

$$ D_a D_b u_c = \gamma_a{}^d \gamma^e_b \gamma^f_c \nabla_d (\gamma^g_e \gamma^g_f \nabla_g u_h) $$

$$ \gamma_{ab} = g_{ab} + n_a n_b $$

$$ {}^{(3)}R^c{}_{dab} = \gamma^e_a \gamma^f_b \gamma^c_h \gamma^g_d R^h{}_{gef} - K^c_a K_{db} + K^c_b K_{ad} $$

$$ {}^{(3)}R_{db} + K K_{db} - K^a_b K_{ad} = \gamma^f_b \gamma^e_h \gamma^g_d R^h{}_{gef} \quad (\lozenge) $$

$$ {}^{(3)}R + K^2 - K_{ab} K^{ab} = R + 2 R_{ab} n^a n^b \quad (*) $$

$$R^c{}_{dab} n^d = (\nabla_a \nabla_b - \nabla_b n_a) n^c$$

$$\gamma^c{}_d n^e \gamma^f_a \gamma^g_b R^d{}_{efg} = D_b K^c_a - D_a K^c_b$$

$$n^e \gamma^g_b R_{eg} = D_b K - D_a K^a_b$$

$h)$

$+ K^c{}_b K_{ad}$

$(\diamond)$

$(*)$

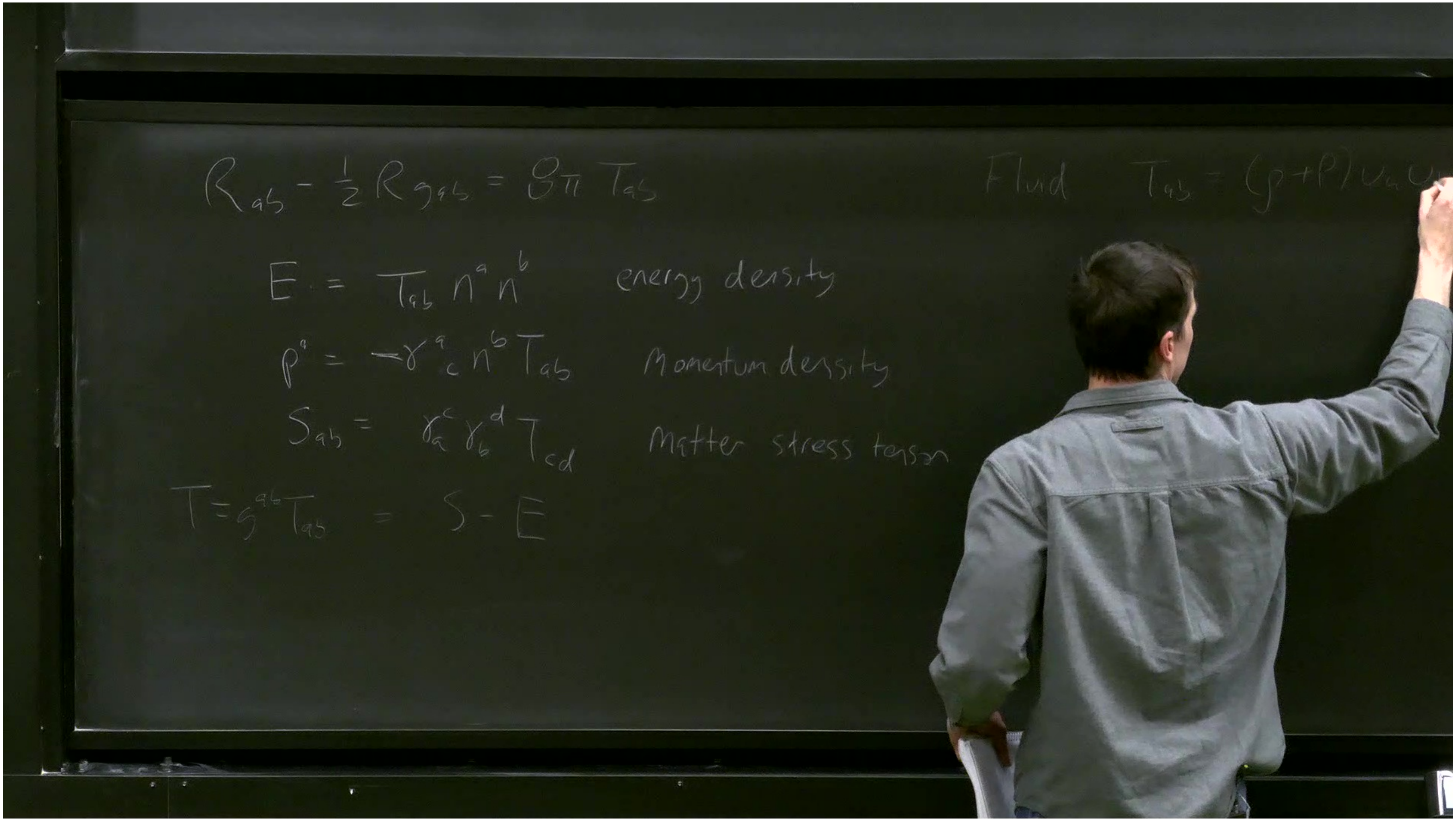
$\gamma^4 R$ ✓ $\gamma^3 n R$ ✓ $\gamma^2 n^2 R$ $\gamma^? R - \mathcal{D}$

$$\gamma_{ac}\, n^d \gamma^e_b\, R^{\ c}_{fed}\, n^f = \gamma_{ac}\, n^d \gamma^e_b \left(\nabla_e \nabla_d n^c - \nabla_d \nabla_e n^c \right)$$

$$\left[\nabla_b n_a = -K_{ab} - D_a \ln\alpha \ n_b \right]$$

$$= -K_{af} K^f_b + \frac{1}{\alpha} D_b D_a \alpha + \gamma^c_a \gamma^e_b n^f \nabla_f K_{ce}$$

$$\gamma_{ac}\, n^a \gamma^e_b\, R^{\ c}_{fed}\, n^f = \frac{1}{\alpha}\mathcal{L}_m K_{ab} + \frac{1}{\alpha} D_a D_b \alpha + K_{af} K^f_b \qquad (\Delta)$$


$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}$
Fluid $T_{ab} = (\rho + P) u_a u$
$E = T_{ab} n^a n^b$
energy density
$p^a = -\gamma^a{}_c n^b T_{ab}$
Momentum density
$S_{ab} = \gamma_a^c \gamma_b^d T_{cd}$
Matter stress tensor
$T = g^{ab} T_{ab} = S - E$

M is globally hyperbolic

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}$$

$E = T_{ab} n^a n^b$ energy density

$p^a = -\gamma^a{}_c n^b T^c{}_b$ Momentum density

$S_{ab} = \gamma_a^c \gamma_b^d T_{cd}$ Matter stress tensor

$T = g^{ab} T_{ab} = S - E$

Fluid $T_{ab} = (\rho + P) u_a u_b + P g_{ab}$

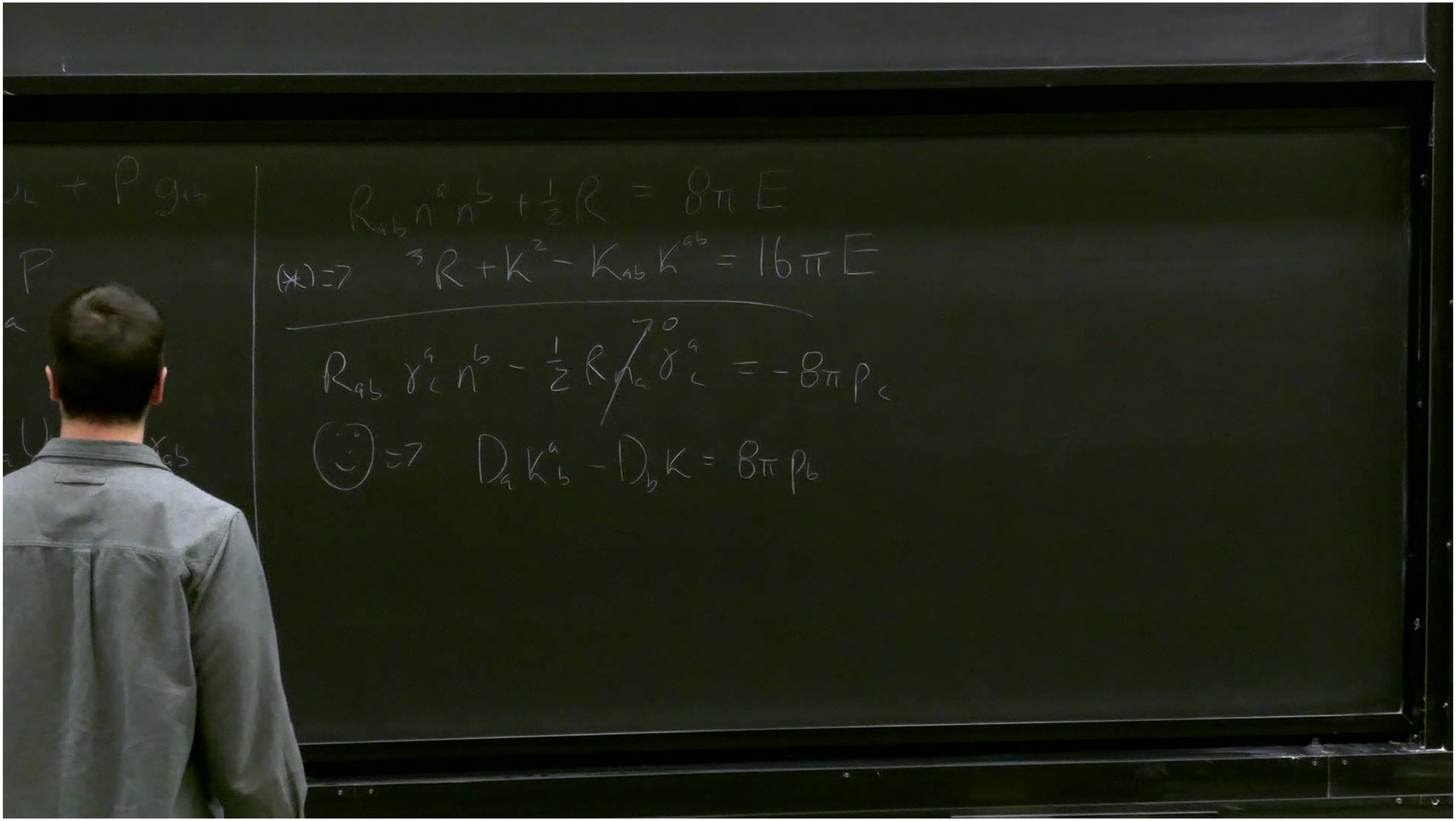
$E = (\rho + P) W^2 - P$

$p^a = (E + P) U^a$

$S_{ab} = (E + P) U_a U_b + P \gamma_{ab}$

$W = -n_a u^a$

$U^a = \frac{1}{W} \gamma^a_b u^b$ Eulerian velocity


$R_{ab} n^a n^b + \frac{1}{2} R = 8\pi E$
$(*) \Rightarrow {}^3R + K^2 - K_{ab} K^{ab} = 16\pi E$
$R_{ab} \gamma^a_c n^b - \frac{1}{2} R \cancel{n_a}^{\,0} \gamma^a_c = -8\pi p_c$
$\Rightarrow D_a K^a_b - D_b K = 8\pi p_b$

$$R_{ab} = 8\pi \left(T_{ab} - \tfrac{1}{2} T g_{ab} \right)$$

$$\gamma^a{}_c \gamma^b{}_d R_{ab} = 8\pi \left[S_{cd} - \tfrac{1}{2}(S-E)\gamma_{cd} \right]$$

$$(\Diamond, \Delta)$$

$$-\tfrac{1}{2}\mathcal{L}_m K_{cd} - \tfrac{1}{\alpha} D_c D_d \alpha + {}^3R_{cd} + K K_{cd} - 2K_{ca}K^a{}_d = 8\pi[\ldots]$$

$$\mathcal{L}_m \gamma_{ab} = -2\alpha K_{ab}$$

$$E^{ab} = G^{ab} - 8\pi T^{ab}$$

Constraints: $E^{ab} n_b = 0$ $\quad E^{tb} = 0$

$$\nabla_a E^{ab} = \nabla_a \cancel{G^{ab}}^{\,0} - 8\pi \cancel{\nabla_a T^{ab}}^{\,0} = 0$$

$$\nabla_t E^{tb} = \text{terms involving } E_{ab},\ \partial_i E_{ab}$$

Maxwell Eqns

$\nabla\cdot\vec{E} = \rho$, $\nabla\cdot\vec{B} = 0$ constraints

$\partial_t \vec{E} = \nabla\times\vec{B} - \vec{J}$, $\partial_t \vec{B} = -\nabla\times\vec{E}$

evolution eqns

$$\partial_t(\nabla\cdot\vec{B}) = 0$$

$$\partial_t(\nabla\cdot\vec{E}) = -\nabla\cdot\vec{J}$$

$$\partial_t\rho + \nabla\cdot\vec{J} = 0$$

$$\partial_t(\nabla\cdot\vec{E} - \rho) = 0$$