Title: Lecture - Strong Gravity, PHYS 777

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Collection/Series: Strong Gravity (Elective), PHYS 777, February 24 - March 28, 2025

Subject: Strong Gravity

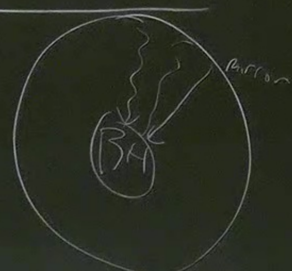
Date: March 06, 2025 - 10:15 AM

URL: https://pirsa.org/25030046

Superradiance

$0 < \omega < m\Omega_H$

Press & Teukolsky



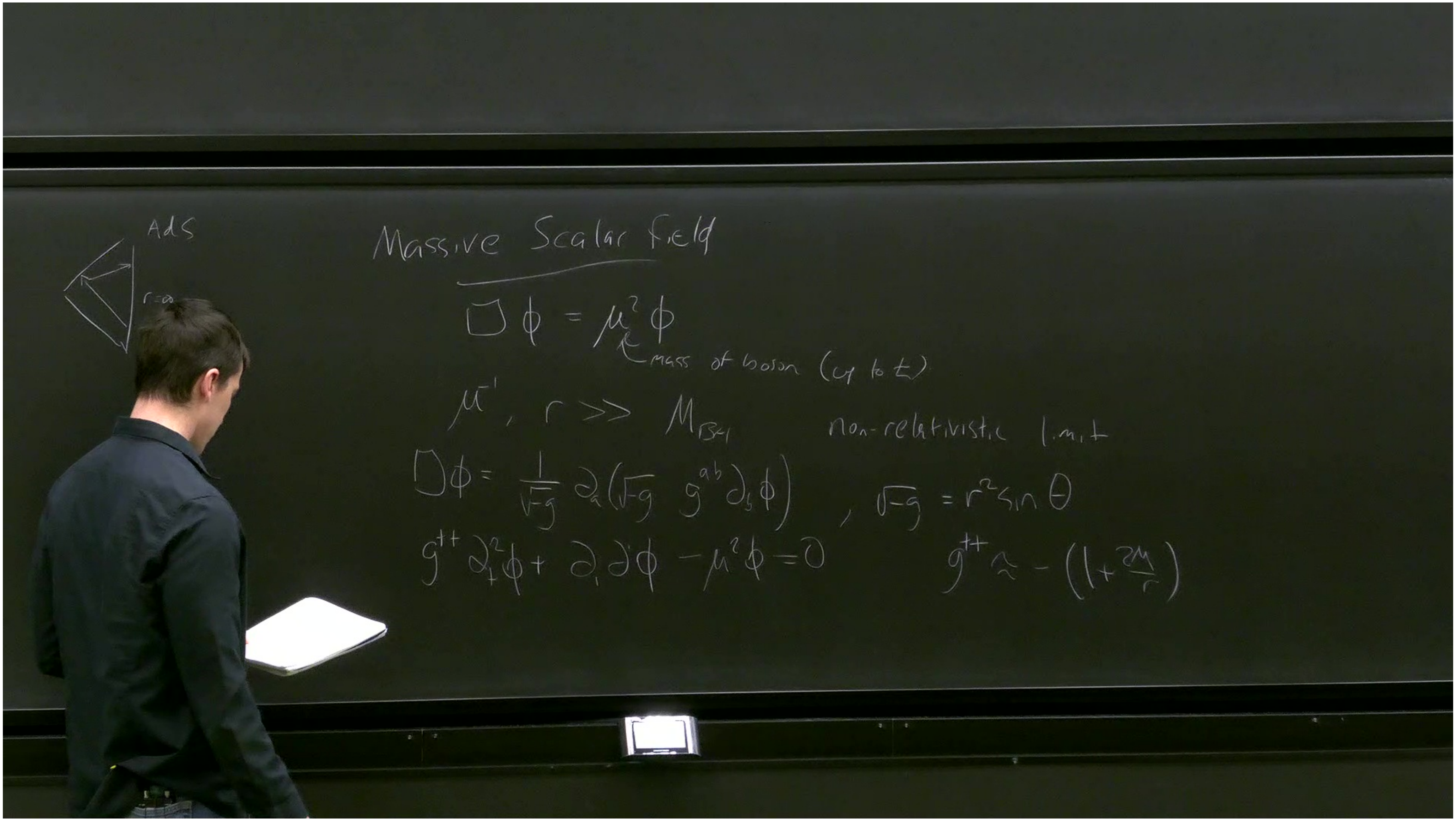
Massive Scalar field

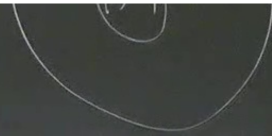
$\Box \phi = \mu^2 \phi$

mass of boson (up to ħ)

$\mu^{-1}, r \gg M_{BH}$ non-relativistic

$\Box \phi = \frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g}\, g^{ab} \partial_b \phi \right), \quad \sqrt{-g} =$


AdS
Massive Scalar field
$\Box\phi = \mu^2\phi$
mass of boson (up to $\hbar$)
$\mu^{-1}, r \gg M_{BH}$
non-relativistic limit
$\Box\phi = \frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}\, g^{ab}\partial_b\phi)$, $\sqrt{-g} = r^2\sin\theta$
$g^{tt}\partial_t^2\phi + \partial_i\partial^i\phi - \mu^2\phi = 0$
$g^{tt} \approx -(1+\frac{2M}{r})$



Ansatz $\phi = \frac{1}{\sqrt{2\mu}}\left[\psi(x)e^{-i\omega t} + \psi^*(x)e^{i\omega t}\right]$

$$\left[-\left(1+\frac{2M}{r}\right)(-i\omega)^2 + \partial_i\partial^i - \mu^2\right]\psi = 0$$

$$(\omega^2-\mu^2)\psi = \left(-\partial_i\partial^i - \frac{2M\omega^2}{r}\right)\psi \qquad \omega \simeq \mu$$

$$(\omega-\mu)\psi = \left(-\frac{1}{2\mu}\partial_i\partial^i - \frac{\mu M}{r}\right)\psi$$

"fine

$$\partial_i \partial^i \phi - \mu^2 \phi = 0 \qquad g^{tt} \simeq -\left(1 + \frac{2M}{r}\right)$$

"fine structure constant" $\quad \alpha = M\mu \simeq 0.1 \left(\frac{\mu}{10^{-12}\,eV}\right)\left(\frac{M_{BH}}{10\,M_\odot}\right)$

Energy $\quad E = \omega - \mu \simeq -\frac{\alpha^2}{2n^2}$

Bohr Radius $\quad r_B = (\mu\alpha)^{-1}$

$$\phi \sim e^{-r/2r_B}\, Y^{\ell=1}(\theta)\, e^{-i(\omega t - m\varphi)}$$

$$|Im(\omega)| \ll |Re(\omega)|$$

3+1 Decomposition of Spacetime

GR, Wald Chap 10.2

3+1 Formalism Gourgoulhon arXiv gr-qc/0703035

Hypersurfaces of Spacetime

4-dim spacetime manifold M

3-dim manifold $\hat{\Sigma}$

Embedding of $\hat{\Sigma}$ in M

$\Phi : \hat{\Sigma} \to M$, 1:1 , Φ,

$\Sigma = \Phi(\hat{\Sigma})$ is called a hypers

Level Set $\quad \Sigma = \{ x^a \in \mathcal{M} \mid F(x^a) = 0 \}$

$\partial_a F$ is normal to Σ

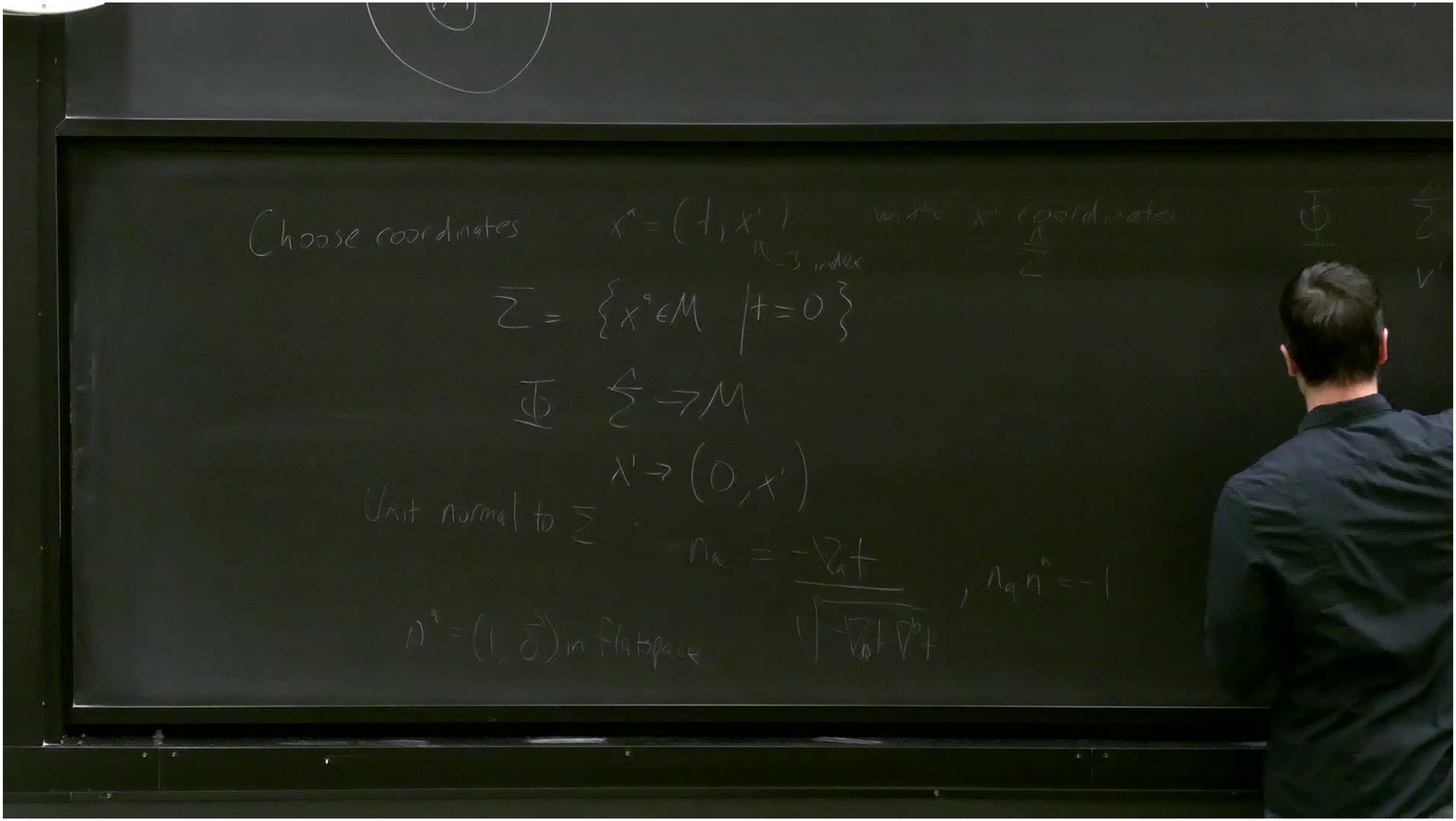
$\iff (\partial_a F) v^a = 0$ for any v^a tangent to Σ

$(\partial_a F)$ timelike, call Σ spacelike

$\partial_a F$ space like, " " timelike

$\partial_a F$ null, " " null

Restrict to Σ spacelike


Choose coordinates x^a = (t, x^i) with x^i coordinates
3-index
Σ = {x^a ∈ M | t = 0}
Φ: Σ → M
x^i → (0, x^i)
Unit normal to Σ: n_a = −∇_a t / √(−∇_a t ∇^a t), n_a n^a = −1
n^a = (1, 0⃗) in flatspace

$x^a = (t, x^i)$ with x^i coordinates $\hat{\Sigma}$

$\uparrow$ 3 index

$\{ x^a \in M \mid t = 0 \}$

$\hat{\Sigma} \to M$

$x^i \to (0, x^i)$

$$n_a = \frac{-\nabla_a t}{\sqrt{-\nabla_b t \, \nabla^b t}}, \quad n_a n^a = -1$$

flatspace

$\Phi : \hat{\Sigma} \to M$

$v^i \to (0, v^i)$ push-forward mapping

$\Phi^* : M \to \hat{\Sigma}$

$(u_t, u_i) \to u_i$ pull back

In M

$(0, v^i)(u_t, u_i) = v^i u_i$

$\hat{\Sigma}$

$v^i u_i$

$n^a = (1, \vec{0})$ in flatspace $\quad \nabla_a t \, \nabla^b t$

For g_{ab} metric in M, induced metric γ_{ij} on Σ

$\gamma_{ij} = g_{ij}$ in our adapted coordinates

Doff hats $\Sigma \cong \hat{\Sigma}$

$$V^a = \underbrace{-V^b n_b n^a}_{\text{orthogonal}} + \overbrace{\left(V^a + V^b n_b n^a\right)}^{\text{tangent}} \qquad n_a \gamma^a_b V^b = n^b V_b - n_b V^b = 0$$

$$= \gamma^a_b V^b$$

$$\gamma^a_b = \delta^a_b + n_b n^a$$

$M \to \Sigma$

$V^a \to \gamma^i_b V^b$

$\gamma_{ab} = \gamma^i_a \gamma^j_b \gamma_{ij} = g_{ab} + n_a n_b$

$-n_b V^b = 0$ $\quad U^a n_a = V^a n_a = 0, \quad \gamma_{ab} U^a U^b = g_{ab} U^a U^b$

$U^a = \lambda n^a \quad \gamma_{ab} U^a V^b = 0$

Intrinsic Curvature same $(g_{ab}, M) \to (\gamma_{ij}, \Sigma)$

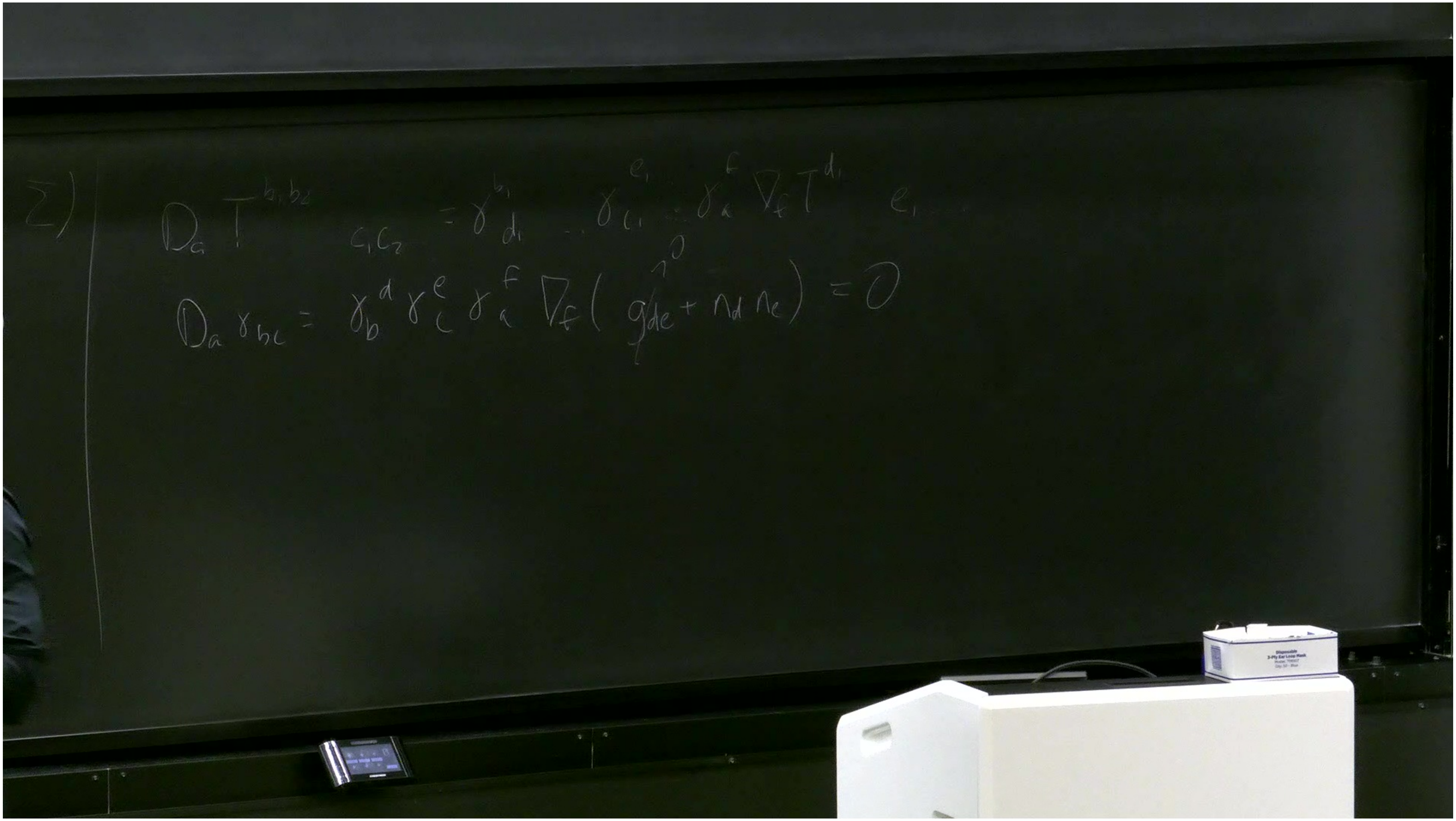
$D_i \gamma_{jk} = 0$ (also assume torsion free)

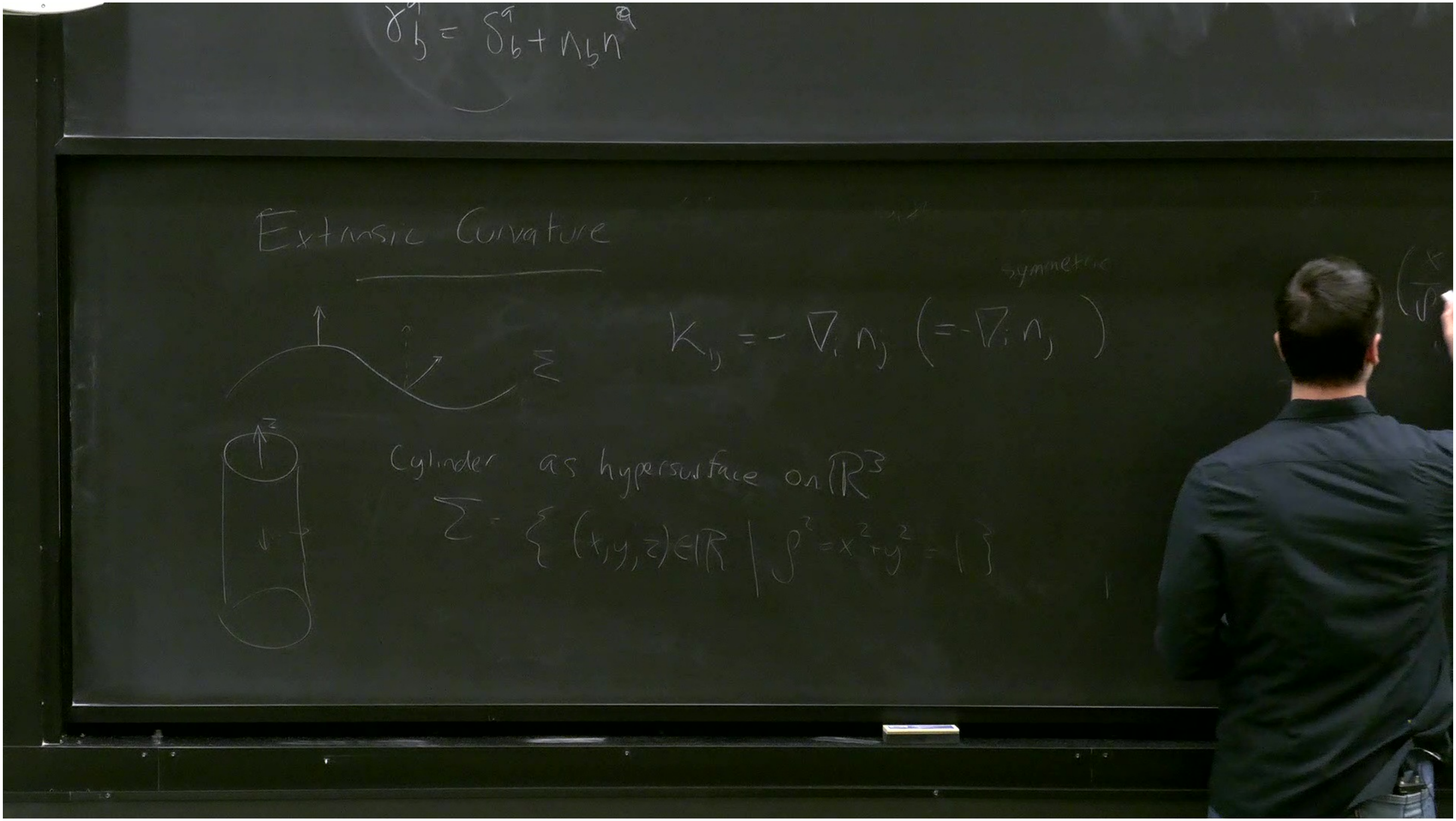
${}^{(3)}\Gamma^i_{jk}$

$$(D_i D_j - D_j D_i) V^k = {}^{(3)}R^k{}_{lij} V^l$$

$${}^{(3)}R_{ij} = {}^{(3)}R^k{}_{ikj}$$

scalar $\;{}^{(3)}R = \gamma^{ij}\,{}^{(3)}R_{ij}$


2)
$D_a T^{b_1 b_2}{}_{c_1 c_2} = \gamma^{b_1}{}_{d_1} \cdots \gamma_{c_1}{}^{e_1} \cdots \gamma_a{}^f \nabla_f T^{d_1 \cdots}{}_{e_1 \cdots}$
$D_a \gamma_{bc} = \gamma_b{}^d \gamma_c{}^e \gamma_a{}^f \nabla_f (g_{de} + n_d n_e) = 0$


$\gamma^a_b = \delta^a_b + n_b n^a$
Extrinsic Curvature
symmetric
$K_{ij} = -\nabla_i n_j \; (= -\nabla_j n_i)$
Σ
Cylinder as hypersurface on $\mathbb{R}^3$
$\Sigma = \{ (x,y,z) \in \mathbb{R} \mid \rho^2 = x^2 + y^2 = 1 \}$

$n^a v_b - n_b v^a = 0$

$v^a n_a = v^a n_a = 0, \quad \gamma_{ab} v^a v^b = g_{ab} v^a v^b$

$v^a = \lambda n^a \quad \gamma_{ab} v^a v^b = 0$

$$n_a = \left(\frac{x}{\rho}, \frac{y}{\rho}, 0\right),$$

$$\nabla_b n_a = \begin{pmatrix} \frac{y^2}{\rho^3} & \frac{-xy}{\rho^3} & 0 \\ \frac{-xy}{\rho^3} & \frac{x^2}{\rho^3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J^a_i = \frac{dx^a}{dx^i} \qquad i = \{\phi, z\} \qquad a = \{x, y, z\} \qquad \tan\phi = \frac{y}{x}$$

$$K_{ij} = -\nabla_b n_a J^a_i J^b_j$$

$$K_{\phi\phi} = \frac{-(x^2+y^2)^2}{\rho^3}\Bigg|_{\rho=1} \Rightarrow 1$$

$$n^a v_b - n_b v^a = 0$$

$$v^a n_a = v^a n_a = 0, \quad \gamma_{ab} v^a v^b = g_{ab} v^a v^b$$

$$v^a = \lambda n^a \qquad \gamma_{ab} v^a v^b = 0$$

$$n_a = \left(\frac{x}{\rho}, \frac{y}{\rho}, 0\right),$$

$$\nabla_b n_a = \begin{pmatrix} \frac{y^2}{\rho^3} & \frac{-xy}{\rho^3} & 0 \\ \frac{-xy}{\rho^3} & \frac{x^2}{\rho^3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$K_{ij} = -\nabla_b n_a J^a_i J^b_j$$

$$K_{\phi\phi} = \frac{-(x^2+y^2)^2}{\rho^3}\Big|_{\rho=1} = -1$$

$$J^a_i = \frac{dx^a}{dx^i} \qquad i = \{\phi, z\} \quad a = \{x, y, z\} \quad \tan\phi = \frac{y}{x}$$

$$K_{ab} = \gamma^i_a \gamma^j_b K_{ij}$$

$$= -\nabla_b n_a - \underbrace{(n^c \nabla_c n_a)}_{= a_a} n_b$$

acceleration