Title: Lecture - Strong Gravity, PHYS 777

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Subject: Strong Gravity

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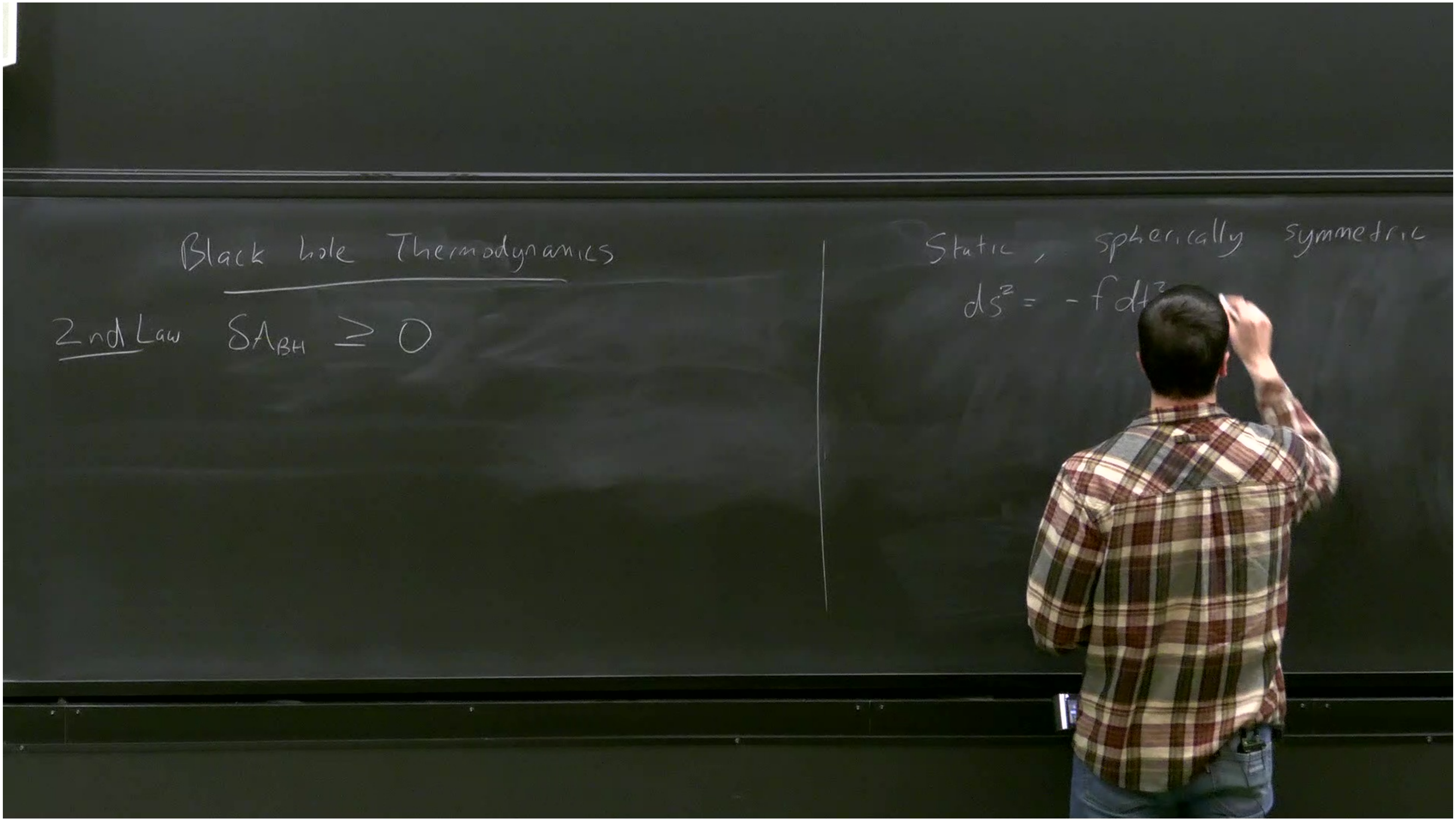
Black hole Thermodynamics
2nd Law δA_BH ≥ 0
Static, spherically symmetric
ds² = - f dt²

Static, spherically symmetric

$$ds^2 = -f\,dt^2 + \frac{1}{f}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$f = f(r)$$

$$u^a = f^{-1/2} t^a$$

$$a^a = u^b \nabla_b u^a$$

$$a^r = \frac{1}{2}\frac{df}{dr}$$

$$A = \sqrt{a^a a_a} = \frac{1}{2} f^{-1/2}\frac{df}{dr} = \frac{M}{r^2\sqrt{1-\frac{2M}{r}}}$$

Static, spherically symmetric

$$ds^2 = -f\,dt^2 + \frac{1}{f}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

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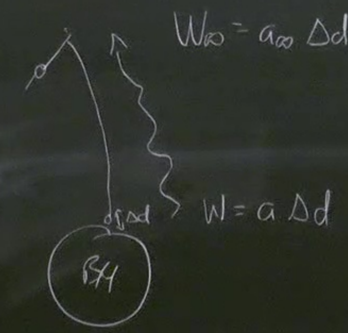
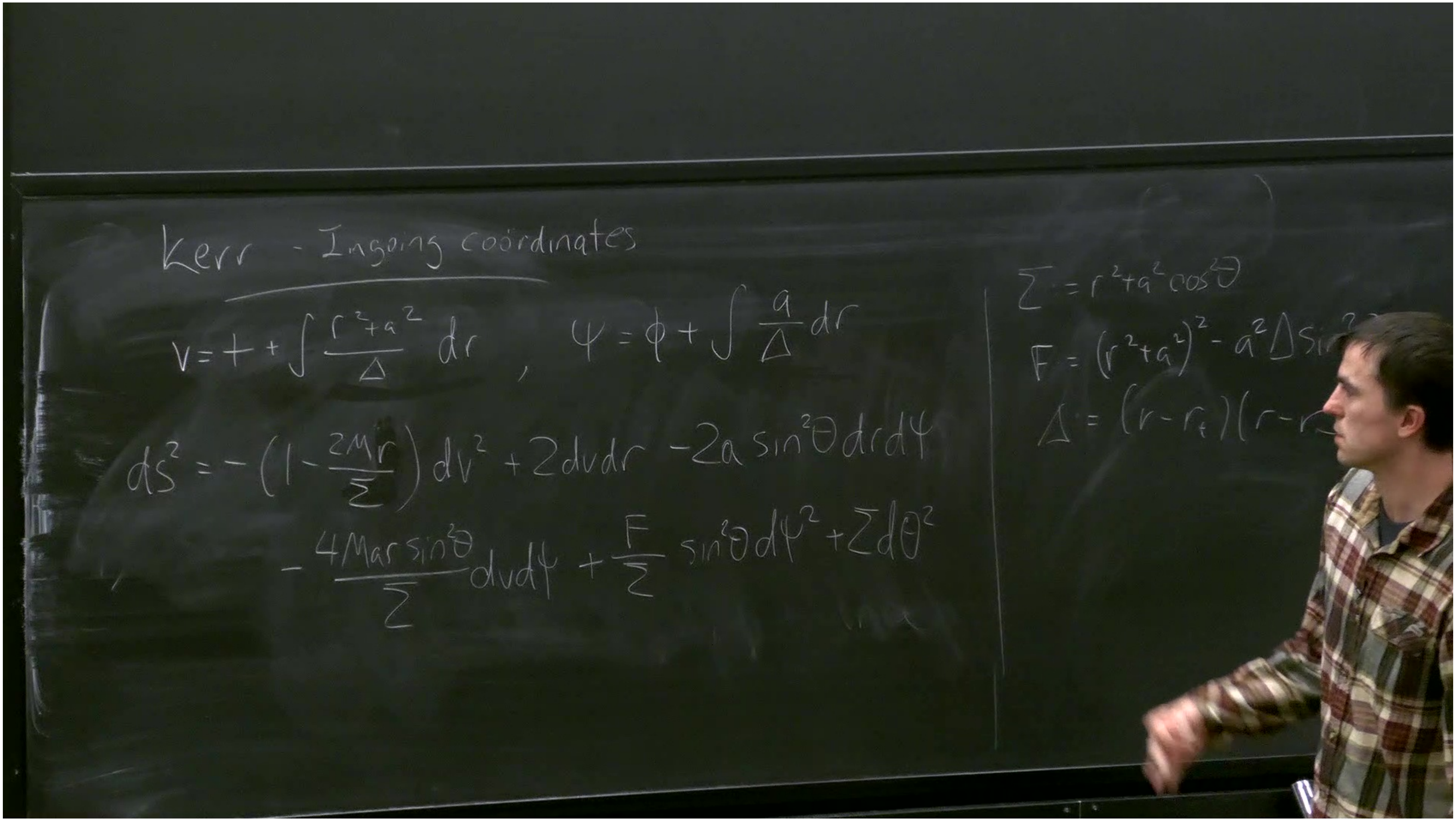
$$A_\infty = f^{1/2} A = \frac{M}{r^2} \xrightarrow{r\to 2M} \frac{1}{4M} := \kappa$$ surface gravity



$$A_\infty = f^{1/2} A$$

$$\nabla_a(\chi_b \chi^b) = -2\kappa\, \chi_a$$

$$\chi^a = \xi^a + \Omega_H\, \phi^a$$


Kerr - Ingoing coördinates
$v = t + \int \frac{r^2+a^2}{\Delta} dr \, , \quad \psi = \phi + \int \frac{a}{\Delta} dr$
$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dv^2 + 2 dv dr - 2a \sin^2\theta \, dr d\psi$
$- \frac{4Mar\sin^2\theta}{\Sigma} dv d\psi + \frac{F}{\Sigma} \sin^2\theta \, d\psi^2 + \Sigma d\theta^2$
$\Sigma = r^2 + a^2 \cos^2\theta$
$F = (r^2+a^2)^2 - a^2 \Delta \sin$
$\Delta = (r - r_+)(r - r$

$$A = \sqrt{a^a a_a} = \frac{1}{2} f^{-1/2} \frac{df}{dr} = \frac{M}{r^2\sqrt{1-\frac{2M}{r}}}$$

$$A_\infty = f^{1/2} A = \frac{M}{r^2} \xrightarrow{r\to 2M} \frac{1}{4M} := \kappa \quad \text{surface gravity}$$

$$\chi^a = \left(\frac{\partial}{\partial v}\right)^a + \Omega_H \left(\frac{\partial}{\partial \psi}\right)^a$$

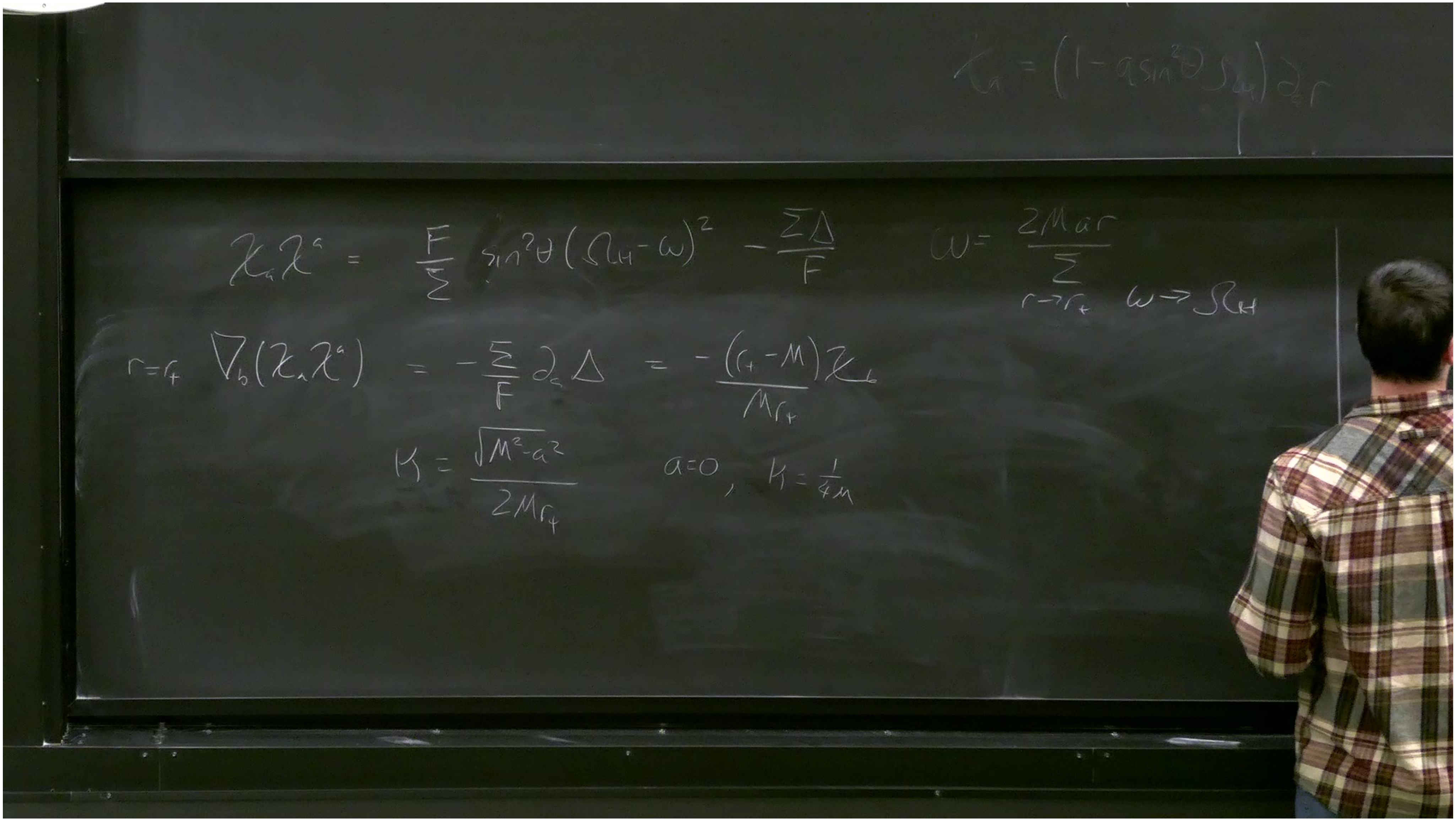
$$\chi_a = \left(-\Sigma + 2Mr - \frac{2Mar\,\Omega_H \sin^2\theta}{\Sigma}\right)\partial_a v + \frac{\sin^2\theta}{\Sigma}\left(F\Omega_H - 2Mar\right)\partial_a\psi + \left(1 - a\sin^2\theta\,\Omega_H\right)\partial_a r$$

$r \to r_+$

$$\Sigma \to 2Mr_+\left(1 - a\,\Omega_H \sin^2\theta\right)$$

$$F \to 4M^2 r_+^2$$

$$\chi_a = \left(1 - a\sin^2\theta\,\Omega_H\right)\partial_a r$$



$\chi_a = (1 - a\sin^2\theta\,\Omega_H)\,\partial_r$
$\chi_a \chi^a = \frac{F}{\Sigma}\sin^2\theta\,(\Omega_H - \omega)^2 - \frac{\Sigma\Delta}{F}$
$\omega = \frac{2Mar}{\Sigma}$
$r \to r_+ \quad \omega \to \Omega_H$
$r = r_+ \quad \nabla_b(\chi_a\chi^a) = -\frac{\Sigma}{F}\partial_a\Delta = -\frac{(r_+ - M)\chi_b}{M r_+}$
$\kappa = \frac{\sqrt{M^2 - a^2}}{2M r_+}$
$a = 0, \quad \kappa = \frac{1}{4M}$

0th law: Surface gravity is constant on stationary black hole horizon

1st Law:
$$\delta M = \frac{\kappa}{8\pi}\,\delta A + \Omega_H\,\delta J$$

c.f.
$$dE = T\,dS - p\,dV$$

3rd Law: Can't achieve $\kappa = 0$ by some physical process

$\kappa = 0 \Rightarrow a = M$

General Penrose Process

Matter T_{ab} (ignore back reaction)

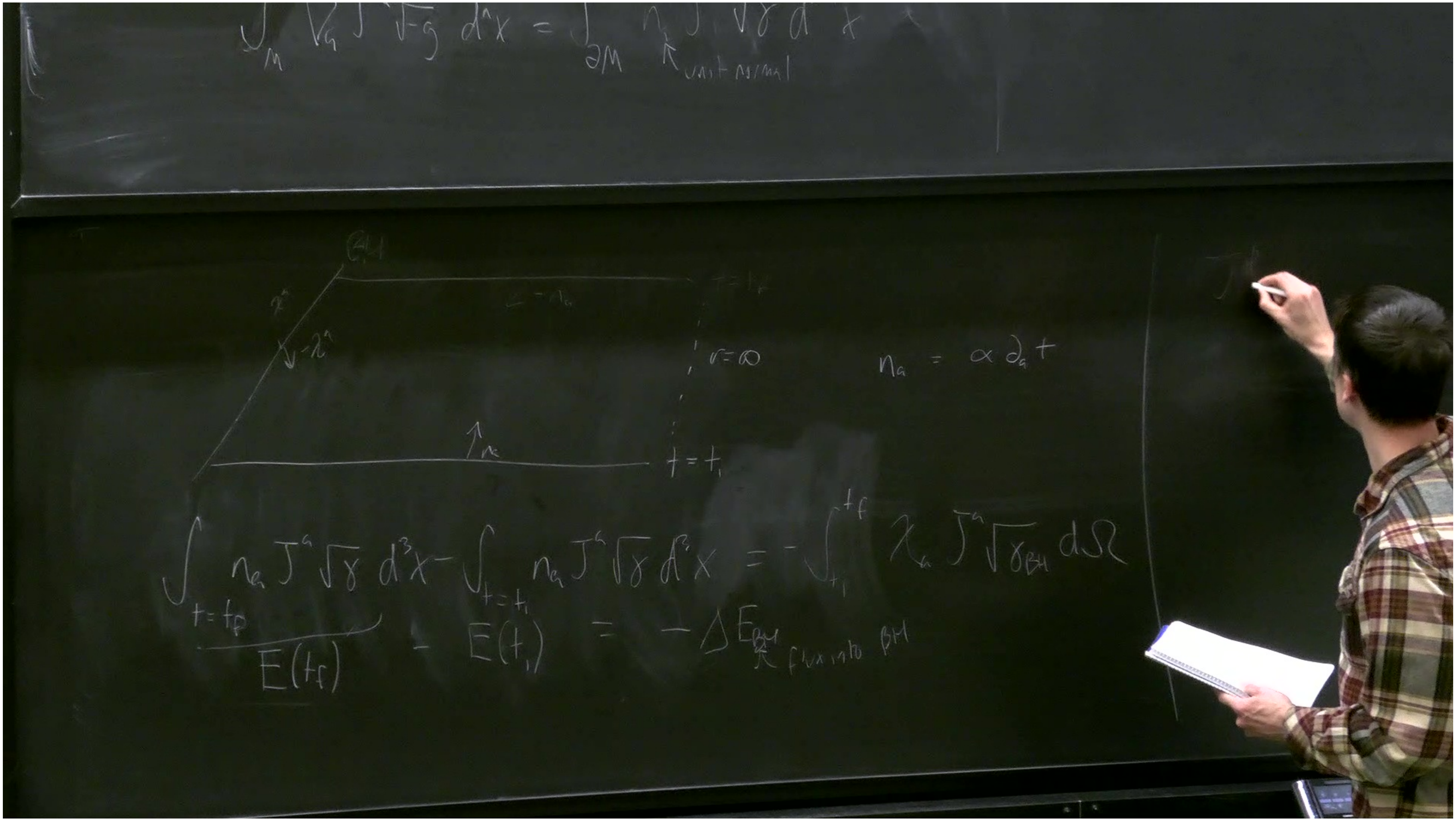
$$J_a = -T_{ab}\,\xi^b$$

$$\nabla_a J^a = -\cancel{\nabla_a\left(T^a{}_b\right)}^{\,0}\,\xi^b - T^a{}_b\,\cancel{\nabla_a \xi^b}^{\,0} = 0$$

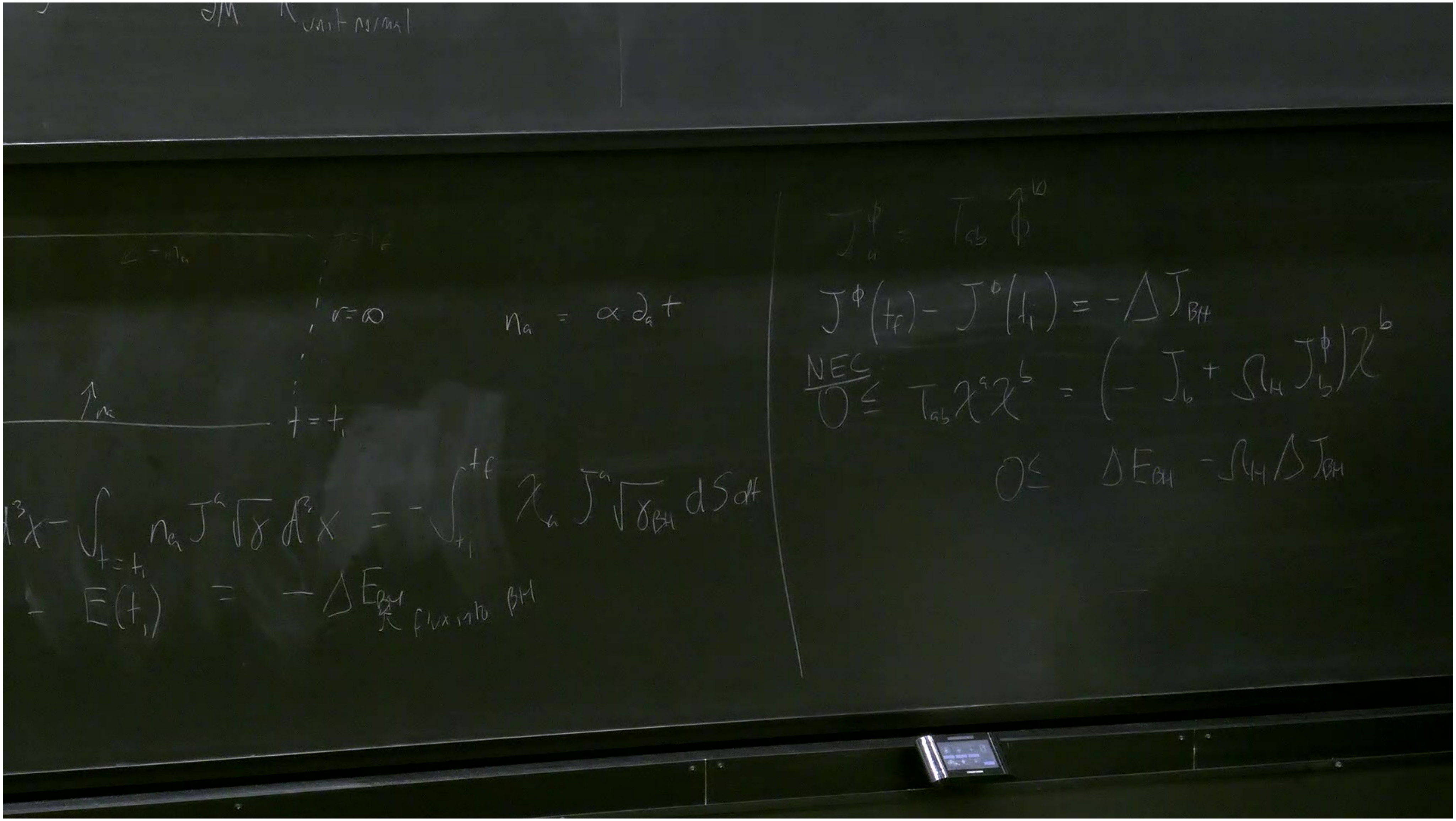
$$\nabla_{(a}\xi_{b)} = 0$$

$$\int_M \nabla_a J^a \sqrt{-g}\, d^n x = \int_{\partial M} n_a J^a \sqrt{\gamma}\, d^{n-1} x$$

n_a: unit normal



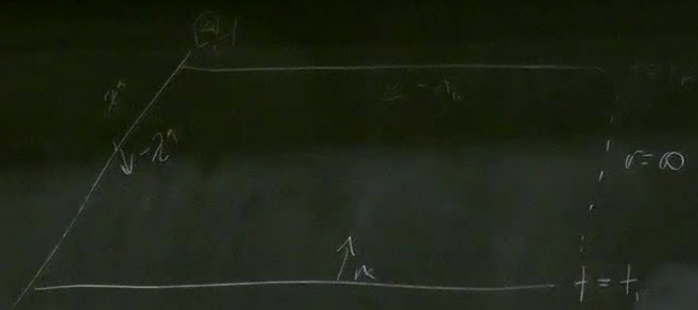
$\int_M \nabla_a J^a \sqrt{-g}\, d^n x = \int_{\partial M} n_a J^a \sqrt{\gamma}\, d^{n-1} x$
unit normal
BH
$t = t_f$
$r = \infty$
$n_a = \alpha \partial_a t$
$t = t_i$
$\int_{t=t_f} n_a J^a \sqrt{\gamma}\, d^3x - \int_{t=t_i} n_a J^a \sqrt{\gamma}\, d^3x = -\int_{t_i}^{t_f} \chi_a J^a \sqrt{\gamma_{BH}}\, d\Omega$
$E(t_f) - E(t_i) = -\Delta E_{BH}$
flux into BH



∂M unit normal
$r = \infty$
$t = t_f$
$t = t_i$
$n_a = \alpha \partial_a t$
$d^3x - \int_{t=t_i} n_a J^a \sqrt{\gamma}\, d^3x = -\int_{t_i}^{t_f} \chi_a J^a \sqrt{\gamma_{BH}}\, dS\, dt$
$- E(t_i) = -\Delta E_{BH}$ flux into BH
$J^\phi_a = T_{ab} \phi^b$
$J^\phi(t_f) - J^\phi(t_i) = -\Delta J_{BH}$
NEC
$0 \leq T_{ab} \chi^a \chi^b = (-J_b + \Omega_H J^\phi_b) \chi^b$
$0 \leq \Delta E_{BH} - \Omega_H \Delta J_{BH}$

$$\int_M \nabla_a J^a \sqrt{-g}\, d^n x = \int_{\partial M} n_a J^a \sqrt{\gamma}\, d^{n-1}x$$

unit normal



$$n_a = \alpha\, \partial_a t$$

$$\underbrace{\int_{t=t_f} n_a J^a \sqrt{\gamma}\, d^3x}_{E(t_f)} - \int_{t=t_i} n_a J^a \sqrt{\gamma}\, d^3x = -\int_{t_i}^{t_f} \chi_a J^a \sqrt{\gamma_{BH}}\, dS\, dt$$

$$E(t_f) - E(t_i) = -\Delta E_{BH}$$

flux into BH

$$J^\phi_a = T_{ab}\, \phi^b$$

$$J^\phi(t_f) - J^\phi(t_i) = -\Delta J_{BH}$$

NEC

$$0 \le T_{ab}\chi^a\chi^b = \left(-J_b + \Omega_H J^\phi_b\right)\chi^b$$

$$0 \le \Delta E_{BH} - \Omega_H \Delta J_{BH}$$

$$E(t_f) - E(t_i) = -\Delta E_{BH}$$ flux into BH

Superradiance

Scalar field $\quad \Box\psi = \nabla_a \nabla^a \psi = 0$

$$T_{ab} = \nabla_a \psi \nabla_b \psi - \frac{1}{2} g_{ab} \nabla_c \psi \nabla^c \psi$$

$$\psi = \bar{\Psi}(r,\theta)\, \mathrm{Re}\left[e^{-i(\omega t - m\phi)} \right]$$

ω: freq, $\quad m$: azim. num.

$$\Delta E_{BH} = \int dt \int dS_{BH} \left(-\chi^a\right) T_{ab} \hat{t}^b$$

$$= -\int dt \int_{r=r_+} \left[(\chi^a)(\nabla_a\psi)(\nabla_b\psi)\hat{t}^b - \frac{1}{2}\chi_a \cancel{\hat{t}^a \nabla_c\psi \nabla^c\psi} \right] dS_{BH}$$

$$= -\int dt \int dS_{BH} \, ($$

$$= -\int dt \int dS_{BH} \left(\partial_t \psi + \Omega_H \partial_\phi \psi\right) \partial_t \psi$$

$$= (\omega - m\Omega_H)\,\omega \underbrace{\int dt \int dS_{BH} |\bar{\Psi}|^2 \sin^2(\omega t - m\phi)}_{\geq 0}$$

$\psi = 0$

$- \frac{1}{2} g_{ab} \nabla_c \psi \nabla^c \psi$

$(\omega t - m\phi)]$

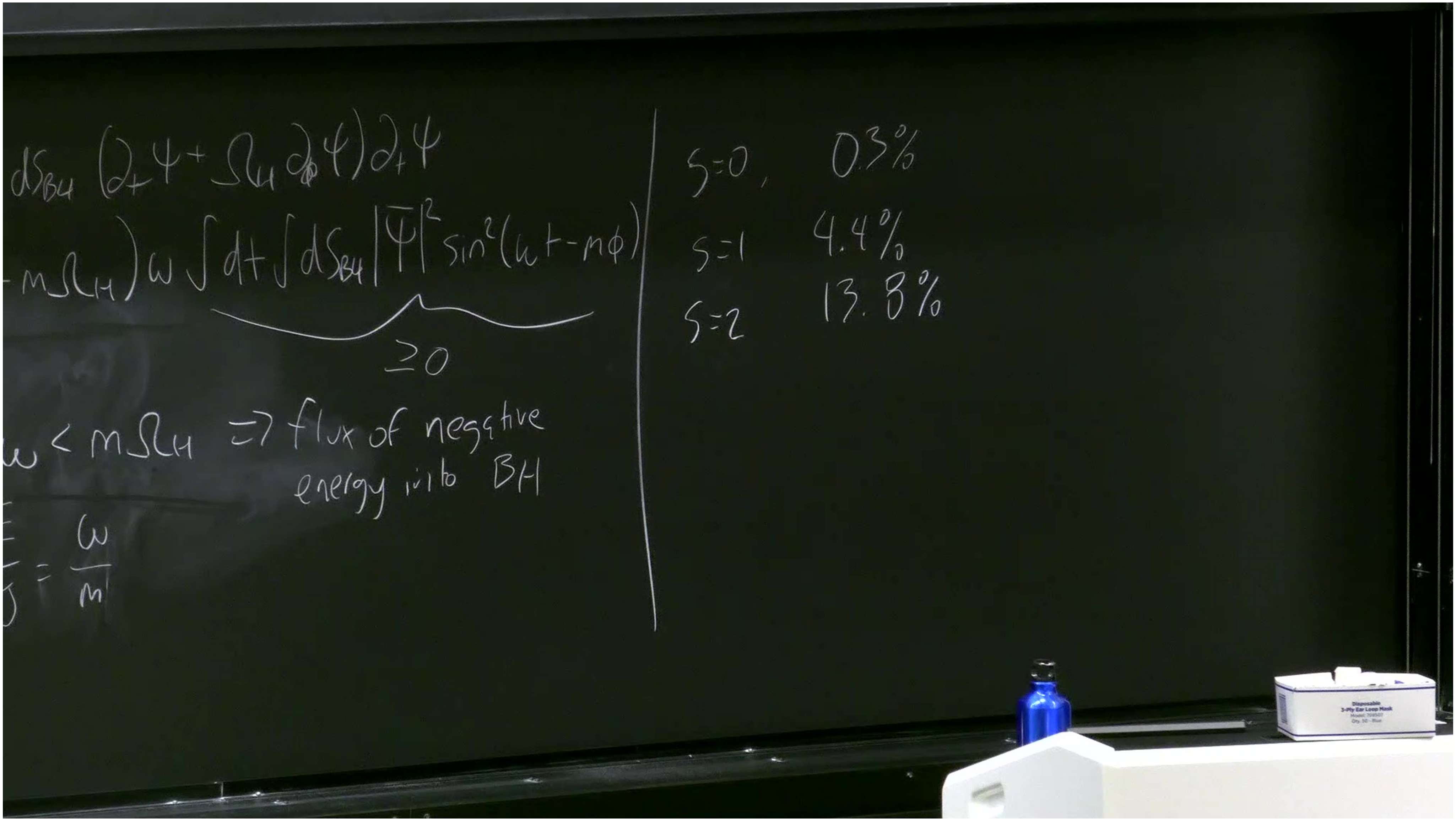
freq

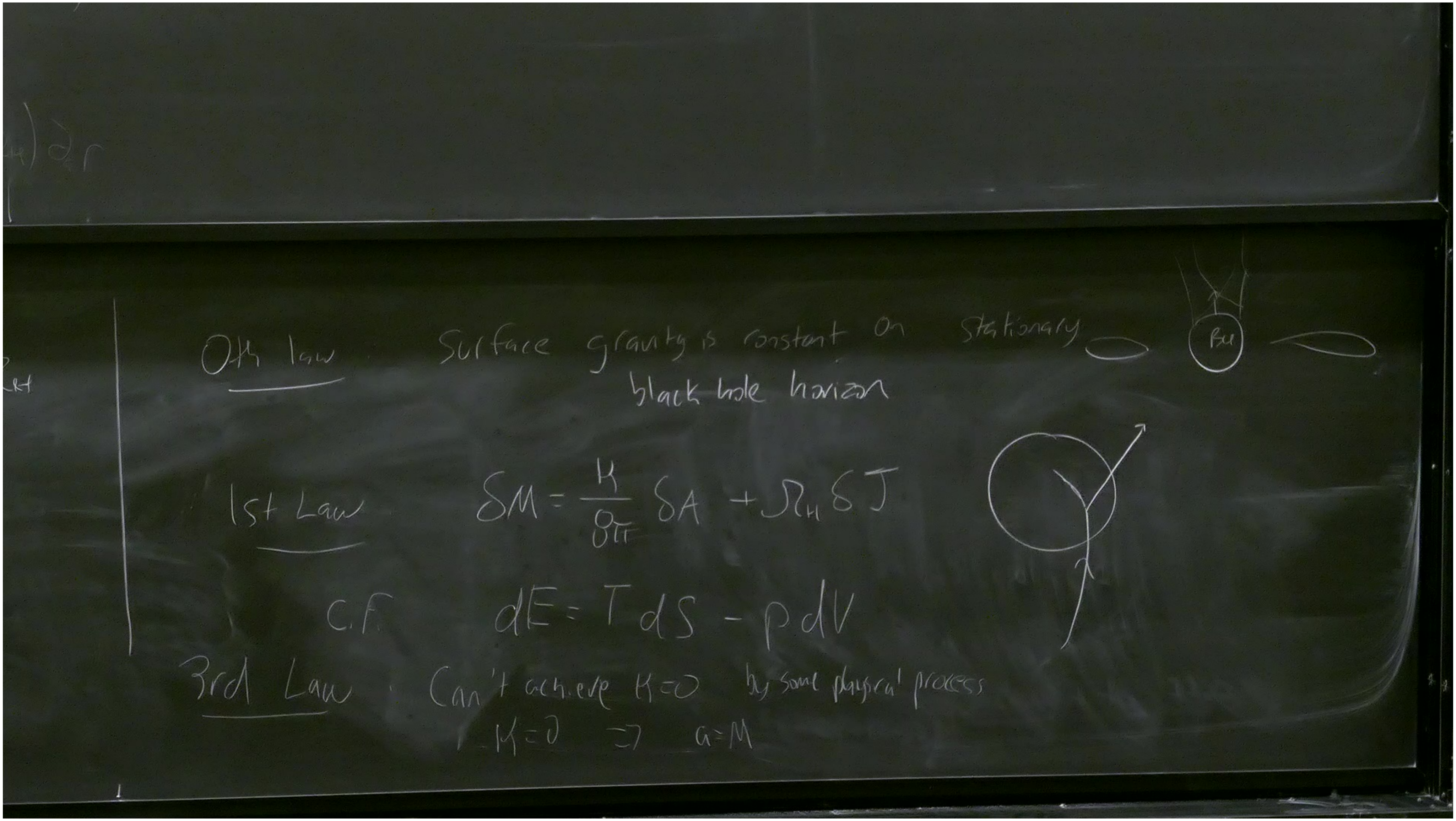
azim. num.

$0 < \omega < m\Omega_H \Rightarrow$ flux of negative energy into BH

$$\frac{\delta E}{\delta J} = \frac{\omega}{m}$$

$$\frac{1}{2} \chi_a \cancel{\xi^a}^{\,\to 0} \nabla_c \psi \nabla^c \psi]\, dS_{BH}$$


$dS_{BH} (\partial_t \psi + \Omega_H \partial_\phi \psi) \partial_t \psi$
$- m\Omega_H) \omega \int dt \int dS_{BH} |\bar{\Psi}|^2 \sin^2(\omega t - m\phi)$
≥ 0
$\omega < m\Omega_H \Rightarrow$ flux of negative energy into BH
$\frac{E}{J} = \frac{\omega}{m}$
$s=0$, 0.3%
$s=1$ 4.4%
$s=2$ 13.8%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BH
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