Title: Lecture - Quantum Gravity, PHYS 644

Speakers: Aldo Riello

Collection/Series: Quantum Gravity (Elective), PHYS 644, February 24 - March 28, 2025

Subject: Quantum Gravity

Date: March 24, 2025 - 9:00 AM

URL: https://pirsa.org/25030032

General Relativity

$$\mathring{i}_{\rho(\xi)}\,\Omega_\Sigma = -dQ_\Sigma(\xi) + \int_\Sigma i_\xi \underline{E} + \oint_{\partial\Sigma} i_\xi \Theta$$

$$\Omega_\Sigma = d\Theta_\Sigma\,,\quad \Theta_\Sigma = -\frac{1}{2}\int_\Sigma \sqrt{h}\, n_a\left(\nabla_b\, dg^{ba} - \nabla^a dg\right)$$

$$= -\frac{1}{2}\int_\Sigma \sqrt{h}\left(\left(\nabla_a n_b\right) dg^{ab} + 2\,d\left(\nabla_a n^a\right) - \nabla_a \beta^a\right)$$

where: $\beta^a := dn^a + g^{ab}\, d(n_b) \rightarrow n_a\beta^a = d(\underbrace{n^a n_a}_{-1}) \equiv 0$

Rec

Σ

Recall:

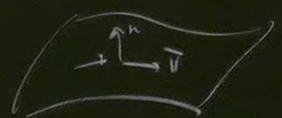
$\Sigma \hookrightarrow M$, $\bar{v}^i$ a v.f. tangent to Σ i.e $n \cdot \bar{v} \equiv 0$

$\bar{\nabla}_i$ cov. deriv. wrt induced metric h_{ij}

$$\underbrace{\bar{\nabla}_i \bar{v}^i}_{3d} = \left(\delta_a^b + n_a n^b\right) \nabla_b \bar{v}^a = \nabla_a \bar{v}^a - \bar{v}^a a_a$$

$$a_a = n^b \nabla_b n_a \quad , \quad n^a a_a = \nabla_n n^2 \equiv 0$$

M



Σ

$\equiv 0$

Assume choice of Σ is independent of g_{ab}

$\Sigma = \{ X^0 \equiv t = 0 \}$, $\delta X^a \equiv 0$

$$n_a = \frac{\partial_a t}{\sqrt{-g^{ab} \partial_a t \, \partial_b t}}$$

$\delta n_a \propto \partial_a t \propto n_a$ b.c. $\delta t = 0$

$n \cdot v \equiv 0$

∂_a

$\equiv 0$

<u>Assume</u> choice of Σ is independent of g_{ab}

$$\Sigma = \{x^0 \equiv t = 0\}, \quad \delta x^a \equiv 0$$

$$n_a = \frac{\partial_a t}{\sqrt{-g^{ab}\partial_a t\, \partial_b t}}$$

$$\delta n_a \propto \partial_a t \propto n_a \quad \text{b.c. } \delta t = 0 \; (*)$$

Then: $\nabla_a \beta^a = \bar{\nabla}_i \beta^i + \partial_a \beta^a$

$$= \bar{\nabla}_i \beta^i + \partial_a \delta n^a + \underbrace{\partial_a g^{ab}\, \delta(n_b)}_{=0\ (*)} \overset{(*)}{=} \bar{\nabla}_i \beta^i + \partial_a n_b\, \delta g^{ab}$$

$$\Theta_\Sigma = -\frac{1}{2}\int_\Sigma \sqrt{h}\,\Big(\big(\underbrace{\nabla_{(a} n_{b)}}_{\frac{1}{2} L_n g_{ab}} + n_{(a} a_{b)}\big)\, \delta g^{ab} + 2\,\delta(\nabla_a n^a) - \underbrace{\bar{\nabla}_i \beta^i}_{\text{CORNER TERM}} \Big)$$

$$g_{ab} = h_{ab} - n_a n_b$$

$$\delta g_{ab} = \delta h_{ab} - 2 n_{(a} \delta n_{b)}$$

"projection" along Σ of $\frac{1}{2} L_n g_{ab}$

(it gives zero when contracted w/ n^a)

$$= \frac{1}{2} h_i^{\ a'} h_j^{\ b'} L_n h_{a'b'} \equiv K_{ij}$$

$$\ldots n^a) - \underbrace{\bar{\nabla}_i \beta^i}_{\text{CORNER TERM}} \Big)$$

$$= -\frac{1}{2}\int_\Sigma \sqrt{h}\left(K_{ij}\,\delta h^{ij} + 2\,\delta K\right) - \cancel{\oint_{\partial\Sigma} \sqrt{\gamma}\; s_i \beta^i}$$

$$\partial\Sigma = \emptyset$$

$$\delta\sqrt{h} = \frac{1}{2}\sqrt{h}\; h^{ij}\,\delta h_{ij}$$

$$= -\frac{1}{2}\int_\Sigma \sqrt{h}\left(K_{ij} - K h_{ij}\right)\delta h^{ij} + \delta \underbrace{\int_\Sigma \sqrt{h}\,(2K)}_{\text{York Gibbons Hawking boundary term}}$$

$$K_{ij} \sim \partial_t h_{ij}$$

$\ldots) - \oint_{\partial\Sigma} \sqrt{\gamma}\, s_i \beta^i$

$\underbrace{\quad}_{\partial\Sigma = \phi}$

$\ldots h^{ij} + \mathbb{d} \underbrace{\int_\Sigma \sqrt{h}\,(2K)}$

York
Gibbons
Hawking
boundary term

$\rightsquigarrow$ presymplectic reduction yields

$$(\mathcal{P}_{\mathrm{can}}, \omega_{\mathrm{can}}) \simeq T^* \mathrm{Riem}(\Sigma)$$

$$\omega_{\mathrm{can}} = -\frac{1}{2}\int_\Sigma \mathbb{d}\Pi^{ij} \wedge \mathbb{d} h_{ij}$$

"Legendre transform" relating can mom to bulk fields:

$$\Pi^{ij} = \boxed{\sqrt{h}}\,(K^{ij} - K h^{ij})$$

momentum DENSITY

"velocity"

$$g^{ab} + 2\,\delta(\underbrace{\nabla_a n^a}_{K}) - \underbrace{\bar{\nabla}_i \beta^i}_{\text{CORNER TERM}})$$

$$= -\frac{1}{2}\int_\Sigma \sqrt{h}\left(K_{ij}\,\delta h^{ij} + 2\,\delta K\right) - \underbrace{\cancel{\oint_{\partial\Sigma} \sqrt{\gamma}\; s_i \beta^i}}_{\partial\Sigma = \phi}$$

$$\delta\sqrt{h} = \frac{1}{2}\sqrt{h}\; h^{ij}\,\delta h_{ij}$$

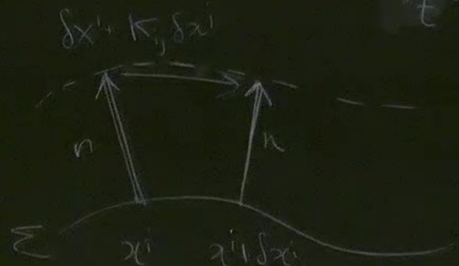
$$= -\frac{1}{2}\int_\Sigma \sqrt{h}\left(K_{ij} - K h_{ij}\right)\delta h^{ij} + \delta \underbrace{\int_\Sigma \sqrt{h}\,(2K)}_{\text{York Gibbons Hawking boundary term}}$$

$K_{ij} \sim \partial_t h_{ij}$

...ntracted w/ n^a)

$$\det M = \prod \lambda = e^{\sum \log \lambda} = e^{\mathrm{Tr}\log M}$$

$$\ldots n_a \ell^{b}{}' \equiv K_{ij}$$



$\rightsquigarrow$ pr...

(P...

"Legen...

bulk

Π...

Questions:

1) Do symmetries descend from $\mathcal{F}$ to $\mathcal{P}_{can}$?

2) $Q_\Sigma(\xi) \approx 0 \rightarrow$ constraints on $\mathcal{P}_{can}$?

3) What to make of $\int_\Sigma i_\xi E$?

[(4) ignore corner term]

① Take g_{ab}, perform a diffeo,

deduce how h_{ij} and Π^{ij} transform.

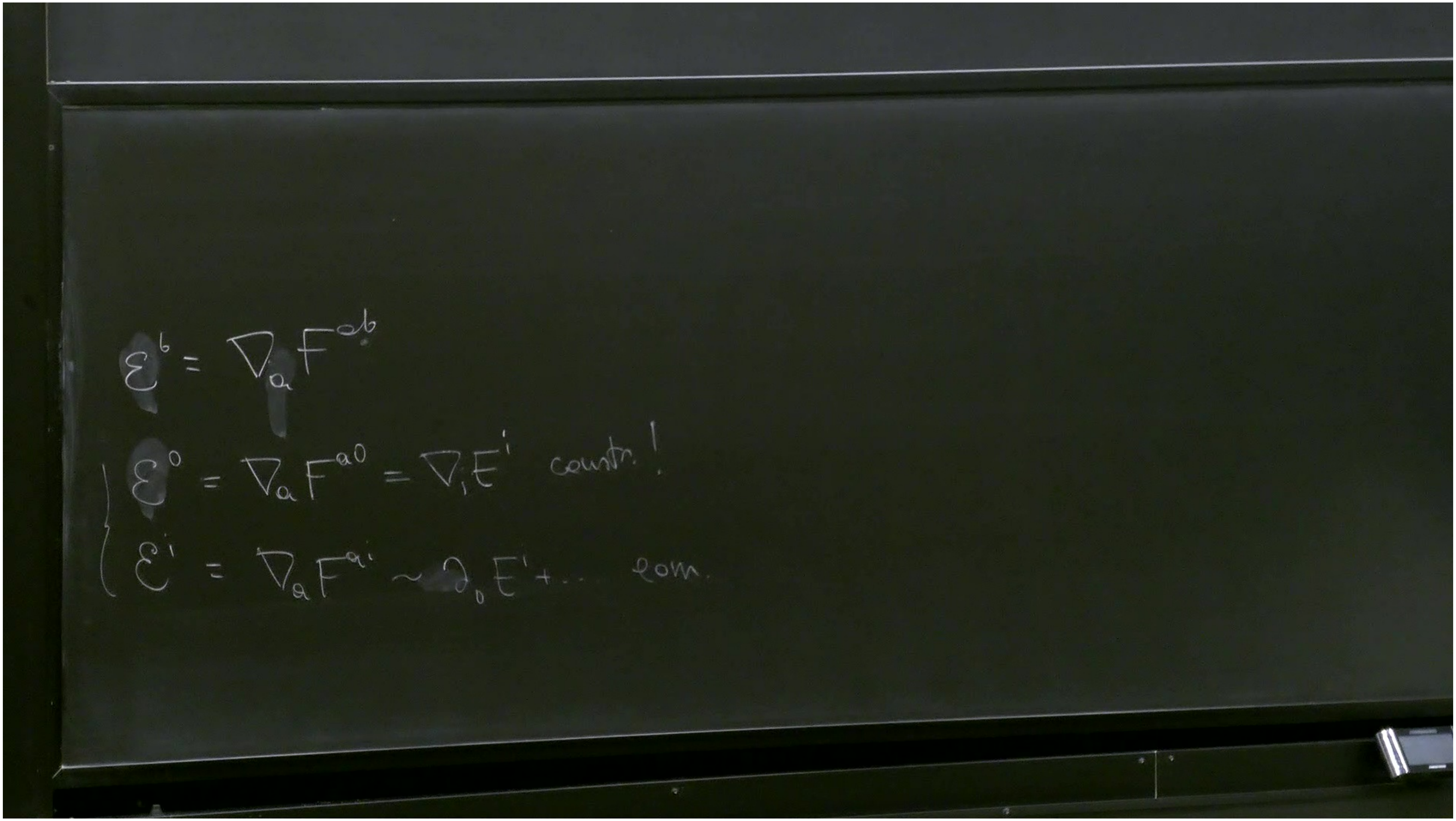
The result is:

$$\delta_\xi h_{ij} = \underbrace{2\alpha \frac{1}{\sqrt{h}}\left(\Pi_{ij} - \frac{1}{2} h_{ij}\Pi\right)}_{K_{ij}} + 2\bar{\nabla}_{(i}\beta_{j)} \sim \alpha \mathring{h} + L_\beta h$$

$$\delta_\xi \Pi^{ij} = \alpha\sqrt{h}\; h^{ia}h^{jb}\left(G_{ab} + \Lambda g_{ab}\right) - \alpha\sqrt{h}\left(\bar{R}^{ij} - \frac{1}{2}\bar{R}h^{ij}\right)$$
$$+ \frac{1}{2}\alpha\frac{1}{\sqrt{h}}\left(\text{"}\Pi^2\text{"}\right)$$
$$+ \sqrt{h}\left(\text{"}\bar{\nabla}\bar{\nabla}\alpha\text{"}\right)$$
$$+ \partial_k\left(\beta^k \pi^{ij}\right) - 2\pi^{k(i}\bar{\nabla}_k\beta^{j)}$$

where: $\alpha = -n_a \xi^a$ ← normal comp.

$\beta^i = + h^i{}_a \xi^a$ ← tangential comp.


$\mathcal{E}^b = \nabla_a F^{ab}$
$\mathcal{E}^0 = \nabla_a F^{a0} = \nabla_i E^i$ constr.!
$\mathcal{E}^i = \nabla_a F^{ai} \sim \partial_0 E^i + \dots$ eom.

boundary term

momentum DENSITY

$+ L_\beta h$

$\bar{R} h^{ij})$

$2\pi^{k(i}\bar{\nabla}_k \beta^{j)}] \leftarrow L_\beta \pi$

(*) = tangential components of E eq. contain the dynamics, i.e. time derivatives

$\sim \partial_t \pi$

↑ not accessible in phase space (not constraint)

$\mathcal{E}^b =$

$\mathcal{E}^0 =$

$\mathcal{E}^i =$

boundary term

momentum

$\boxed{\text{DENSITY}}$

"velocity"

$(*)$ = tangential components of E eq. contain the dynamics, i.e. time derivatives

$\sim \partial_t \pi$

not accessible in phase space (not constraint)

$\Rightarrow$ Diffeos normal to Σ do not descend to P_{can}.

$(i\beta j) \sim \alpha \dot{h} + L_\beta h$

$\alpha\sqrt{h}\left(\bar{R}^{ij} - \frac{1}{2}\bar{R}h^{ij}\right)$

$\frac{1}{2}\alpha\frac{1}{\sqrt{h}}\left(''\pi^2''\right)$

$\sqrt{h}\left(''\bar{\nabla}\bar{\nabla}\alpha''\right)$

$\left[\partial_k(\beta^k\pi^{ij}) - 2\pi^{k(i}\bar{\nabla}_k\beta^{j)}\right] \leftarrow L_\beta \pi$

density