Title: Lecture - Quantum Gravity, PHYS 644

Speakers: Aldo Riello

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Subject: Quantum Gravity

Date: March 20, 2025 - 9:00 AM

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From COVARIANT to CANONICAL ph. sp.

$$(\mathcal{F}, \Omega_\Sigma) \overset{\text{presympl red}}{\rightsquigarrow} (P_{can}, \omega_{can})$$

$(\mathcal{F}, \Omega_\Sigma)$: off shell, presympl.

(P_{can}, ω_{can}): off shell, sympl. ← usually cotangent bundle

$$(\mathcal{F}, \Omega_\Sigma) \overset{\boxed{\text{on-shell}}}{\rightsquigarrow} (\mathcal{F}_{EL}, \omega)$$

$$(P_{can}, \omega_{can}) \overset{\boxed{\text{on shell}}}{\rightsquigarrow} (\mathcal{C}, \iota_{\mathcal{C}}^* \omega_{can})$$

$\boxed{\text{on shell}}$ ← impose "constraints"

$(\mathcal{F}_{EL}, \omega)$: onshell, presympl if gauge

$(\mathcal{C}, \iota_{\mathcal{C}}^* \omega_{can})$: onshell, presympl.

$$(\mathcal{F}_{EL}, \omega) \overset{\boxed{\text{mod out gauge}}}{\rightsquigarrow} \text{"}(\mathcal{F}_{EL}, \omega)/\text{Gauge"}$$

$$(\mathcal{C}, \iota_{\mathcal{C}}^* \omega_{can}) \overset{\boxed{\text{mod out gauge}}}{\rightsquigarrow} \text{"}(\mathcal{C}, \iota_{\mathcal{C}}^* \omega_{can})/\text{Gauge"}$$

$$\text{"}(\mathcal{F}_{EL}, \omega)/\text{Gauge"} \cong \text{"}(\mathcal{C}, \iota_{\mathcal{C}}^* \omega_{can})/\text{Gauge"}$$

onshell, symplectic   onshell, symplectic

E&M

$\mathcal{F} \sim$ …

$F(A)$ …

$\mathcal{G} \sim$ …

$\rho(\xi)$ …

$M = \mathbb{R}$ …

to CANONICAL ph. sp

usually cotangent bundle

presympl red $\rightsquigarrow (P_{can}, \omega_{can})$

off shell, sympl.

impose "constraints"

$\boxed{\text{on shell}}$

$(\mathcal{C}, \iota_C^* \omega_{can})$

on shell, presympl.

gauge

$\boxed{\text{mod out gauge}}$

" $\cong$ "$(\mathcal{C}, \iota_C^* \omega_{can}) / \text{Gauge}$"

on shell, symplectic

gauge

E&M

$\mathcal{F} \sim \Omega^1(M) \ni A$

$F(A) = dA$ field strength (gauge inv.)

$\mathcal{G} \sim C^\infty(M)$ Abelian

$\rho(\xi) A = d\xi$

$M = \mathbb{R}^{1,3}, \ \Sigma = \{t = 0\}$

$$\Omega_\Sigma = \int dF^{0i}(A) \wedge dA_i$$
$$= \int d(\dot{A}_i - \partial_i A_0) \wedge dA_i$$

$\mathbb{X} \in \mathrm{Ker}(\Omega_\Sigma)$ $\qquad \mathbb{X} = \int X_i \frac{\delta}{\delta A_i} + X_0 \frac{\delta}{\delta A_0}$

$\Updownarrow$

$$0 \overset{!}{=} i_{\mathbb{X}} \Omega_\Sigma = \int_\Sigma \underbrace{(\dot{X}^i - \partial_i X_0)}_{=0} dA_i - \underbrace{X_i}_{=0} d\dot{A}_i - \underbrace{\nabla^i X_i}_{=0} dA_0$$

$\Rightarrow$ $X_i|_\Sigma = 0$
X_0 is unconstrained
as long as $\dot{X}_i|_\Sigma = \partial_i X_0$

$$\longleftrightarrow [A_\mu] \sim (A_i, E^i) = (A_i|_\Sigma, \dot{A}_i - \partial_i A_0|_\Sigma)$$

$$E^i = F^{0i}|_\Sigma$$

$S^3)$

$$\rightsquigarrow \left[\begin{array}{l} \Omega_{Con} = \int_\Sigma dE^i \wedge dA_i \\ P_{Con} \sim T^* \Omega(\Sigma) \ni (E^i, A_i) \end{array} \right.$$

GAUGE: $\delta_\xi A_\mu = \partial_\mu \xi + [A_\mu, \xi]$

$[\delta_\xi A_\mu] \overset{?}{=} \delta_\xi [A_\mu]$

it works b.c. $\begin{cases} \delta_\xi A_i = \partial_i \xi + [A_i, \xi] \\ \delta_\xi E^i = 0 + [E^i, \xi] \end{cases}$ indep. of $A_0, \dot{A}_i$ ✓ except through E^i

$\Rightarrow$ GAUGE tr. descend to (P_{Con}, ω_{Con})

$\rho_{Con}(\xi)(A, E) = (d\xi, 0)$

<u>EOM</u>: are there residual eom that constrain initial data?

$n_\nu \nabla_\mu F^{\mu\nu} = 0 \sim \nabla_i F^{i0} = 0$

$\downarrow$ on Σ

Gauss constr.: $\boxed{\nabla_i E^i = 0}$

GENERAL RE

More conceptually:

on $\mathcal{F}$: $\quad \mathring{\mathbb{I}}_{\rho(\xi)} \Omega_\Sigma = -\mathbb{d}\, Q_\Sigma(\xi) \quad (YM)$

$\quad\quad Q_\Sigma(\xi) \approx 0 \quad (\partial\Sigma = \phi)$

on $\mathcal{P}_{can}$: $\quad \mathring{\mathbb{I}}_{\rho_{can}(\xi)} \omega_{can} = -\mathbb{d}\int_\Sigma E^i \partial_i \xi \underset{ibp.}{=} + \mathbb{d}\int_\Sigma (\nabla_i E^i)\xi \approx 0$

Gauss constraint

$\rightarrow (\mathcal{P}_{can}, \rho_{can}, Q \equiv H)$
$\quad\quad\quad\quad \overset{?}{\approx} 0$

give us the setup for sympl. reduction

$\mathcal{P}_{can} \rightsquigarrow \mathcal{P}_{reduced} = \mathcal{C}/\mathcal{G}_{gauge}$

$\Longrightarrow$ Constraints are the Ham. generators of gauge transf. on $(\mathcal{P}_{can}, \omega_{can})$

$$\{H(\xi), H(\eta)\} = H([\xi,\eta])$$

$\quad H(\xi) \equiv Q_\Sigma(\xi)$

constraint algebra represents the gauge algebra (off shell.)

& closes on-shell. $\quad H \equiv Q_\Sigma \approx 0$

GENERAL RELATIVITY

$$\mathcal{F} = \mathrm{Riem}_{1,3}(M) \ni g_{ab}$$

$$\mathfrak{g} = \mathrm{diff}(M) \simeq \left(\mathfrak{X}^1(M), [\,,\,]\right)$$

$$\mathcal{L} = \frac{1}{2}\sqrt{g}\left(\mathrm{Ricci}(g) - 2\Lambda\right)$$

$$\tilde{E}_{ab} = \#\left(G_{ab} + \Lambda g_{ab}\right) \approx 0$$

$$\frac{1}{2} g^{-1}(\partial g \cdots)$$

$$\downarrow$$

$$\tilde{\Theta}^a = -\frac{1}{2}\left(\nabla_b \, \mathbb{d} g^{ba} - \nabla^a \, \mathbb{d} g\right) = -\frac{1}{2}\left(g^{bc}\, \mathbb{d}\Gamma^a_{bc} - g^{ab}\, \mathbb{d}\Gamma^c_{bc}\right)$$

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$$\Theta_\Sigma = -\frac{1}{2}\int_\Sigma \sqrt{h}\left(n_a \left(\underset{\approx}{d\Gamma^a_{bc}}\, g^{bc} - \underline{\underline{n^a\, d\Gamma^c_{ac}}}\right)\right)$$

$$d(\nabla_a n^a) = \nabla_a dn^a + \underline{\underline{d\Gamma^a_{ab}\, n^b}}$$

$$g^{ab}\, d(\nabla_a n_b) = \nabla^a dn_a - g^{ab}\, \underset{\approx}{d\Gamma^c_{ab}}\, n_c$$

$$\beta^a = dn^a + g^{ab}\, dn_b \qquad n_a \beta^a = n_a dn^a + dn_a n^a = d(\overset{-1}{\overset{\parallel}{n^2}}) \equiv 0$$

$$\Theta_\Sigma = -\frac{1}{2}\int_\Sigma \sqrt{h}\left(\nabla_{(a} n_{b)}\, dg^{ab} + 2\, \underset{\substack{\text{drops out}\\ \text{in } \Omega_\Sigma}}{d}\, (\underbrace{\nabla n}_{K\ (\text{York})}) - \underbrace{\nabla_a \beta^a}_{\text{corner term} \to 0}\right)$$

$\Gamma^c_{bc})$

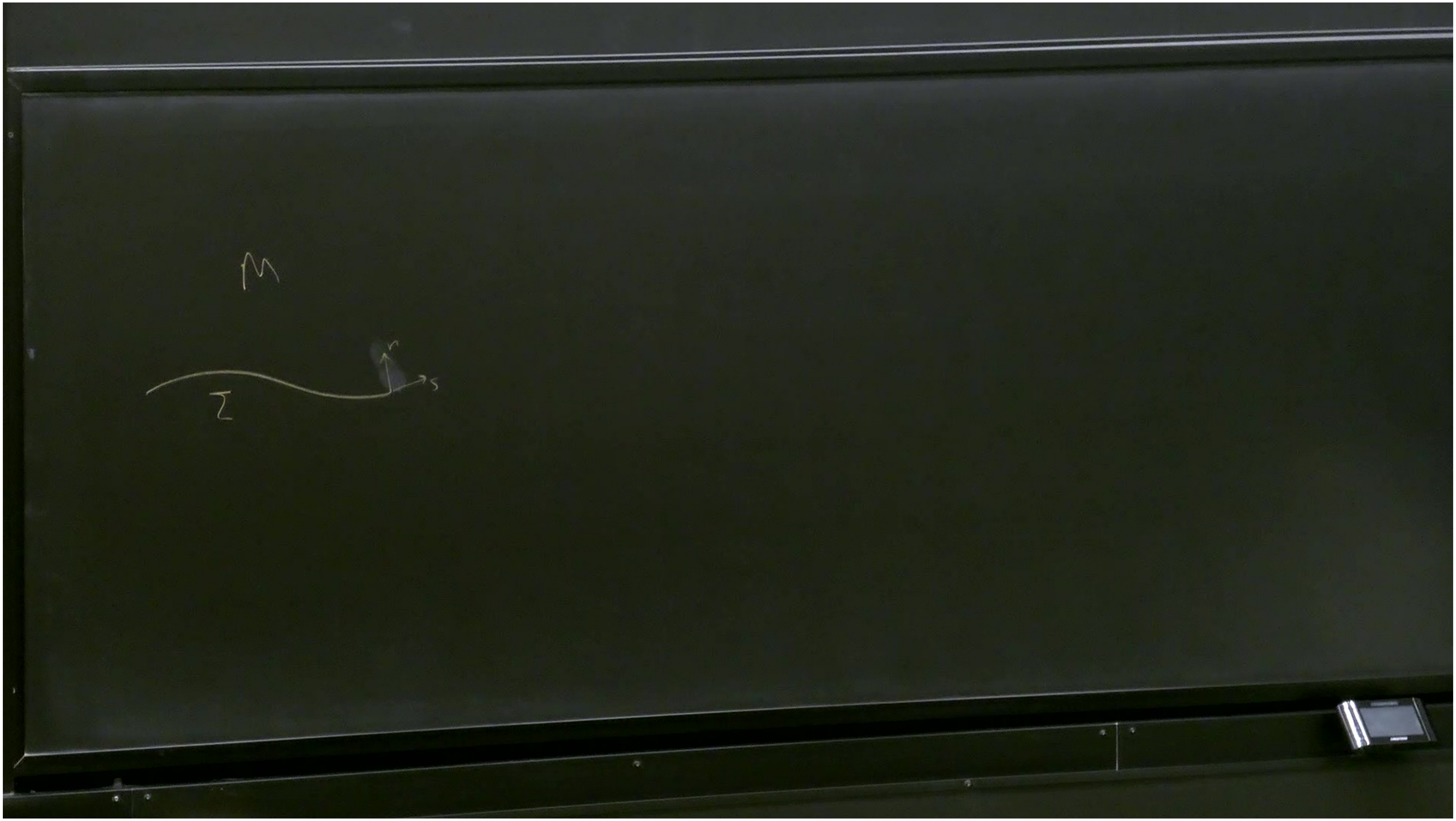
$$\nabla_{(a} n_{b)} = \frac{1}{2} L_n g_{ab}$$

$$\Theta_\Sigma \sim \int_\Sigma \underbrace{\dot{g}\, \delta g}_{\downarrow\ \Omega} + \underbrace{d(\quad) + d(\quad)}_{\downarrow\ \text{drop out } (\partial\Sigma = \phi)}$$

$= d(n^2) \equiv 0$

$\nabla_e \beta^e$

corner term $\to 0$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$\ldots d\Gamma^c_{ac} \Big)$

$\nabla_{(a} n_{b)} = \frac{1}{2} L_n g_{ab}$

$$\Theta_\Sigma \sim \int_\Sigma \underbrace{\dot g \, \delta g}_{\Omega} + \underbrace{d(\ ) + d(\ )}_{\text{drop out } (\partial\Sigma = \phi)}$$

n_c

$n_a dn^a + dn_a n^a = d(\overset{-1}{\overset{\shortparallel}{n^2}}) \equiv 0$

$$\int_{\partial\Sigma} s_a \beta^a \sim s_a dn^a + s^a dn_a$$

$$d\big(\underbrace{\nabla n}_{K \text{ (York)}}\big) - \underbrace{\nabla_a \beta^a}_{\text{corner term} \to 0}$$