Title: Lecture - Quantum Gravity, PHYS 644

Speakers: Aldo Riello

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Subject: Quantum Gravity

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Recap

Def. Lagrangian sym

$\rho : \mathfrak{g} \longrightarrow \mathfrak{X}^1(\mathcal{F})$

such that $\mathbb{L}_{\rho(\xi)} \underline{L} = d\,\underline{R}(\xi)$ ← boundary!

Thm • $\mathbb{L}_{\rho(\xi)} E_I \approx 0 \leftrightarrow \rho(\xi)$ descends to $(\mathcal{F}_{EL}, \omega)$

(N1) → • $\nabla_a \tilde{J}^a \approx 0$ for $\underline{J}(\xi) = \mathbb{i}_{\rho(\xi)} \underline{\theta} - \underline{R}(\xi)$

Noether current

• $\partial\Sigma = \phi$ on $\mathcal{F}_{EL}$ $\quad \mathbb{i}_{\rho(\xi)} \omega \approx - \mathbb{d} Q_\Sigma(\xi) \quad Q_\Sigma := \int_\Sigma \underline{J}$

Ex $\mathfrak{g}$ = Poincaré syms of ...

LOCAL (GAUGE) ...

$\mathcal{G} = (\Gamma(\Xi \to M) ...$

(ξ)

...ndary !

$)$ descends to $(\mathcal{F}_{EL}, \omega)$

$(\xi) = i_{\rho(\xi)}\,\omega - \underline{R}(\xi)$

Noether current

$= -dQ_\Sigma(\xi) \qquad Q_\Sigma := \int_\Sigma \underline{J}$

②

<u>Ex</u> $\mathfrak{g}$ = Poincaré syms of scalar th.

LOCAL (GAUGE) SYMMETRIES

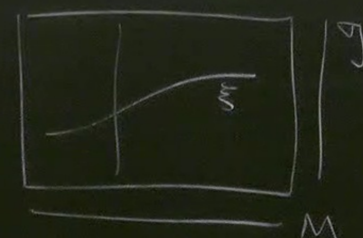
$$\mathcal{G} = \Big(\Gamma(\underbrace{\Xi \longrightarrow M}_{\text{vector bndle}}),\ [\,\cdot,\cdot\,]\Big)$$

infinite dimensional Lie algebra

<u>Ex</u>: ① YM $\quad \Xi \sim M \times \mathfrak{g}$

$$\xi \in \mathcal{G} \sim C^\infty(M, \mathfrak{g})$$

$$[\xi, \eta]_{\mathcal{G}}(x) = [\xi(x), \eta(x)]_{\mathfrak{g}}$$



...of scalar th.

...) SYMMETRIES

$(\ldots M) , [\cdot , \cdot])$

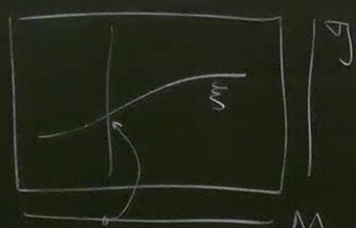
...dle ,

...nd Lie algebra

$\Xi \sim M \times \mathfrak{g}$

$\in \mathcal{G} \sim C^\infty(M, \mathfrak{g})$

$[\xi, \eta]_{\mathcal{G}}(x) = [\xi(x), \eta(x)]_{\mathfrak{g}}$



② $\mathcal{G} \simeq \mathrm{diff}(M)$

$\Xi = TM$

$\xi \in \mathcal{G} = \Gamma(TM) = \mathfrak{X}^1(M)$

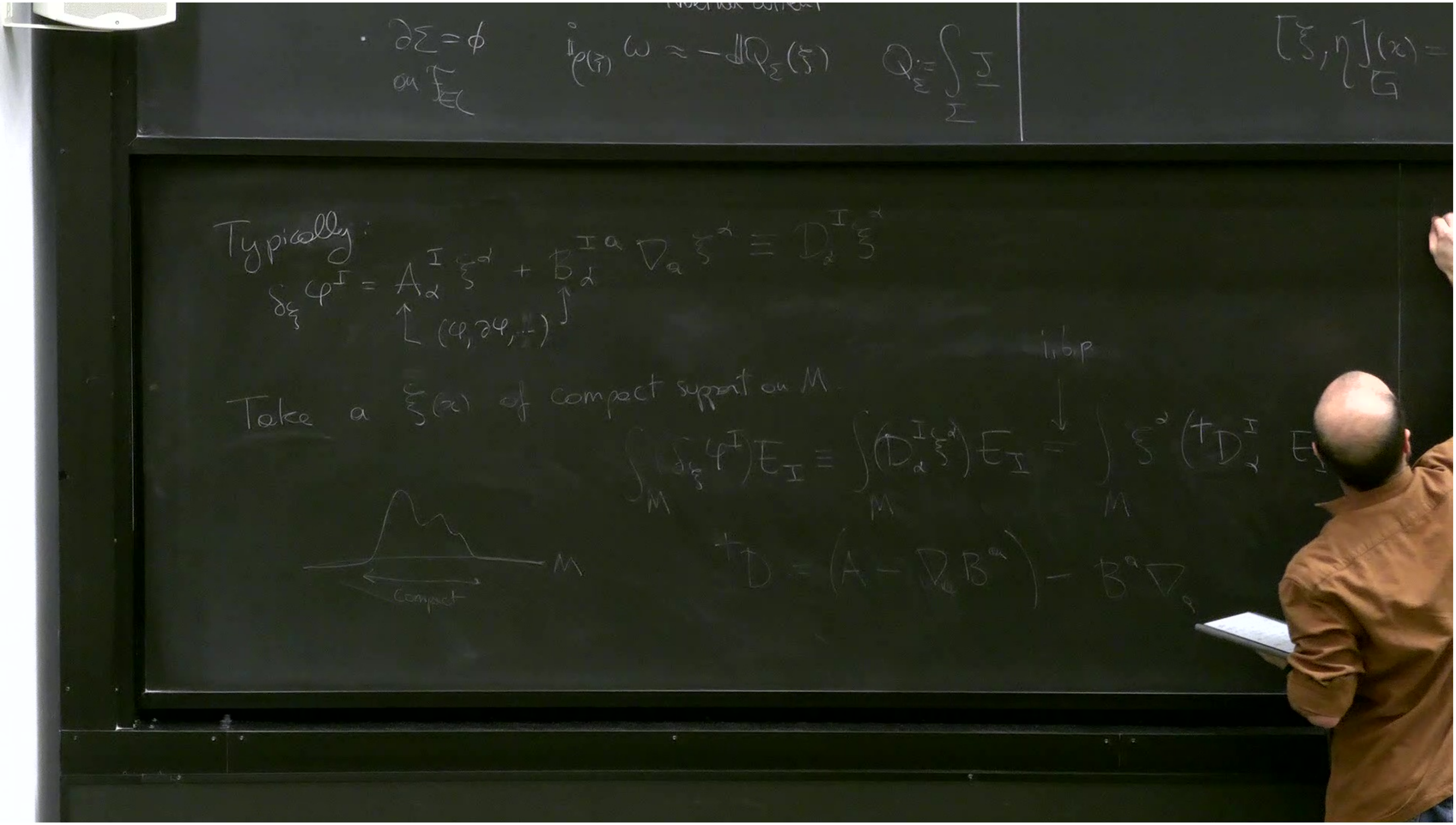
$[\xi, \eta]_{\mathcal{G}} = L_\xi \eta = -L_\eta \xi$

Def (local action)

$\rho : \mathcal{G} \longrightarrow \mathfrak{X}^1(F)$

$\xi \longmapsto \rho(\xi) = \int_M \left(\delta_\xi \varphi^I(x)\right) \frac{\delta}{\delta \varphi^I(x)}$

depend only on ξ, φ and a finite nb of deriv. at x.


$\partial\Sigma = \phi$ on F_{EL}
$i_{\rho(\xi)}\,\omega \approx -dQ_\Sigma(\xi)$
$Q_\Sigma := \int_\Sigma J$
$[\xi, \eta]_G(x) =$
Typically:
$\delta_\xi \varphi^I = A^I_\alpha \xi^\alpha + B^{Ia}_\alpha \nabla_a \xi^\alpha \equiv D^I_\alpha \xi^\alpha$
$(\varphi, \partial\varphi, \ldots)$
Take a $\xi(x)$ of compact support on M
i.b.p
$\int_M (\delta_\xi \varphi^I) E_I \equiv \int_M (D^I_\alpha \xi^\alpha) E_I = \int_M \xi^\alpha ({}^\dagger D^I_\alpha E_I$
M
compact
${}^\dagger D = (A - \nabla_a B^a) - B^a \nabla_a$

G $\{\xi(x), \eta(x)\}_{g}$

on ξ, φ and a finite nb of deriv. at x.

Thm (Noether 2)

If G is a local Lagrangian sym

then $^{\dagger}D^{I}_{\alpha} E_{I} \equiv 0$.

Pf: ξ of cpt support

$$0 \overset{\text{cpt supp.}}{=} \int_M d\, \underline{J}(\xi) \underset{\text{Noether 1}}{=} \int_M (\delta_\xi \varphi^I) E_I = \int \xi^\alpha (^{\dagger}D^{I}_{\alpha} E_I) \quad \forall \xi \Rightarrow {}^{\dagger}D^{I}_{\alpha} E_I \equiv 0 \quad \blacksquare$$

(ξ: arbitrary local function)

Physical interpretation: in the presence of local sym there are LESS eom than DoF. (by $\# \alpha$)

M

compact

$D = (A - \nabla_a B^a) - B^a \nabla_a$

Physical

$\rightarrow$ lose deterministic evolution

UNLESS

we somehow "declare" physically indistinguishable

$\#\alpha$ configurations (related by action of G!)

↳ REDUNDANCY of GAUGE

But I can choose ξ to vanish at Σ_{in}

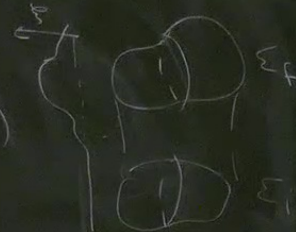
with $\xi|_{\Sigma_f}$ arbitrary!

$\rightarrow Q_{\Sigma_f}(\xi) \approx 0$ ☒

Proposition $\boxed{\partial\Sigma = \phi} \Rightarrow Q_\Sigma(\xi) \approx 0 \quad \forall \xi(x)$

Pf $\quad 0 \approx \int_M d\,\underline{J}(\xi) = Q_{\Sigma_f}(\xi) - Q_{\Sigma_i}(\xi)$

(no cpt support, but onshell + N1)



Physical interpretation. in the presence of local sym there are LESS com than DoF. (by # α)

...e

...t Σ in

...bitrary!

≈ 0

Corollary

From arbitrariness of Σ & ξ

$$Q_\Sigma(\xi) \approx 0 \Rightarrow \underline{J}(\xi) \approx d\underline{j}(\xi)$$

And in fact:

$$\tilde{J}^a(\xi) = \xi^\alpha \tilde{C}^a_\alpha + D_e \tilde{j}^{ab}(\xi)$$

s.t. on shell: $\boxed{\tilde{C}^a_\alpha \approx 0}$ and $j^{ab} \approx j^{[ab]}$

$$\leadsto Q_\Sigma(\xi) = \int_\Sigma \boxed{n_a \tilde{C}^a_\alpha} \xi^\alpha \approx 0$$

$\partial\Sigma = \phi$

Constraints! Ex: $n_a \tilde{C}^a \sim$ Gauss $D_i F^{i0}$

Thm local sym, on $(\mathcal{F}_{EL}, \omega)$, $\partial\Sigma = \phi$

$$\mathring{\mathbb{I}}_{\rho(\xi)}\, \omega \underset{\uparrow}{\approx} - dQ_{\Sigma}(\xi) \approx 0$$

"old thm"

$\Longrightarrow$ the "Cov. Ph. Sp." $(\mathcal{F}_{EL}, \omega)$

has a degenerate sympl. structure

in the "gauge directions"

$$\ker(\omega^{\flat}) \sim \mathrm{Im}(\rho)$$

m, on $(\mathcal{F}_{EL}, \omega)$, $\partial\Sigma = \emptyset$

$- dQ_\Sigma(\xi) \approx 0$

thm"

ov. Ph. Sp." $(\mathcal{F}_{EL}, \omega)$

degenerate sympl structure

e "gauge directions":

$\ker(\omega^\flat) \sim \mathrm{Im}(\rho)$

ienting out gauge fixes

two problems:

1) makes $\mathcal{F}_{EL}/\mathcal{G}$ symplectic

2) makes eom deterministic!

Thm local sym, on $(\mathcal{F}_{EL}, \omega)$, $\partial\Sigma = \phi$

$$\mathbb{I}_{\rho(\xi)}\, \omega \approx - dQ_\Sigma(\xi) \approx 0$$

"old thm"

$\Longrightarrow$ the "Cov. Ph. Sp." $(\mathcal{F}_{EL}, \omega)$ has a degenerate sympl structure in the "gauge directions"

$$\ker(\omega^\flat) \sim \mathrm{Im}(\rho)$$

$\rightsquigarrow$ Quotienting out gauge fixes

two problems:

1) makes $\mathcal{F}_{EL}/\mathcal{G}$ symplectic
2) makes eom deterministic!

Example: parametrized particle

"time reparametrization" is a local Lagr. sym.

$\underline{J}(\xi) = 0$ off shell ☺

General Relativity

$$S = \int \sqrt{g}\left(\frac{1}{2}R - \Lambda\right) d^4x = \int_M \mathcal{L}$$

$\mathcal{F} = \{ g_{ab} \text{ metric on } M \}$

notation:

$$\delta g^{ab} := g^{aa'} g^{bb'} \delta g_{a'b'} \left[= -\delta(g^{ab}) \right]$$

$$\delta g := g^{ab} \delta g_{ab} = 2 \frac{\delta \sqrt{g}}{\sqrt{g}}$$

$G = \text{diff}(M) = \mathfrak{X}^1(M)$

$$\boxed{L_{\rho(\xi)} g_{ab} \equiv \delta_\xi g_{ab}} = \boxed{L_\xi g_{ab} = 2\nabla_{(a}\xi_{b)}}$$

change in field conf. "Active diffeos" ~ Lie transport

$\underline{\mathcal{L}}$

$dg_{a'b'} \left[= - d(g^{ab}) \right]$

$= 2 \frac{d\sqrt{g}}{\sqrt{g}}$

$\boxed{\mathcal{L}_\xi g_{ab} = 2\nabla_{(a}\xi_{b)}}$

"Active diffeo" ~ Lie transport

$[\rho(\xi), \rho(\eta)] = -\rho([\xi,\eta])$ ⚠

PROP $\mathbb{L}_{\rho(\xi)} \underline{\mathcal{L}} = L_\xi \underline{\mathcal{L}}$

BACKGROUND INDEPENDENCE

Counter ex: scalar field on (M, g_{ab})

$\mathcal{F} = C^\infty(M) \ni \phi \qquad \mathcal{L} = \frac{1}{2} g^{ab}\nabla_a\phi\nabla_b\phi\sqrt{g}\,d^4x$

$\mathbb{L}_{\rho(\xi)} g = 0$

$\mathbb{L}_{\rho(\xi)}\phi = L_\xi \phi = \xi^a\nabla_a\phi$

$$\boxed{\mathbb{L}_{\rho(\xi)} g_{ab} \equiv \delta_\xi g_{ab}} = \boxed{\mathcal{L}_\xi g_{ab} = 2\nabla_{(a}\xi_{b)}}$$

change in field conf. "Active diffeos" ~ Lie transport

→ Cor $\mathbb{L}_{\rho(\xi)} \underline{L} = \mathcal{L}_\xi \underline{L} = d(\underbrace{i_\xi \underline{L}}_{R(\xi)})$

↳ diffeos are Lagn sym

$$\tilde{\Theta}^a \sim \nabla \delta g$$

$$\tilde{R}^a = \xi^a L$$

$$\Longrightarrow \tilde{J}^a(\xi) = i_{\rho(\xi)} \tilde{\Theta}^a - \tilde{R}^a(\xi)$$

$$= \tilde{C}^a{}_b \xi^b + \nabla_b \tilde{j}^{ab}(\xi)$$

$$G^a{}_b + \Lambda \delta^a{}_b \approx 0$$

KOMAR 2-CURRENT

$$\boxed{\tilde{j}^{ab} = -\frac{1}{2}\left(\nabla^a \xi^b - \nabla^b \xi^a\right)} \quad \text{or} \quad \underline{j} = -\frac{1}{2} * d\xi^b$$

$B^a) - B^a \nabla_a$

Physical interpretation: in the presence of local sym there are LESS com than DoF. (by # α)

Thm

$$\mathbb{i}_{\rho(\xi)} \Omega = -\mathbb{d} J(\xi) + i_\xi E^{ab}\, \mathbb{d} g_{ab} + d\, i_\xi \Theta$$

Pf: only need bcgrnd indep!

$$\mathbb{L}_{\rho(\xi)} \Theta = \mathbb{i}_{\rho(\xi)} \Omega + \mathbb{d}\, \mathbb{i}_{\rho(\xi)} \Theta$$

$\xrightarrow{\text{b. ind}}$ $\|$

$$L_\xi \Theta = d\, i_\xi \Theta + \boxed{i_\xi\, d\Theta}$$

$R(\xi)$ b. ind

but $i_\xi\, d\Theta = i_\xi\left(-\mathbb{d} L + E\right) = +\mathbb{d}(i_\xi L) + i_\xi E$