Title: Lecture - Quantum Gravity, PHYS 644

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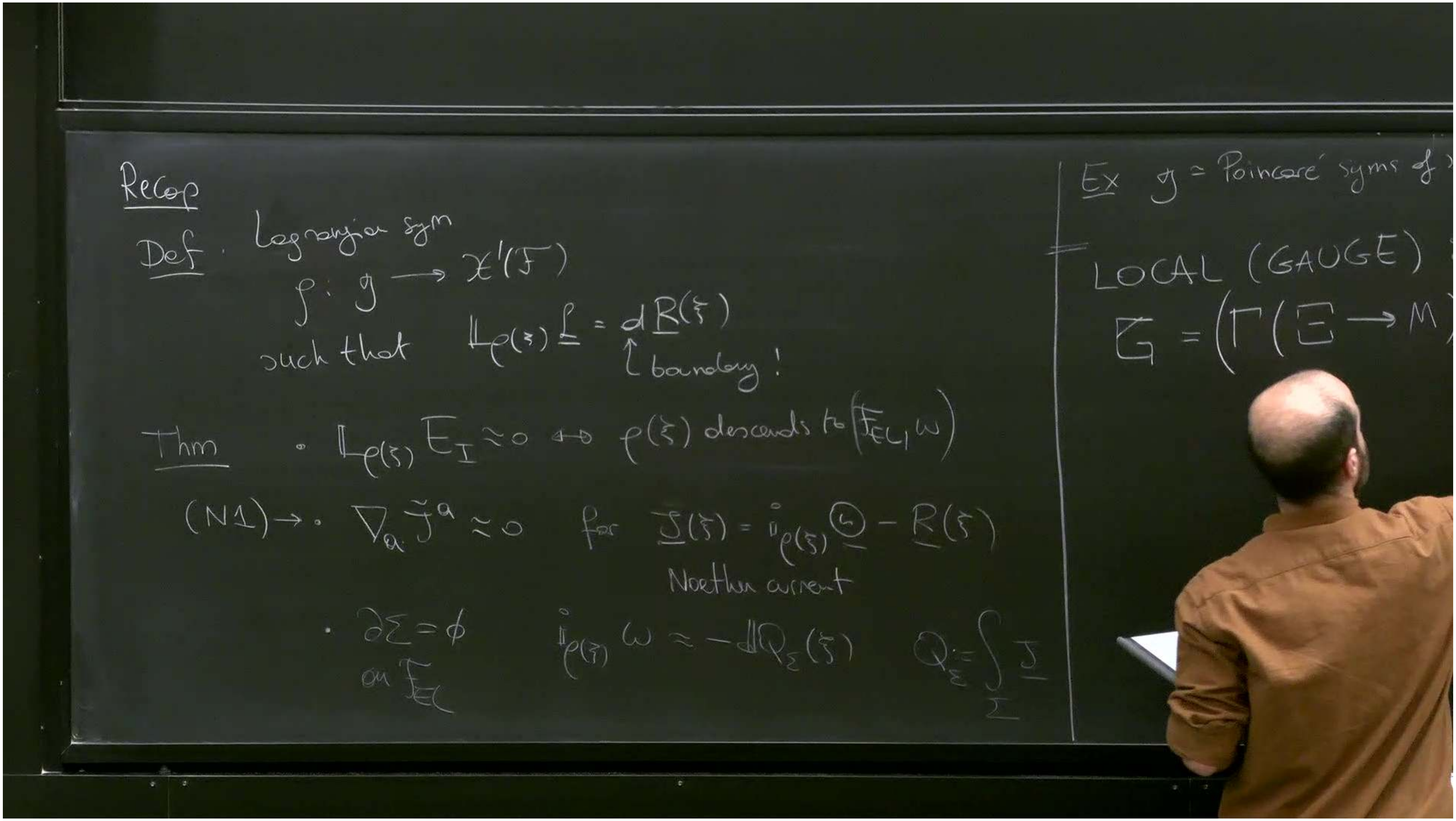


Recap

Def. Lagrangian sym

$\rho : \mathfrak{g} \longrightarrow \mathfrak{X}^1(\mathcal{F})$

such that $\mathbb{L}_{\rho(\xi)} \underline{L} = d\,\underline{R}(\xi)$

$\uparrow$ boundary!

Thm • $\mathbb{L}_{\rho(\xi)} E_I \approx 0 \leftrightarrow \rho(\xi)$ descends to $(\mathcal{F}_{EL}, \omega)$

(N1) → • $\nabla_a \tilde{J}^a \approx 0$ for $\underline{J}(\xi) = \overset{\circ}{\mathbb{i}}_{\rho(\xi)} \underline{\theta} - \underline{R}(\xi)$

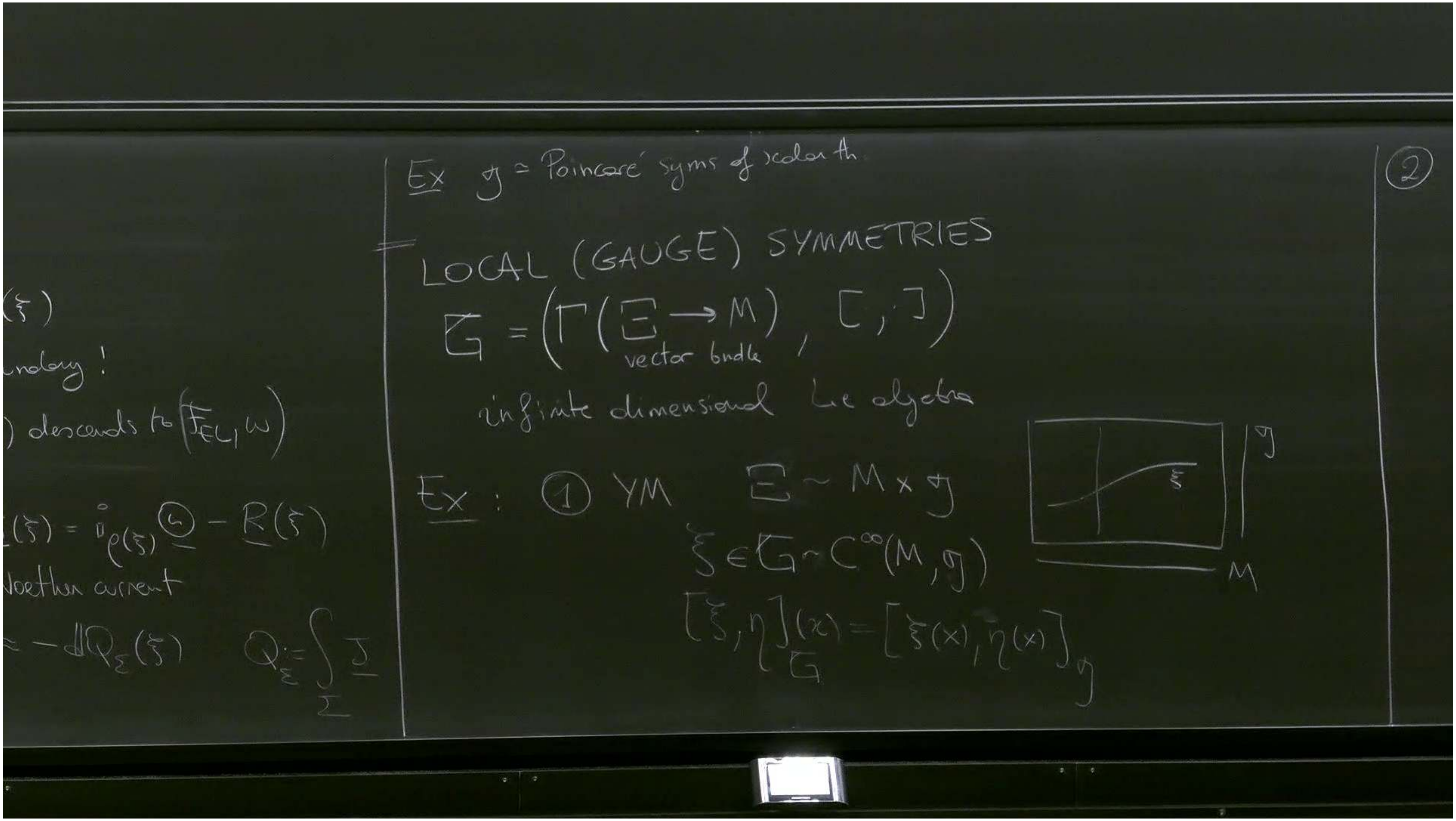
Noether current

• $\partial\Sigma = \phi$ on $\mathcal{F}_{EL}$ $\quad \overset{\circ}{\mathbb{i}}_{\rho(\xi)} \omega \approx - \mathbb{d} Q_\Sigma(\xi)$ $\quad Q_\Sigma := \int_\Sigma \underline{J}$

Ex $\mathfrak{g}$ = Poincaré syms of

LOCAL (GAUGE)

$\mathcal{G} = \Big(\Gamma(\Xi \rightarrow M$



(ξ)

ndary !

$)$ descends to $(\mathcal{F}_{EL}, \omega)$

$(\xi) = i_{\rho(\xi)} \underline{\omega} - \underline{R}(\xi)$

Noether current

$= -dQ_\Sigma(\xi) \quad Q_\Sigma := \int_\Sigma \underline{J}$

<u>Ex</u> $\mathfrak{g} \simeq$ Poincaré syms of scalar th.

LOCAL (GAUGE) SYMMETRIES

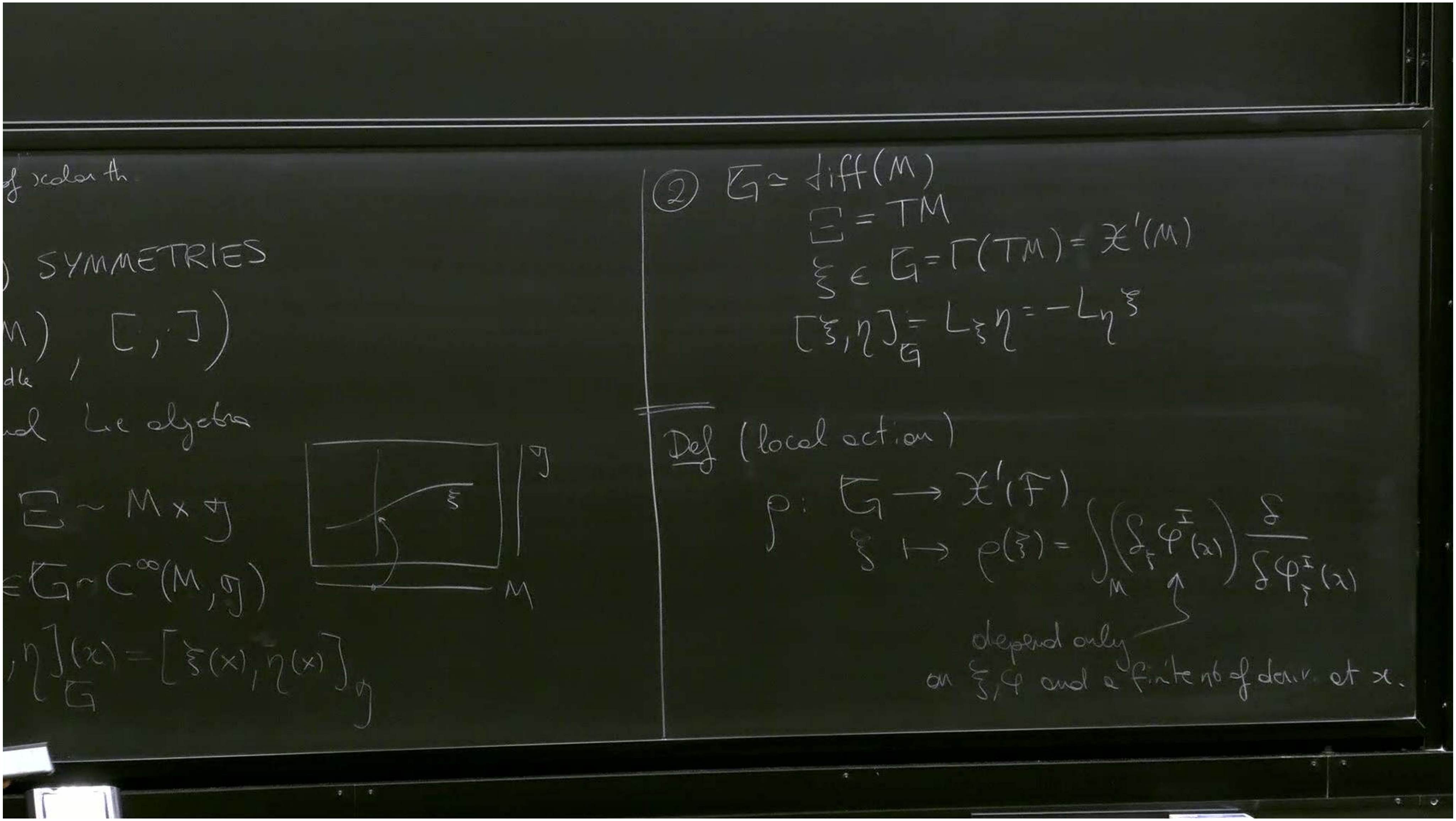
$$\mathfrak{G} = \Big(\Gamma(\underbrace{\Xi \longrightarrow M}_{\text{vector bndle}}), [\,,\,]\Big)$$

infinite dimensional Lie algebra

<u>Ex</u> : ① YM $\quad \Xi \sim M \times \mathfrak{g}$

$$\xi \in \mathfrak{G} \sim C^\infty(M, \mathfrak{g})$$

$$[\xi, \eta]_{\mathfrak{G}}(x) = [\xi(x), \eta(x)]_{\mathfrak{g}}$$



of scalar th.

) SYMMETRIES

$M)$, $[\,,\,]\big)$

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nd Lie algebra

$\Xi \sim M \times \mathfrak{g}$

$\in \mathbb{G} \sim C^{\infty}(M, \mathfrak{g})$

$, \eta]_{\mathbb{G}}(x) = [\xi(x), \eta(x)]_{\mathfrak{g}}$

② $\mathbb{G} \simeq \mathrm{diff}(M)$

$\Xi = TM$

$\xi \in \mathbb{G} = \Gamma(TM) = \mathfrak{X}^{1}(M)$

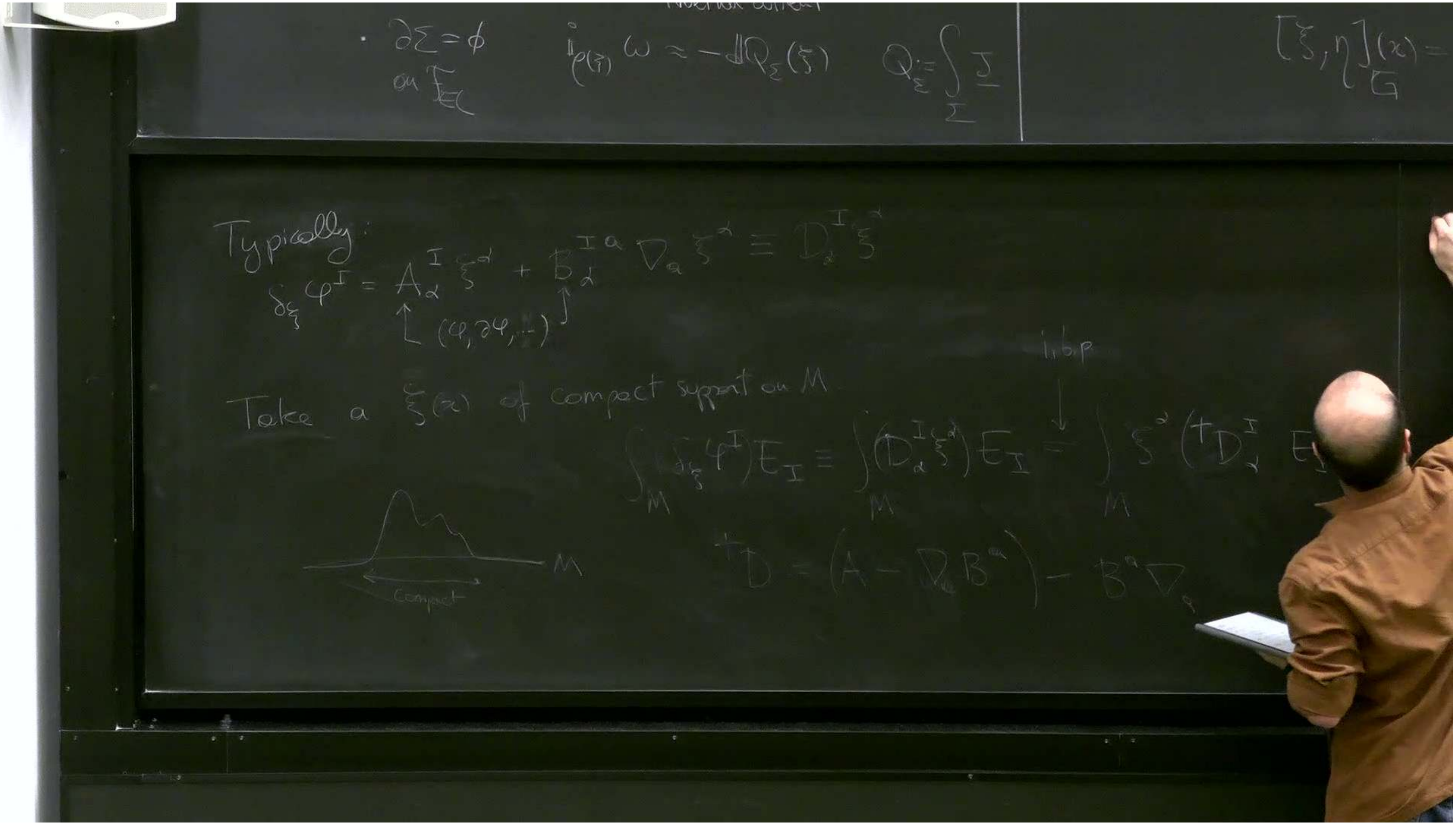
$[\xi, \eta]_{\mathbb{G}} = L_{\xi}\eta = -L_{\eta}\xi$

<u>Def</u> (local action)

$$\rho : \mathbb{G} \longrightarrow \mathfrak{X}^{1}(F)$$

$$\xi \longmapsto \rho(\xi) = \int_M \left(\delta_{\xi} \varphi^{I}(x)\right) \frac{\delta}{\delta \varphi^{I}_{\xi}(x)}$$

depend only on ξ, φ and a finite nb of deriv. at x.



$\cdot\ \partial\Sigma = \phi$ on $\mathcal{EL}$

$$\mathbb{i}_{\rho(\xi)}\,\omega \approx -\mathbb{d} Q_\Sigma(\xi) \qquad Q_\Sigma := \int_\Sigma \underline{J}$$

$$[\xi,\eta]_G(x) =$$

Typically:

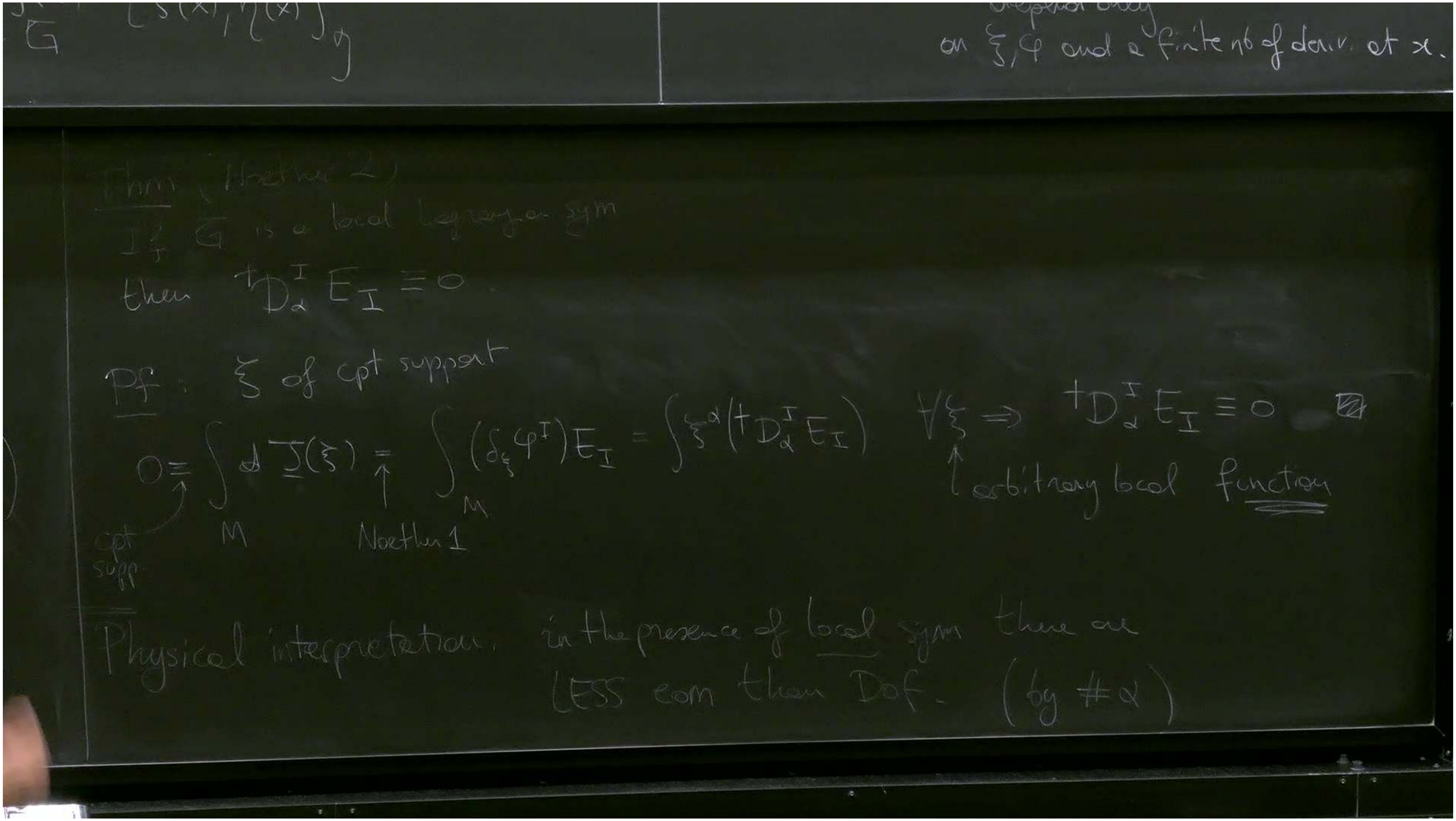
$$\delta_\xi \varphi^I = A^I_\alpha \xi^\alpha + B^{Ia}_\alpha \nabla_a \xi^\alpha \equiv D^I_\alpha \xi^\alpha$$

$$(\varphi, \partial\varphi, \dots)$$

Take a $\xi(x)$ of compact support on M

$$\int_M (\delta_\xi \varphi^I) E_I \equiv \int_M (D^I_\alpha \xi^\alpha) E_I \overset{\text{i.b.p}}{=} \int_M \xi^\alpha ({}^{\dagger}D^I_\alpha\ E_I$$

$$^{\dagger}D = (A - \nabla_a B^a) - B^a \nabla_a$$



$G \quad [\xi(x), \eta(x)]_{\mathfrak{g}}$

on ξ, φ and a finite nb of deriv. at x.

Thm (Noether 2)

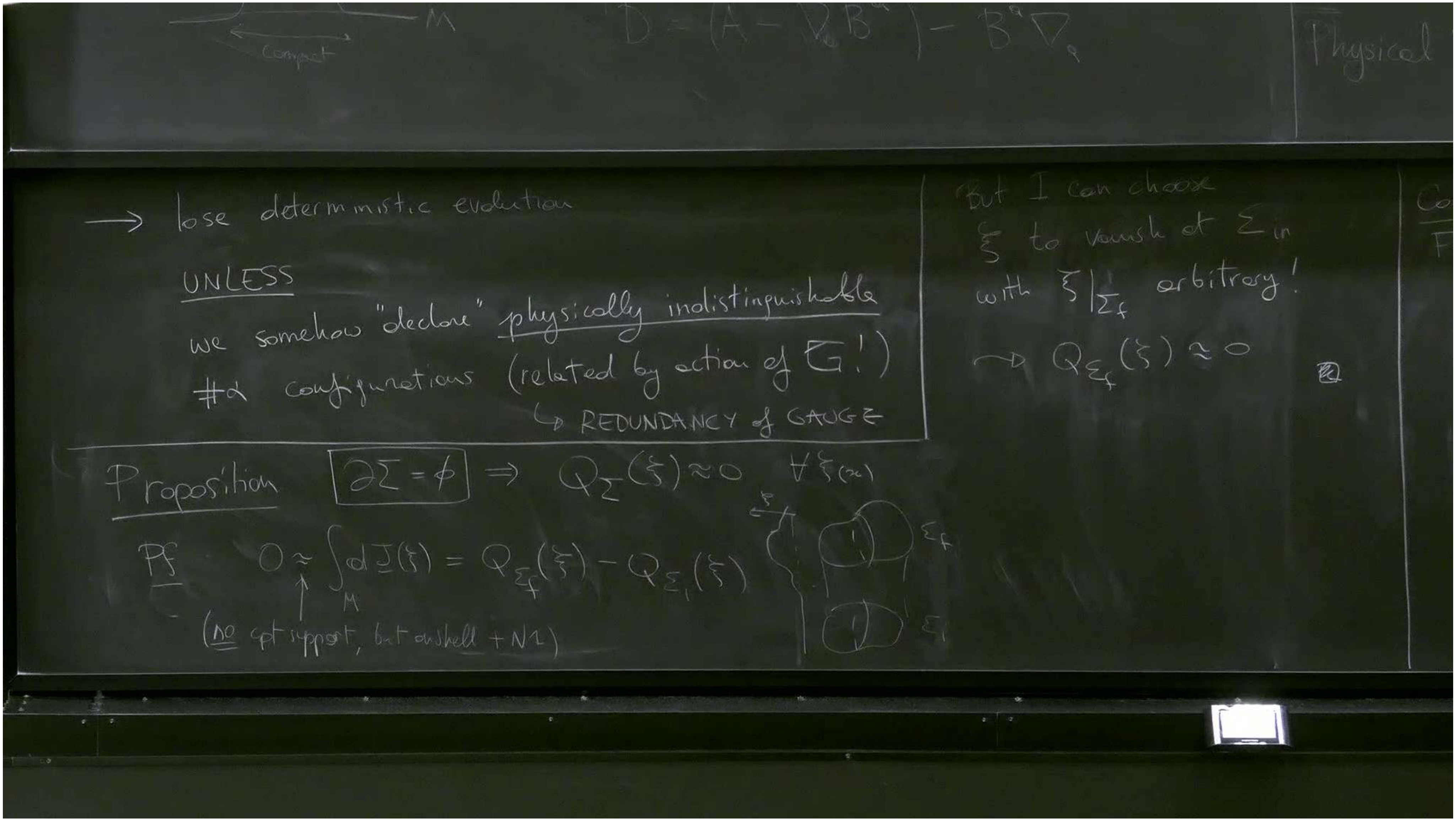
If G is a local Lagrangian sym

then $\;{}^{+}D^{I}_{\alpha} E_{I} \equiv 0$

Pf: ξ of cpt support

$$0 \underset{\text{cpt supp}}{\equiv} \int_M d\,\underline{J}(\xi) \underset{\text{Noether 1}}{\overset{\uparrow}{=}} \int_M (\delta_\xi \varphi^I) E_I = \int \xi^\alpha \left({}^{+}D^{I}_{\alpha} E_I\right) \quad \forall \underset{\text{arbitrary local function}}{\xi} \Rightarrow {}^{+}D^{I}_{\alpha} E_I \equiv 0 \quad ▨$$

Physical interpretation: in the presence of local sym there are
LESS eom than Dof. (by $\#\alpha$)



M

compact

$D = (A - \nabla_a B^a) - B^a \nabla_a$

Physical

→ lose deterministic evolution

UNLESS

we somehow "declare" physically indistinguishable

#α configurations (related by action of $\mathcal{G}$!)

↪ REDUNDANCY of GAUGE

Proposition $\boxed{\partial\Sigma = \phi} \Rightarrow Q_\Sigma(\xi) \approx 0 \quad \forall \xi(x)$

Pf $0 \overset{\uparrow}{\approx} \int_M d\,\underline{J}(\xi) = Q_{\Sigma_f}(\xi) - Q_{\Sigma_i}(\xi)$

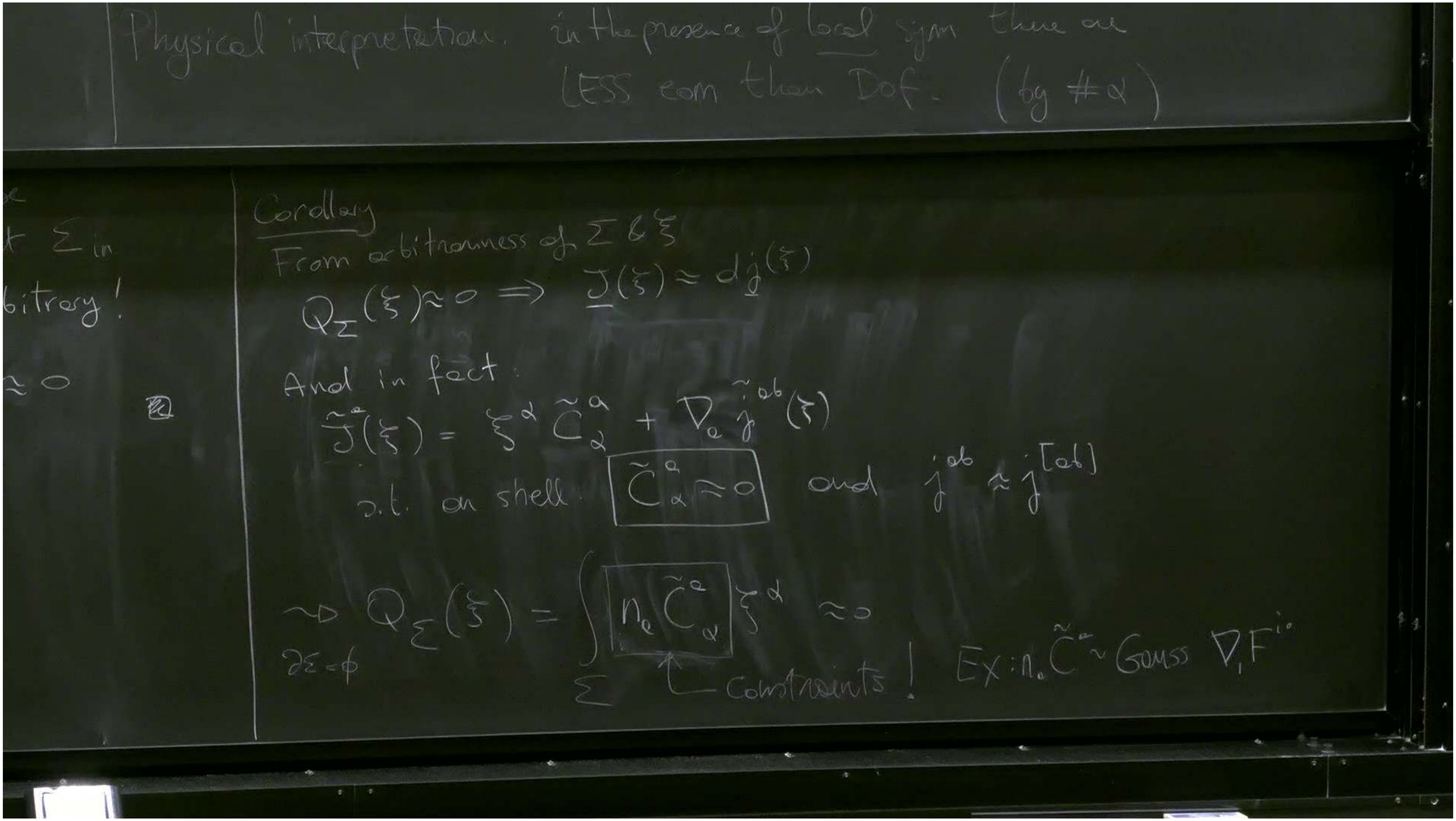
(no cpt support, but on shell + N1)

But I can choose ξ to vanish at Σ_{in} with $\xi|_{\Sigma_f}$ arbitrary!

$\rightarrow Q_{\Sigma_f}(\xi) \approx 0$ ☒

Co

F



Physical interpretation. in the presence of <u>local</u> sym there are LESS com than DoF. (by #α)

e

t Σ in

bitrary!

≈ 0

☒

<u>Corollary</u>

From arbitrariness of Σ & ξ

$$Q_\Sigma(\xi) \approx 0 \Longrightarrow \underline{J}(\xi) \approx d\underline{j}^{(\xi)}$$

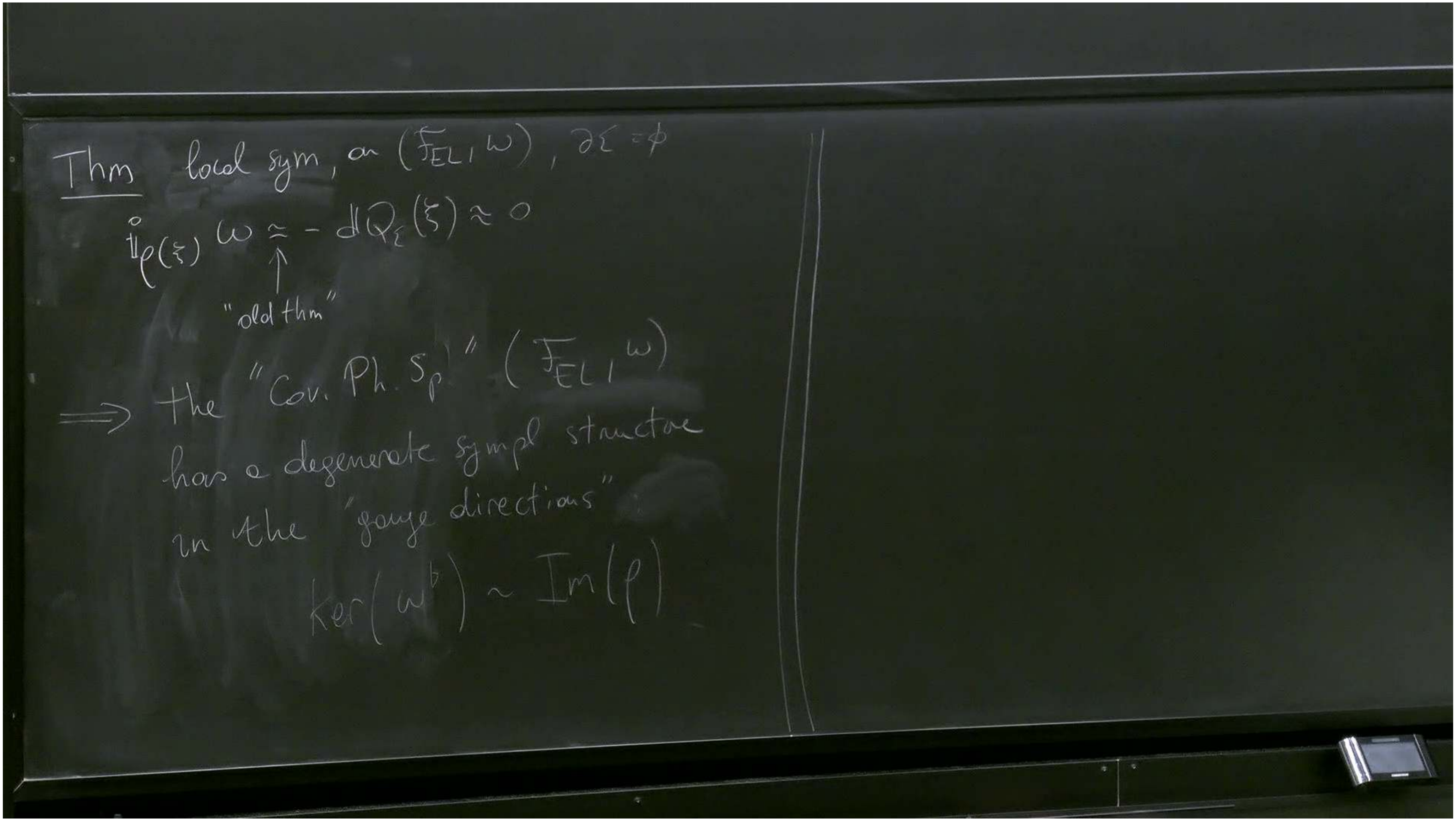
And in fact:

$$\tilde{J}^a(\xi) = \xi^\alpha \tilde{C}^a_\alpha + \nabla_a \tilde{j}^{ab}(\xi)$$

s.t. on shell: $\boxed{\tilde{C}^a_\alpha \approx 0}$ and $j^{ab} \approx j^{[ab]}$

$$\underset{\partial\Sigma = \phi}{\rightsquigarrow} Q_\Sigma(\xi) = \int_\Sigma \boxed{n_a \tilde{C}^a_\alpha} \xi^\alpha \approx 0$$

Constraints! Ex: $n_a \tilde{C}^a \sim$ Gauss $\nabla_i F^{i0}$

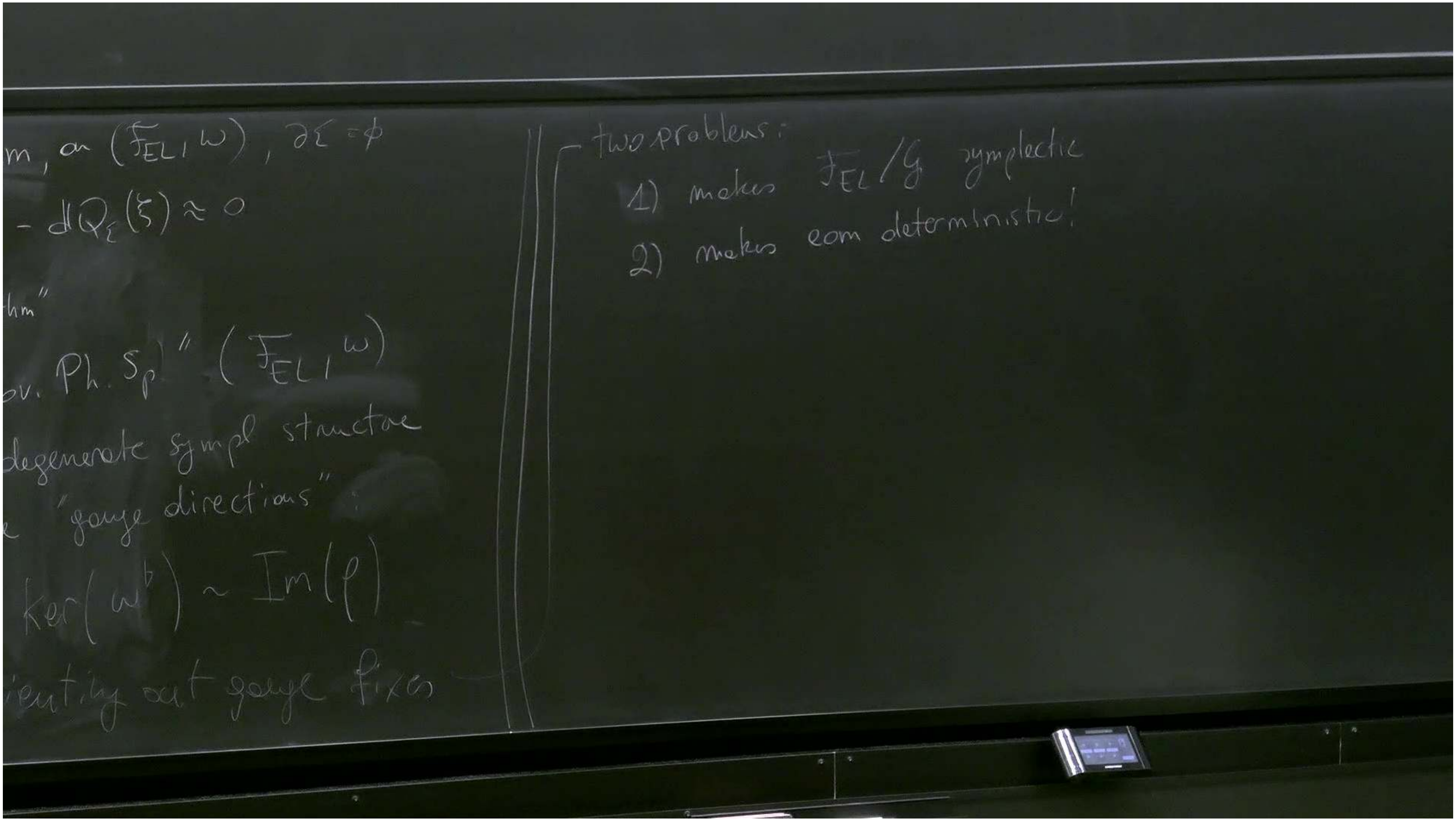


Thm local sym, on $(\mathcal{F}_{EL}, \omega)$, $\partial\Sigma = \phi$

$$\mathring{\mathbb{I}}_{\rho(\xi)}\,\omega \underset{\substack{\uparrow \\ \text{"old thm"}}}{\approx} - dQ_{\Sigma}(\xi) \approx 0$$

$\Longrightarrow$ the "Cov. Ph. Sp." $(\mathcal{F}_{EL}, \omega)$ has a degenerate sympl structure in the "gauge directions"

$$\ker(\omega^{\flat}) \sim \mathrm{Im}(\rho)$$



m, on $(\mathcal{F}_{EL}, \omega)$, $\partial\Sigma = \phi$

$- dQ_\Sigma(\xi) \approx 0$

hm"

ov. Ph. Sp." $(\mathcal{F}_{EL}, \omega)$

degenerate sympl structure

e "gauge directions":

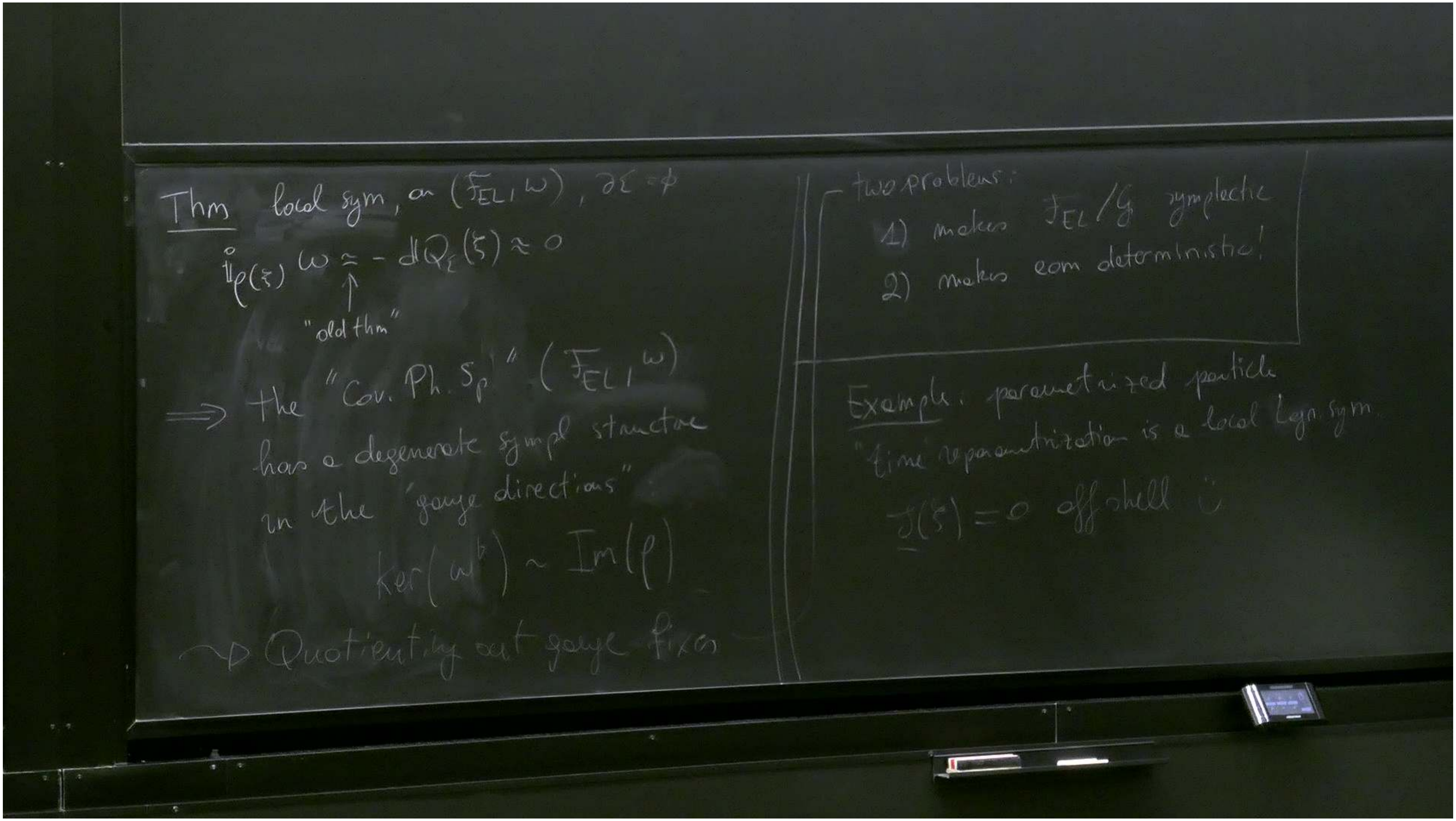
$\ker(\omega^\flat) \sim \mathrm{Im}(\rho)$

ienting out gauge fixes

two problems:

1) makes $\mathcal{F}_{EL}/\mathcal{G}$ symplectic

2) makes eom deterministic!



Thm local sym, on $(\mathcal{F}_{EL}, \omega)$, $\partial\Sigma = \phi$

$$\mathring{\mathbb{I}}_{\rho(\xi)}\,\omega \underset{\substack{\uparrow \\ \text{"old thm"}}}{\simeq} -\,dQ_{\Sigma}(\xi) \approx 0$$

$\Longrightarrow$ the "Cov. Ph. Sp" $(\mathcal{F}_{EL}, \omega)$ has a degenerate sympl structure in the "gauge directions"

$$\ker(\omega^{\flat}) \sim \mathrm{Im}(\rho)$$

$\rightsquigarrow$ Quotienting out gauge fixin

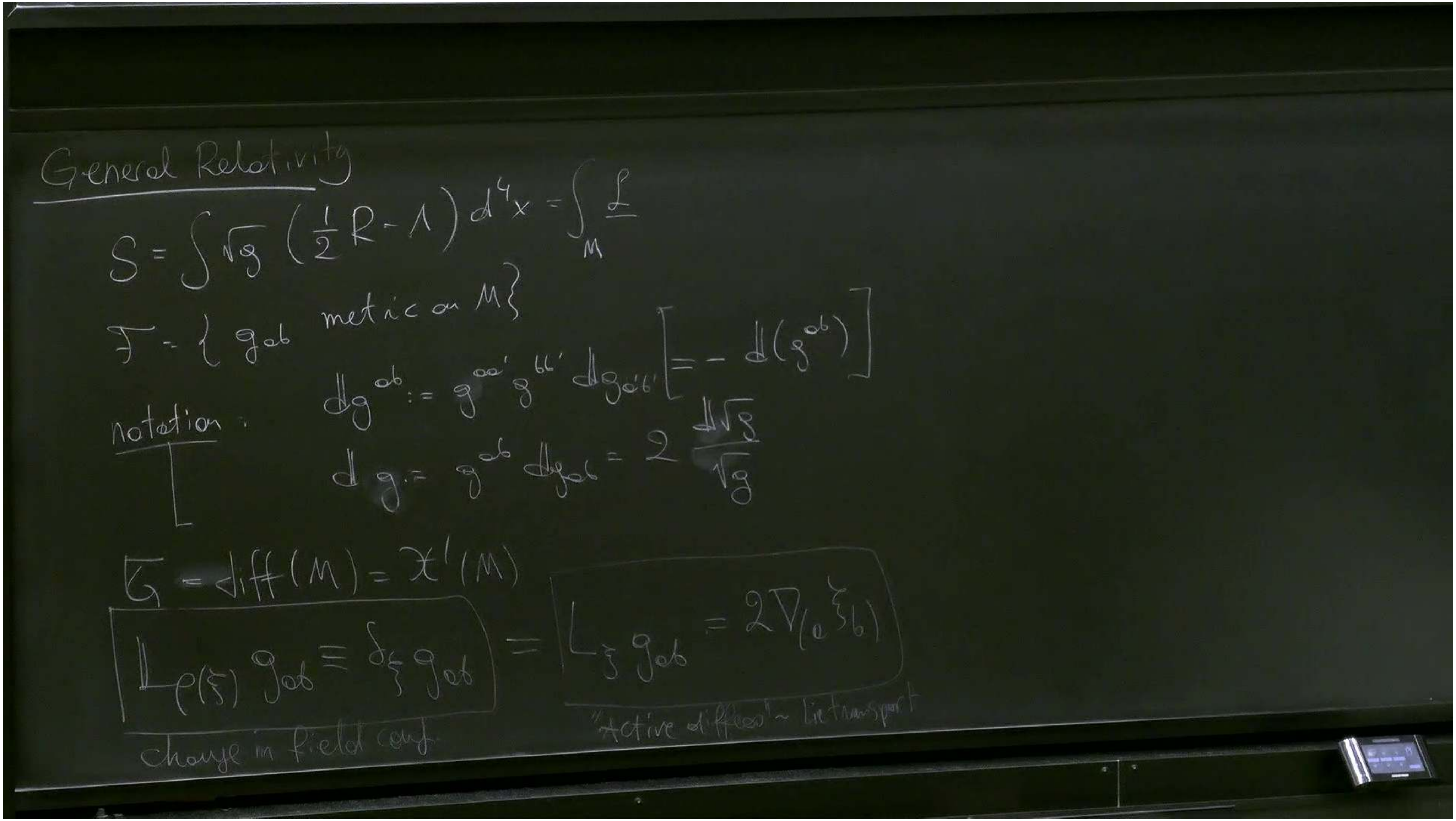
two problems:

1) makes $\mathcal{F}_{EL}/\mathcal{G}$ symplectic
2) makes eom deterministic!

Example: parametrized particle

"time reparametrization" is a local Lagr. sym.

$\underline{J}(\xi) = 0$ off shell ☺



General Relativity

$$S = \int \sqrt{g} \left(\frac{1}{2} R - \Lambda \right) d^4 x = \int_M \mathcal{L}$$

$$\mathcal{F} = \{ g_{ab} \text{ metric on } M \}$$

notation :

$$\mathbb{d} g^{ab} := g^{aa'} g^{bb'} \mathbb{d} g_{a'b'} \left[= - \mathbb{d}(g^{ab}) \right]$$

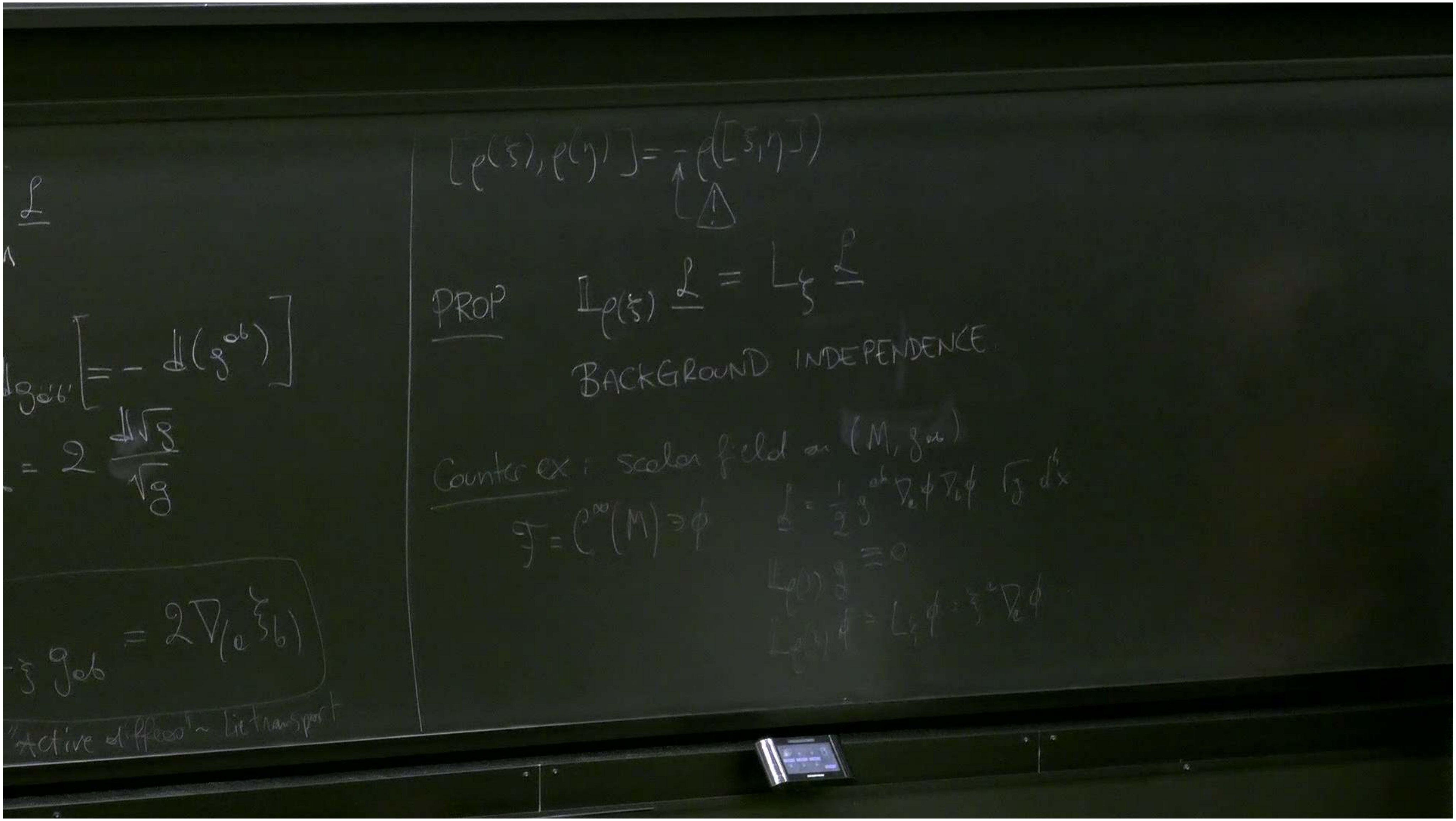
$$\mathbb{d}\, g := g^{ab} \mathbb{d} g_{ab} = 2 \frac{\mathbb{d}\sqrt{g}}{\sqrt{g}}$$

$$\mathcal{G} = \text{diff}(M) = \mathfrak{X}^1(M)$$

$$\boxed{\mathbb{L}_{\rho(\xi)} g_{ab} \equiv \delta_{\xi} g_{ab}} = \boxed{L_{\xi} g_{ab} = 2 \nabla_{(a} \xi_{b)}}$$

change in field conf.

"Active diffeos" ~ Lie transport



$\underline{\mathcal{L}}$

$\mathrm{d}g_{a'b'}\left[= -\,\mathrm{d}(g^{ab})\right]$

$= 2\,\frac{\mathrm{d}\sqrt{g}}{\sqrt{g}}$

$\boxed{\xi\, g_{ab} = 2\nabla_{(a}\xi_{b)}}$

"Active diffeo" ~ Lie transport

$[\rho(\xi), \rho(\eta)] = \overset{\uparrow ⚠}{-}\rho([\xi,\eta])$

PROP $\mathbb{L}_{\rho(\xi)}\underline{\mathcal{L}} = L_\xi\underline{\mathcal{L}}$

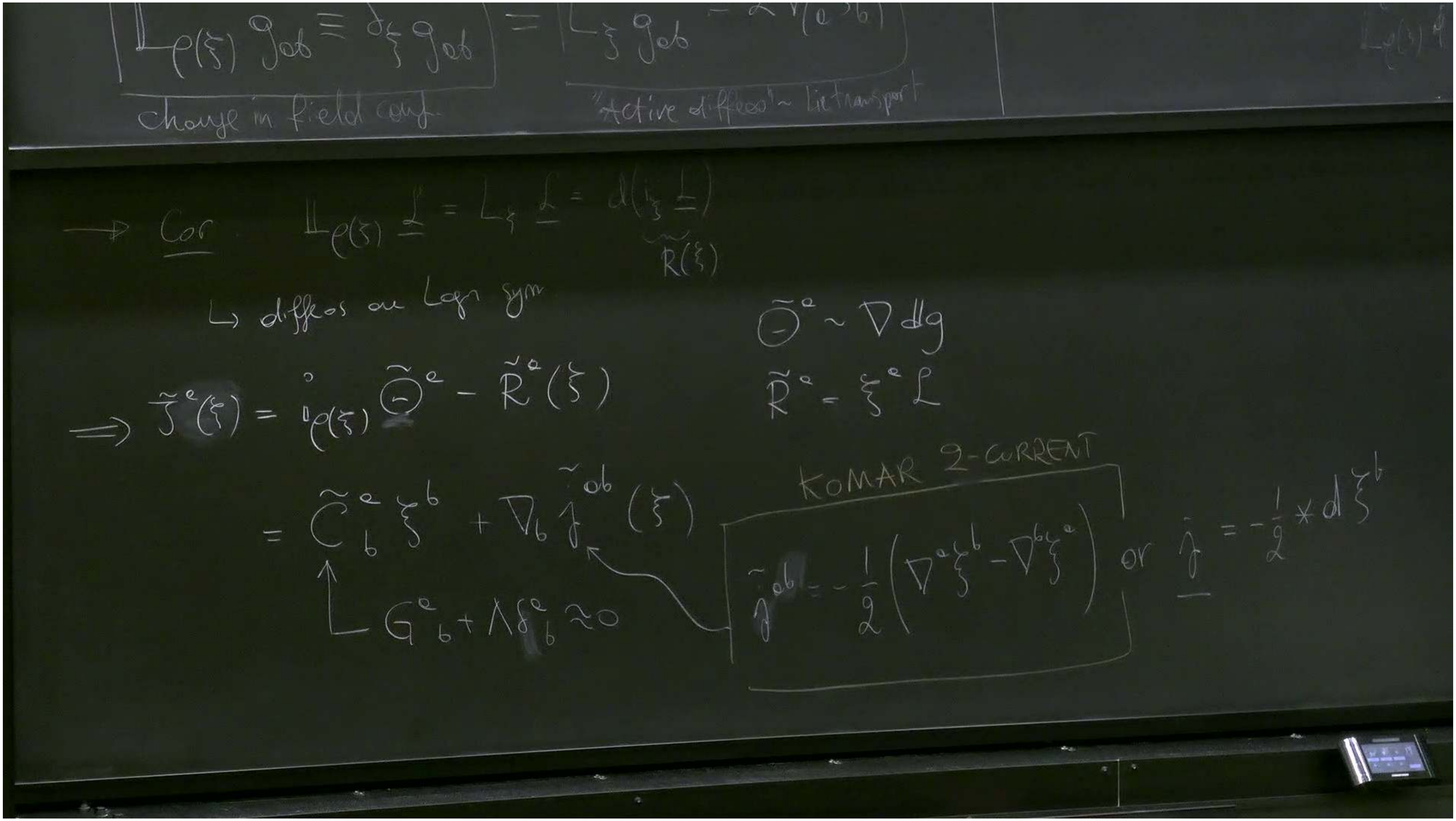
BACKGROUND INDEPENDENCE

Counter ex: scalar field on (M, g_{ab})

$\mathcal{F} = C^\infty(M) \ni \phi$ $\quad \underline{\mathcal{L}} = \frac{1}{2} g^{ab}\nabla_a\phi\nabla_b\phi\sqrt{g}\,d^4x$

$\mathbb{L}_{\rho(\xi)}\, g \equiv 0$

$\mathbb{L}_{\rho(\xi)}\,\phi = L_\xi\phi = \xi^a\nabla_a\phi$



$$\boxed{\mathbb{L}_{\rho(\xi)} g_{ab} \equiv \delta_\xi g_{ab}} = \mathcal{L}_\xi g_{ab} - 2\nabla_{(a}\xi_{b)}$$

change in field conf.

"Active diffeo" ~ Lie transport

$\mathbb{L}_{\rho(\xi)}$

→ Cor $\mathbb{L}_{\rho(\xi)} \underline{L} = \mathcal{L}_\xi \underline{L} = d(\underbrace{i_\xi \underline{L}}_{R(\xi)})$

↳ diffeos are Lagn sym

$\tilde{\Theta}^a \sim \nabla \delta g$

$\Longrightarrow \tilde{J}^a(\xi) = \mathbb{I}_{\rho(\xi)} \tilde{\Theta}^a - \tilde{R}^a(\xi)$

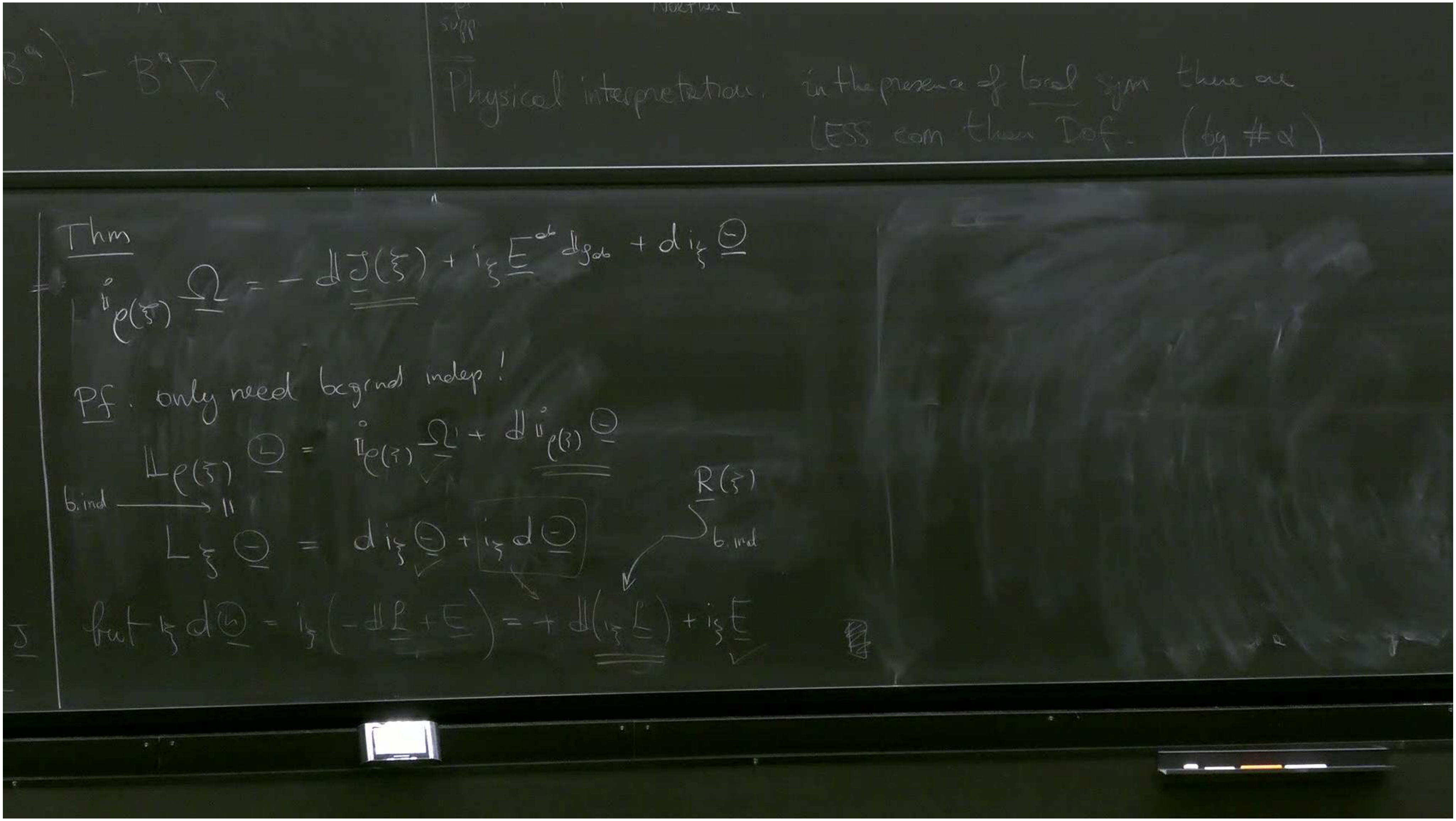
$\tilde{R}^a = \xi^a L$

$= \tilde{C}^a{}_b \xi^b + \nabla_b \tilde{j}^{ab}(\xi)$

$\quad G^a{}_b + \Lambda \delta^a{}_b \approx 0$

KOMAR 2-CURRENT

$$\boxed{\tilde{j}^{ab} = -\frac{1}{2}\left(\nabla^a \xi^b - \nabla^b \xi^a\right)} \text{ or } \underline{j} = -\frac{1}{2} * d\xi^b$$



Noether I

supp

$B^a) - B^a \nabla_a$

Physical interpretation. in the presence of local sym there are LESS com than DoF. (by # α)

Thm

$$\overset{\circ}{\mathbb{I}}_{\rho(\xi)} \underline{\Omega} = - \mathbb{d} \underline{\underline{\underline{J}(\xi)}} + i_\xi \underline{E}^{ab} \, \mathbb{d} g_{ab} + d i_\xi \underline{\Theta}$$

Pf. only need bcgrnd indep!

$$\mathbb{L}_{\rho(\xi)} \underline{\Theta} = \overset{\circ}{\mathbb{I}}_{\rho(\xi)} \underline{\Omega} + \mathbb{d} \underline{\underline{\overset{\circ}{\mathbb{I}}_{\rho(\xi)} \underline{\Theta}}}$$

b. ind $\longrightarrow$ $\|$

$$L_\xi \underline{\Theta} = d i_\xi \underline{\Theta} + \boxed{i_\xi d \underline{\Theta}}$$

$\underline{R}(\xi)$ b. ind

$$\text{but } i_\xi d \underline{\Theta} = i_\xi \left(- \mathbb{d} \underline{L} + \underline{E} \right) = + \mathbb{d} \underline{\underline{(i_\xi \underline{L})}} + i_\xi \underline{E}$$

$\underline{J}$