Title: Lecture - Quantum Gravity, PHYS 644

Speakers: Aldo Riello

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Subject: Quantum Gravity

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Recap

$$\mathbb{d}\underline{L} = \underline{E}_I \, \mathbb{d}\varphi^I - d\,\underline{\Theta}$$

↑ Eul-Lag eom ↑ presympl pot current

DEF $\rho : \mathfrak{g} \xrightarrow{\simeq} \mathfrak{X}^1(\mathcal{F})$ Lagrangian sym. if

$\exists R : \mathfrak{g} \to \Omega^{top-1,0}(M \times \mathcal{F})$: $\mathbb{L}_{\rho(\xi)} \underline{L} = d\,\underline{R}(\xi)$ ← off shell

↑ $\boxed{\text{tot deriv.}}$

Def: Noether current: $\langle \underline{J}, \xi \rangle := \mathbb{i}_{\rho(\xi)} \underline{\Theta} - \underline{R}(\xi)$

↑ $\in \Omega^{top-1,0}(M \times \mathcal{F}, \mathfrak{g}^*)$

Def: CPS $(\mathcal{F}_{EL}, \omega)$ $\omega = \iota^*_{EL} \int_\Sigma \mathbb{d}\underline{\Theta}$

THM: ω indep of Σ.

THM

$\mathbb{L}_{\rho(\xi)} \underline{E}_I \approx 0$ Sym. descends to the shell

(iff $\delta_\xi \varphi^I$ solves linearized eom)

THM (Noether 1)

$d\,\underline{J}(\xi) \approx 0 \quad \forall \xi \in \mathfrak{g}$.

Corollary: $Q_\Sigma(\xi) := \int_\Sigma \underline{J}(\xi)$ indep of Σ onshell

↑ Noether charge.

THM $\partial\Sigma = \phi$, then over $(\mathcal{F}_{EL}, \omega)$

$\mathbb{i}_{\rho(\xi)} \omega \approx -\mathbb{d} Q_\Sigma(\xi)$ & Q_Σ unambiguous

No ambiguities

ρ

onshell

nonambiguous

Pf :

$$\mathbb{L}_\rho \, d\!\!I \underline{\mathcal{L}} = \mathbb{L}_\rho \underline{E} - d\, \mathbb{L}_\rho \underline{\Theta}$$

$$= \mathbb{L}_\rho \underline{E} - d\Big(\underline{\underline{d\!\!I\, \mathring{\mathbb{I}}_\rho \underline{\Theta}}} + \mathring{\mathbb{I}}_\rho \underbrace{d\!\!I \underline{\Theta}}_{\underline{\Omega}}\Big)$$

$$= d\!\!I\, \mathbb{L}_\rho \underline{\mathcal{L}} \underset{\uparrow \text{hyp}}{=} d\!\!I\, d\, \underline{R} = - d\, \underline{\underline{d\!\!I \underline{R}}}$$

$$\Rightarrow d\Big(\underbrace{\mathring{\mathbb{I}}_\rho \underline{\Omega} + d\!\!I \underline{\underline{J}}}_{\underline{\alpha}}\Big) = \mathbb{L}_\rho \underline{E} \approx 0 .$$

$\uparrow$ previous thm.

$\Rightarrow$ by technical lemma $\left(\underline{\alpha} \in \Omega^{top-k,\ell} \quad k,\ell \geq 1 , \; d\underline{\alpha} \approx 0 \Rightarrow \underline{\alpha} \approx d\underline{\beta} \right)$

$$\exists\, \underline{S} \in \Omega^{top-2,1}(M \times \mathcal{F}, \mathfrak{g}^*) \; : \; \boxed{\mathring{\mathbb{I}}_\rho \underline{\Omega} \approx - d\!\!I \underline{J} + d\underline{S}}$$

$\Rightarrow$ if $\boxed{\partial\Sigma = \phi}$: integrating $\rightsquigarrow$ $\mathring{\mathbb{I}}_\rho \omega \approx - d\!\!I Q$ $\blacksquare$

No ambiguities:

$$\underline{L} \mapsto \underline{L} + d\underline{\ell}$$

$$\underline{\Theta} \mapsto \underline{\Theta} + \mathbb{d}\underline{\ell} + d\underline{\vartheta}$$

$$\underline{R} \mapsto \underline{R} + \mathbb{I}_\rho \underline{\ell} + d\underline{r}$$

$$\underline{\Omega} \mapsto \underline{\Omega} + d\,\mathbb{d}\underline{\vartheta}$$

$$\omega \mapsto \omega \qquad (\partial\Sigma = \emptyset)$$

$$\mathbb{i}_\rho \underline{\Theta} - \underline{R} = \underline{J} \mapsto \underline{J} + \mathbb{i}_\rho(\mathbb{d}\underline{\ell} + d\underline{\vartheta}) - \mathbb{I}_\rho \underline{\ell} - d\underline{r}$$

$$= \underline{J} + d(\mathbb{i}_\rho \underline{\vartheta} - \underline{r})$$

$$d\underset{\sim}{J} \approx 0 \mapsto d\underline{J} \approx 0$$

$$Q \mapsto Q \qquad (\partial\Sigma = \emptyset)$$

Corollary

if $(\mathcal{F}_{EL}, \omega)$ sympl, then:

$$\{Q(\xi), Q(\eta)\} = Q([\xi,\eta]) + \underset{\text{cocycle}}{K(\xi,\eta)}$$

Example $\quad \underline{L} = -\frac{1}{2}(\nabla_a \varphi \nabla^a \varphi)\, \underline{\epsilon}$

$\underline{E} = d\varphi\, \Box \varphi\, \underline{\epsilon}\,, \quad \tilde{\Theta}^a = -d\varphi (\nabla^a \varphi)$

• Sym 1 : $\quad \varphi(x) \mapsto \varphi(x) - \underset{\text{const}}{\xi}$

$\rho : \mathbb{R} \to \mathfrak{X}^1(\mathcal{F})\,, \quad \rho(\xi)\varphi(x) \equiv \delta_\xi \varphi(x) = -\xi$

Pf:

$$\mathbb{L}_\rho \,\mathrm{d}\underline{\mathcal{L}} = \mathbb{L}_\rho \underline{E} - \mathrm{d}\,\mathbb{L}_\rho \underline{\Theta}$$

$$= \mathbb{L}_\rho \underline{E} - \mathrm{d}\big(\mathrm{d}\,\mathbb{i}_\rho \underline{\Theta} + \mathbb{i}_\rho \underbrace{\mathrm{d}\underline{\Theta}}_{\underline{\Omega}}\big)$$

$$= \mathrm{d}\,\mathbb{L}_\rho \underline{\mathcal{L}} \underset{\uparrow \text{hyp}}{=} \mathrm{d}\,\mathrm{d}\underline{R} = -\mathrm{d}\,\mathrm{d}\underline{R}$$

$$\to \mathrm{d}\big(\underbrace{\mathbb{i}_\rho \underline{\Omega} + \mathrm{d}\underline{J}}_{\underline{\alpha}}\big) = \mathbb{L}_\rho \underline{E} \underset{\uparrow\text{ previous thm.}}{\approx} 0.$$

$\Rightarrow$ by technical lemma $\left(\underline{\alpha} \in \Omega^{\text{top}-k,\ell} \quad k,\ell \geq 1, \ \mathrm{d}\underline{\alpha} \approx 0 \Rightarrow \underline{\alpha} \approx \mathrm{d}\underline{\beta}\right)$

$$\exists\, \underline{S} \in \Omega^{\text{top}-2,1}(M\times\mathcal{F}, \mathfrak{g}^*) \ : \ \boxed{\mathbb{i}_\rho \underline{\Omega} \approx -\mathrm{d}\underline{J} + \mathrm{d}\underline{S}}$$

$$\Rightarrow \text{if } \boxed{\partial\Sigma = \phi} : \text{integrating} \to \mathbb{i}_\rho \omega \approx -\mathrm{d}Q \qquad \blacksquare$$

PROP ρ Lag. sym s.t.

$$\mathbb{L}_\rho \underline{\mathcal{L}} = 0 \quad \& \quad \mathbb{L}_\rho \underline{\Theta} = 0$$

off shell w/out bdry obstruction

then: $\underline{J} = \mathbb{i}_\rho \underline{\Theta} \quad \& \quad \mathbb{i}_\rho \underline{\Omega} = -\mathrm{d}\underline{J}$

Pf $0 = \mathbb{L}_\rho \Theta = \mathbb{i}_\rho \underbrace{\mathrm{d}\Theta}_{\Omega} + \mathrm{d}\underbrace{\mathbb{i}_\rho \Theta}_{J} \quad \blacksquare$

"Best case scenario"

Corollary

if $(\mathcal{F}_{EL}, \omega)$ sympl, then:

$$\{Q(\xi), Q(\eta)\} = Q([\xi,\eta]) + K(\xi,\eta)$$

cocycle.

Example

$$\underline{L} = -\frac{1}{2}(\nabla_a\varphi\nabla^a\varphi)\,\underline{\epsilon}$$

$$\underline{E} = \mathbb{d}\varphi\,\Box\varphi\,\underline{\epsilon}\,, \quad \tilde{\Theta}^a = -\mathbb{d}\varphi(\nabla^a\varphi)$$

• Sym 1: $\varphi(x) \mapsto \varphi(x) - \xi$ ← const

$$\rho: \mathbb{R} \to \mathfrak{X}'(\mathcal{F})\,, \quad \rho(\xi)\varphi(x) \equiv \delta_\xi\varphi(x) = -\xi$$

$$\mathbb{L}_\rho \underline{L} = 0 \quad \& \quad \mathbb{L}_{\rho(\xi)}\tilde{\Theta}^a = -\underbrace{(\mathbb{d}\xi)}_{=0}(\nabla^a\xi) - \mathbb{d}\varphi\underbrace{(\nabla^a\xi)}_{=0} = 0$$

$\underline{L} - d\underline{r}$

$\tilde{\mathcal{J}}$

Corollary

if $(\mathcal{F}_{EL}, \omega)$ sympl, then:

$$\{Q(\xi), Q(\eta)\} = Q([\xi,\eta]) + \underbrace{K(\xi,\eta)}_{\text{cocycle.}}$$

Example $\quad \underline{L} = -\frac{1}{2}(\nabla_a\varphi \nabla^a\varphi)\, \underline{\epsilon}$

$$\underline{E} = \mathbb{d}\varphi\, \Box\varphi\, \underline{\epsilon}\,, \quad \tilde{\Theta}^a = -\mathbb{d}\varphi(\nabla^a\varphi)$$

• Sym 1 : $\quad \varphi(x) \mapsto \varphi(x) - \xi \quad$ ← const.

$$\rho : \mathbb{R} \to \mathfrak{X}^1(\mathcal{F}), \quad \rho(\xi)\varphi(x) \equiv \delta_\xi \varphi(x) = -\xi$$

$$\mathbb{L}_\rho \underline{L} = 0 \quad \& \quad \mathbb{L}_{\rho(\xi)}\tilde{\Theta}^a = -\underbrace{(\mathbb{d}\xi)}_{=0}(\nabla^a\xi) - \mathbb{d}\varphi\underbrace{(\nabla^a\xi)}_{=0} = 0$$

$\underline{L} - d\underline{r}$

Corollary

if (F_{EL}, ω) sympl, then:

cocycle

$$\{Q(\xi), Q(\eta)\} = Q([\xi,\eta]) + K(\xi,\eta)$$

Example $\underline{L} = -\frac{1}{2}(\nabla_a\varphi\nabla^a\varphi)\,\underline{\epsilon}$ $\sqrt{g}\,d^4x$

$\underline{E}_I \,d\!\!\!/\varphi = d\!\!\!/\varphi\,\Box\varphi\,\underline{\epsilon}$, $\tilde{\Theta}^a = -d\!\!\!/\varphi(\nabla^a\varphi)$

- Sym 1 : $\varphi(x) \mapsto \varphi(x) - \xi$ const

$\rho : \mathbb{R} \to \mathfrak{X}^1(F)$, $\rho(\xi)\varphi(x) \equiv \delta_\xi\varphi(x) = -\xi$

$$\mathbb{L}_\rho \underline{L} = 0 \;\;\&\;\; \mathbb{L}_{\rho(\eta)}\tilde{\Theta}^a = -\underset{\underset{0}{\parallel}}{(d\!\!\!/\xi)}(\nabla^a\xi) - d\!\!\!/\varphi\underset{=0}{(\nabla^a\xi)} = 0$$

$\tilde{J}(\xi) = I_{\rho(\xi)}\tilde{\Theta}$

$dJ \approx 0$ since ...

rmk : best case ...

- sym 2   spacet... (i...

e.g. if (M,g) is Mi...

$\mathbb{L}_{\rho(\xi)}\underline{L}$

$\tilde{J}^a(\xi) = \mathbb{I}_{\rho(\xi)} \tilde{\Theta}^a = -\delta_\xi \varphi (\nabla^a \varphi) = \xi \nabla^a \varphi$

$dJ \approx 0$ since $\nabla_a \tilde{J}^a = \xi \Box \varphi \approx 0$.

<u>rmk</u> : best case scenario.

- <u>sym 2</u> spacetime sym. $\chi \in \mathfrak{X}^1(M)$ s.t. $L_\chi g_{ab} = 0$ (killing)
  (isometries)

e.g. if (M, g) is Mink $\rightarrow$ $\mathfrak{g} \simeq$ Poincaré algebra.

$$\mathbb{L}_{\rho(\chi)} \underline{L} = \dots = \nabla_b \Big(\underbrace{-\tfrac{1}{2} \chi^b \nabla\varphi \cdot \nabla\varphi}_{\tilde{R}^b(\chi)} \Big) \quad , \quad \rho(\chi)\varphi = \chi^b \nabla_b \varphi$$

$$\tilde{J}^a(\chi) = \chi^b t^a{}_b \quad , \quad t^a{}_b = \underbrace{-\nabla^a\varphi \nabla_b \varphi}_{\mathbb{I}_\rho \Theta} + \overbrace{\tfrac{1}{2} \delta^a_b \nabla\varphi \cdot \nabla\varphi}^{R}$$

$\varphi(x) = -\xi$

$\xi) = 0$

$=0$

E.M. tensor.

$\mathbb{1}_p \Theta$

$L_\chi g_{ab} = \nabla_a \chi_b + \nabla_b \chi_a = 0$

$\chi \in \mathcal{X}^1(M)$

$\rho(\chi)\varphi(x) = \chi^a \nabla_a \varphi(x)$ $\leftarrow$ translates φ

(rmk $\rho(\chi) g \equiv 0$ b.c. g_{ab} not part of $\mathcal{F}$!)

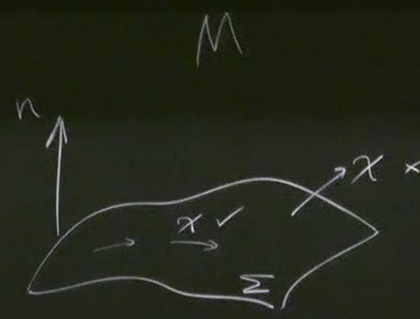
$\hookrightarrow d\tilde{J} = 0 \mapsto dJ = 0$

$\tilde{Q} \mapsto Q \quad (\partial\Sigma = \emptyset)$

$\mathbb{L}_\rho \underline{L} = 0$ & $\mathbb{L}_{\rho(\xi)} \tilde{\Theta}^a = -(\mathbb{d}\xi)(\nabla^a \xi) - \dots$

$$\mathbb{I}_{\rho(\chi)} \tilde{\Omega}^a = -\mathbb{d}\tilde{J}^a(\chi) + \tilde{S}^a(\chi)$$

$$\hookrightarrow -\mathbb{d}\varphi\, \chi^a \underbrace{\Box\varphi}_{\text{eom.}} - \underbrace{\nabla_b \tilde{S}^{[ba]}(\chi)}_{\sim d\tilde{S}}$$



integrate over Σ, $\underline{\partial\Sigma = \emptyset}$

$$\mathbb{I}_{\rho(\chi)} \Omega_\Sigma = -\mathbb{d} Q_\Sigma(\chi) - \int_\Sigma \mathbb{d}\varphi\, (n_a \chi^a)\, \Box\varphi + \underline{\text{zero}}$$

<u>rmk</u>: χ is an "off shell sym" if $n_a \chi^a = 0$ i.e. if $\chi^a \parallel \Sigma$

<u>counterex</u>: time translation always transverse!

E.M. tensor.

$\tilde{S}^{[ba]}(\chi)$

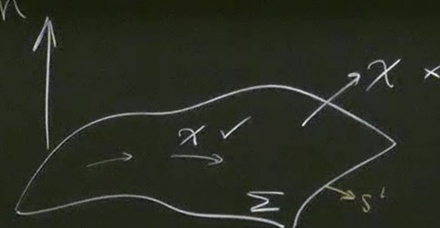
$d\tilde{S}$

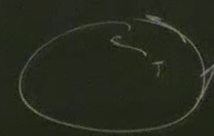
M

$\chi^{[a}\nabla^{b]}\varphi$

$\downarrow$

$\tilde{S}^{ab}(\chi)$





$\ldots^a)\,\Box\varphi + \int_{\partial\Sigma} n_a s_b\, \tilde{S}^{ab}(\chi)$

$\chi^a = 0$ i.e. if $\chi^a \parallel \Sigma$

e!

$\chi \in \mathfrak{X}'(M)$ $\quad L_\chi g_a$

$\rho(\chi)\,\varphi(x) = \chi^a \nabla_a$

(rmk $\quad \rho(\chi)\, g \equiv 0$