Title: Lecture - Quantum Gravity, PHYS 644

Speakers: Aldo Riello

Collection/Series: Quantum Gravity (Elective), PHYS 644, February 24 - March 28, 2025

Subject: Quantum Gravity

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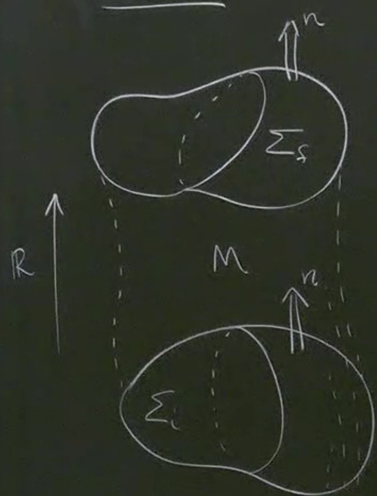
RECAP



- Field space over M

$$\mathcal{F} = \Gamma(F \to M) \ni \varphi$$

- Mixed (local) forms

$$\left(\Omega^{k,\ell}(M\times\mathcal{F}),\ d,\ \mathbb{d}\right)$$

$$d\mathbb{d} + \mathbb{d}d = 0$$

- Tekens thm:

$$\Omega^{top,1} = \Omega^{top,1}_{src} \oplus d\Omega^{top-1,1}$$

$\boxed{\text{no derivatives on } \mathbb{d}\varphi^I}$

COVARIANT PH. SP.

$\mathcal{L} \in \Omega^{top,0}(M\times\mathcal{F})$ Lagrangian density

$$\Omega^{top,1}(M\times\mathcal{F}) \ni \mathbb{d}\mathcal{L} \underset{\text{Tekens}}{=} \underbrace{E_I}_{\text{Euler-Lagrange e.o.m.}} \mathbb{d}\varphi^I - d\,\underbrace{\theta}_{\text{presympl potential current}}$$

presympl potential current $\in \Omega^{top-1,1}(M\times\mathcal{F})$

$$\mathcal{F}_{EL} := \{\varphi : E_I(\varphi) = 0\} \xhookrightarrow{\iota_{EL}} \mathcal{F}$$

the shell (aka "space of physical histories")

Notation: $\approx$ onshell equality (i.e. upon pullback to $\mathcal{F}_{EL}$)

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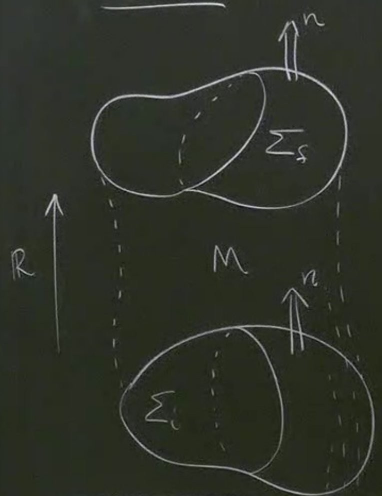

currents $\Omega^{top-1}(M)\otimes W$

$J = \pm\, g_{ab}\tilde{J}^a dx^b$, $dJ = \pm(\nabla_a \tilde{J}^a) \in$

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Takens

Euler-Lagrange e.o.m

presympl potential current $\in \Omega^{top-1,1}(M\times\mathcal{F})$

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Notation: $\approx$ onshell equality (i.e. upon pullback to $\mathcal{F}_{EL}$)

Thm $\underline{\Omega} := d\underline{\Theta}$ presympl. current $\in \Omega^{top-1,2}(M\times\mathcal{F})$

$\boxed{d\underline{\Omega} \approx 0} \Rightarrow \omega := \iota_{EL}^* \int_\Sigma \underline{\Omega}$ indep. of Σ! $\in \Omega^2(\mathcal{F}_{EL})$

$(\partial\Sigma = \phi)$

$\rightarrow$ $\boxed{(\mathcal{F}_{EL}, \omega)}$ COVARIANT PH. SPACE

phys history $\leftrightarrow$ initial cond @ Σ

$\varphi \in \mathcal{F}_{EL}$ $\quad (\phi(\vec{x}), \pi(\vec{x})) = (\varphi, \dot{\varphi})|_{t=0}$

Q: symplectic?

A1: scalar field: yes $\cong$ can. ph. space

A2: [teaser] gauge theory: no, degenerate!

(M×F)

")

pullback

EL

Rmk the shell $[\underline{L}] = [\underline{L} + d\underline{\ell}]$

also $\underline{(+1)} \rightsquigarrow \underline{(4)} + d\underline{\vartheta} + \mathbb{I}\underline{\ell}$

Lemma: $\alpha \in \Omega^{top-k,\ell}(M \times \mathcal{F})$, $k, \underline{\ell} \geq 1$

then if $d\alpha|_\varphi = 0 \Rightarrow \alpha$ is d-exact at φ.

(Lee-Wald 1991)

$\rightsquigarrow$ I will use (improperly) to say that

if $d\alpha \underset{(=)}{\approx} 0 \Rightarrow \exists \beta \in \Omega^{top-k-1,\ell}$; $\alpha \underset{(=)}{\approx} d\beta$

at φ.

$\alpha = d\beta$
$(=)$

Prop ω does not depend on $\underline{\ell}$ or $\underline{\vartheta}$ $(\partial\Sigma = \phi)$

Pf: $\underline{\Omega} \to \underline{\Omega} + dd\underline{\vartheta} + \cancel{d^2\underline{\ell}}$

$\Omega_\Sigma \to \Omega_\Sigma - \cancel{\int_\Sigma d\, d\vartheta}$ $\boxed{\square}$

SYMMETRIES

Lie algebra action on $\mathcal{F}$

is a Lie algebra homomorphism

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{X}^1(\mathcal{F})$$

$$\xi \longmapsto \rho(\xi) = \int_M (\delta_\xi \varphi^I) \frac{\delta}{\delta \varphi^I}$$

i.e. $\rho([\xi, \eta]) = [\rho($

i.e. $\rho([\xi,\eta]) = [\rho(\xi), \rho(\eta)]$

"$\delta_\xi L$ is a tot der"

Def: $\rho : \mathfrak{g} \to \mathfrak{X}^1(\mathcal{F})$ is a Lagrangian sym

if $\mathbb{L}_{\rho(\xi)} \underline{L} = d \underline{R}(\xi)$

for some $\underline{R} : \mathfrak{g} \to \Omega^{top-1,0}(M \times \mathcal{F})$ ($\mathbb{R}$-linear)

Q: is such a ρ Hamiltonian in $(\mathcal{F}_{EL}, \omega)$?

$\frac{\delta}{\delta \varphi^I}$

First, we need to check that
ρ descends / is tangent to the shell $\mathcal{F}_{EL}$:

Thm: $\mathbb{L}_{\rho(\xi)} \underline{E}_I \approx 0$ if ρ is a L. sym.

i.e $\delta_\xi \varphi^I$ is a sol. of the lin eom

Pf (sketch)

$$\mathbb{L}_{\rho(\xi)} \mathbb{d} \underline{L} = \mathbb{L}_{\rho(\xi)} (E_I \mathbb{d}\varphi^I) - d\, \mathbb{L}_{\rho(\xi)} \Theta$$

$$\mathbb{L}_{\rho(\xi)} \mathbb{d} \underline{L} = \mathbb{d}\, \mathbb{L}_{\rho(\xi)} \underline{L} \underset{\text{hyp}}{=} \mathbb{d}\, d \underline{R}(\xi) = -d\, \mathbb{d} \underline{R}(\xi)$$

$$\Rightarrow \underbrace{\left(\mathbb{L}_{\rho(\xi)} E_I\right)}_{\text{source}} d\varphi^I = \underbrace{-E_I\, d\delta_\xi \varphi^I}_{(*)?} + \underbrace{d(\ldots)}_{\text{bdry}}$$

- if $\delta_\xi \varphi^I = \Lambda^I{}_J \varphi^J \quad (d\Lambda = 0)$

  $\Rightarrow (*)$ is source

  $\Rightarrow$ from Takens $\quad \mathbb{L}_{\rho(\xi)} E_J = -E_I \Lambda^I{}_J \approx 0$

- exercise generalize to case with $\delta_\xi \varphi = A^I{}_J \varphi^J + B^{eI}{}_J \nabla_e \varphi^J$

  dependent on φ

$\boxed{}$

$\Rightarrow$

$_I \Lambda^I{}_J \approx 0$

$\Lambda^I{}_J \varphi^J + B^{eI}{}_J \nabla_e \varphi^J$

dependent on φ

$\boxed{2}$

Thm (Noether 1) If ρ is L.sym

then $\underline{J}(\xi) := \mathring{\mathbb{I}}_{\rho(\xi)} \underline{\Theta} - \underline{R}(\xi)$

is an onshell conserved current (Noether c.)

$$\boxed{d \underline{J} \approx 0}$$

hyp.

Pf: $\mathbb{L}_{\rho(\xi)} \underline{L} = d \underline{R}(\xi)$

$$\underbrace{d \mathring{\mathbb{I}}_{\rho(\xi)} \underline{L}}_{\equiv 0 \text{ 0-form on } \underline{L}} + \mathring{\mathbb{I}}_{\rho(\xi)} d\!\!\!d \underline{L} = \mathring{\mathbb{I}}_{\rho(\xi)} E_I d\!\!\!d \varphi^I + d \mathring{\mathbb{I}}_{\rho(\xi)} \underline{\Theta}$$

$= \delta_\xi \varphi^I$

$$\Rightarrow \; d\,\underline{J} = d\big(i_{\rho}\,\Theta - R\big) = -\,E_I\,\delta_\xi \varphi^I \approx 0. \quad \blacksquare$$

$i_{\rho(\xi)}\,\underline{\Theta}$

<u>Corollary</u> (Noether charge)

If $\partial\Sigma = \emptyset$ then $\boxed{Q(\xi) := \iota^*_{E\omega} \int_\Sigma \underline{J}(\xi)}$

is independent of Σ for all $\xi \in \mathfrak{g}$ <u>on shell</u>

Thm ρ: L. sym & $\partial\Sigma = \phi$, then:

$$\mathring{\mathbb{I}}_{\rho(\xi)} \Omega_\Sigma \approx - \mathrm{d} Q_\Sigma(\xi) \quad \text{on } \mathcal{F}_{EL}$$

and

(i) Q_Σ is conserved $\leftarrow \boxed{\checkmark}$

(ii) Q_Σ is free from ambiguities $\leftarrow \boxed{\text{excercise}}$