Title: Lecture - Quantum Gravity, PHYS 644

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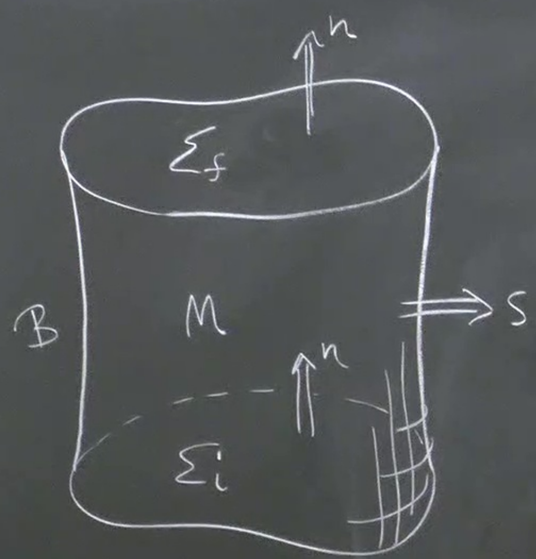
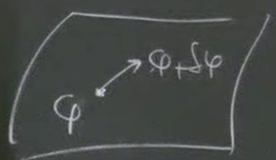




	Spacetime	Field space
mfd	M	$\mathcal{F} = \Gamma(F \to M) \sim C^\infty(M,V)$
coords	$x^a,\ a=1\dots n$	$\varphi^I(x)\ ,\ \begin{cases} I=1,\dots,m \\ x\in M \end{cases} \leftarrow \infty!$
Cartan calculus	$(d, i_\cdot, L_\cdot)$	$(\mathbb{d}, \mathbb{i}_\cdot, \mathbb{L}_\cdot)$
	$df = \sum_a \frac{\partial f}{\partial x^a} dx^a$	$\mathbb{d}S = \int_M \sum_I \frac{\delta S}{\delta \varphi^I(x)} [\mathbb{d}\varphi^I(x)]$ ↑ diff operator! $(\mathbb{d}\partial\varphi = \partial\mathbb{d}\varphi)$
	$X = \sum_a X^a \frac{\partial}{\partial x^a}$	$\mathbb{X} = \int_M \sum_I \left(\delta_{\mathbb{X}} \varphi^I(x)\right) \frac{\delta}{\delta \varphi^I(x)}$
	$L_X = i_X d + d i_X$	$\mathbb{L}_{\mathbb{X}} = \mathbb{i}_{\mathbb{X}} \mathbb{d} + \mathbb{d}\, \mathbb{i}_{\mathbb{X}}$

Mixed forms

$$\Omega^{\bullet}(M \times \mathcal{F}) = \sum_{k=0}^{\dim(M)} \sum_{\ell=0}^{\infty} \Omega^{k,\ell}(M \times \mathcal{F}) \subset \Omega^{k}(M) \otimes \Omega^{\ell}(\mathcal{F})$$

w/ differential $\mathbf{d} := d + \mathbb{d}$

$$0 \overset{!}{=} \mathbf{d}^2 = d\,\mathbb{d} + \mathbb{d}\,d$$

$$0 = i_X \mathbb{d} + \mathbb{d}\, i_X$$

etc.

Eg. $\Omega^{1,1}(M \times \mathcal{F})$

$\ni$

* $\alpha_1(x,\varphi) = \nabla_a \varphi(x)\, \mathbb{d}\varphi(x) \wedge dx^a = -d\varphi \wedge \mathbb{d}\varphi\,(x) \longrightarrow A_1 = \int_{line} \alpha_1 = \int_{line} \mathbb{d}\varphi \wedge d\varphi \in \Omega^1(\mathcal{F})$

$= \mathbb{d}\varphi \wedge d\varphi\,(x)$

* $\alpha_2(x,\varphi) = \nabla_a \mathbb{d}\varphi \wedge dx^a = \mathbb{d}\,d\varphi = -d\,\mathbb{d}\varphi \longrightarrow A_2 = \int_{line} \mathbb{d}\,d\varphi = \mathbb{d}\int_{line} d\varphi$

Takens thm

$$\Omega^{top,1}(M\times F) = \underbrace{\Omega^{top,1}_{src}(M\times F)}_{\text{"source"}} \oplus \underbrace{d\Omega^{top-1,1}(M\times F)}_{\text{"boundary"}}$$

$\boxed{\text{no derivatives on } d\varphi^I}$

E.g. α_1 src if $\dim(M)=1$

α_2 isn't.

[Anderson's homotopy operators]

$\boxed{\begin{array}{l} \underline{\text{Rmk}}\ J \in \Omega^{top-1,k} \\ *J = (-1)^{top}\,\tilde{J}_a\,dx^a \\ *dJ = (-1)^{k+1}(\nabla_a \tilde{J}^a) \end{array}}$

Applic...

$\Omega^{top,\ldots}$

$\Omega^{top,1}\ldots$

Application

$\Omega^{top,0}(M\times F) \ni \underline{L}(\varphi, x)$ assume depends only on $\varphi(x), \partial\varphi(x)$

$$\Omega^{top,1}(M\times F)\ni d\underline{L} = \frac{\partial L}{\partial\varphi^I}\, d\varphi^I + \frac{\partial L}{\partial(\partial_a\varphi^I)}\,\partial_a d\varphi^I$$

$$= \underbrace{\left(\frac{\partial L}{\partial \varphi^I} - \partial_a \frac{\partial L}{\partial(\partial_a\varphi^I)}\right)}_{\text{src}} d\varphi^I + \underbrace{\partial_a\left(\frac{\partial L}{\partial(\partial_a\varphi^I)}\, d\varphi^I\right)}_{\text{bdry}}$$

↳ Euler-Lagrange

$\pi_{src}(dL)$

$\pi_{bdry}(dL)$

$$= \underline{E}_I(\varphi)\, d\varphi^I - d\ 4$$

φ^I

$$\varphi^I + \underbrace{\partial_a\left(\frac{\partial L}{\partial(\partial_a \varphi^I)}\, d\varphi^I\right)}_{\text{bdry}}$$

$$\tilde{n}^a = \pm \frac{\partial L}{\partial(\partial_a \varphi^I)}\, d\varphi^I$$

Lagrange

$\pi_{\text{bdry}}(dL)$

(dL)

COVARIANT PHASE SPACE

anchor

current
$J \in \Omega^{top-1}(M) \otimes W$

$$*J = \pm \left(g_{ab} \tilde{J}^a dx^b \right)$$

$$top \leftarrow dJ = \pm \left(\nabla_a \tilde{J}^a \right) \underline{\epsilon}$$

$$\underline{\epsilon} = \sqrt{g}\, dx^1 \wedge \dots \wedge dx^{top}$$

$$\int_{\partial M} J = \int_M dJ = \int_M \nabla_a \tilde{J}^a \sqrt{g} = \int_{\partial M} n_a \tilde{J}^a \sqrt{h}$$

$$S = \int_M \underline{L}$$

action; Lagrangian $\in \Omega^{top,0}$

$$\mathrm{d}\underline{L} = E_I \,\mathrm{d}\varphi^I - d\underline{\Theta}$$

e.o.m.; pre symplectic current

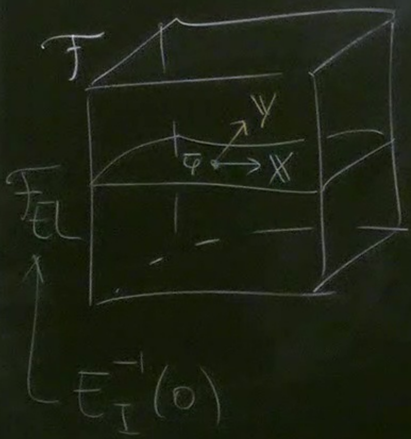
the shell $\mathcal{F}_{EL} := \{\varphi \in \mathcal{F} : E_I(\varphi) = 0\} \xhookrightarrow{\iota_{EL}} \mathcal{F}$

Notation: going on shell means pulling back to $\mathcal{F}_{EL}$

α, β are mixed forms $\quad \alpha \approx \beta$ iff $\iota_{EL}^*(\alpha - \beta) = 0$

What does it mean to pull back $d\varphi^{I}$?

→ It can only be contracted w/ $\mathbb{X} \in T_{\bar\varphi}\mathcal{F}_{EL}$, $\bar\varphi \in \mathcal{F}_{EL}$



Ψ ✗

$\mathbb{X}$ ✓

iff $\mathbb{I}_{\mathbb{X}} E_I|_{\bar\varphi} = 0$

$$\delta_{\mathbb{X}} E_I|_{\bar\varphi} = \frac{\partial E_I}{\partial \varphi^J}(\delta_{\mathbb{X}}\varphi^J) + \frac{\partial E_I}{\partial(\partial_a \varphi^J)}\partial_a(\delta_{\mathbb{X}}\varphi^J) + \dots \Big|_{\bar\varphi}$$

→ $\delta_{\mathbb{X}}\varphi^J$ is a solution of the linearized eom at $\bar\varphi$

aka "on shell perturbation"

$$= E_I(\bar\varphi + \delta_{\mathbb{X}}\varphi) - E_I(\bar\varphi) + O(\delta_{\mathbb{X}}\varphi^2)$$

to pull back $d\varphi^I$?

ontracted w/ $\mathbb{X} \in T_{\bar\varphi}\mathcal{F}_{EL}$, $\bar\varphi \in \mathcal{F}_{EL}$

iff $\mathbb{I}_{\mathbb{X}} E_I|_{\bar\varphi} = 0$

$$\delta_{\mathbb{X}} E_I|_{\bar\varphi} = \underbrace{\frac{\partial E_I}{\partial \varphi^J}}_{-m^2+3\lambda\bar\varphi^2}(\delta_{\mathbb{X}}\varphi^J) + \frac{\partial E_I}{\partial(\partial_a\varphi^J)}\partial_a(\delta_{\mathbb{X}}\varphi^J) + \frac{\partial E_I}{\partial(\partial_a\partial_b\varphi^J)}\partial_a\partial_b\, d\varphi^J + \dots$$

$= 0$ $\quad \eta^{ab}$

$\longrightarrow \delta_{\mathbb{X}}\varphi^J$ is a solution of the linearized eom at $\bar\varphi$

oka "on shell perturbation"

$$= E_I(\bar\varphi + \delta_{\mathbb{X}}\varphi) - E_I(\bar\varphi) + \mathcal{O}(\delta_{\mathbb{X}}\varphi^2)$$

$\mathbb{Y}$ ✗

$\mathbb{X}$ ✓

$E_I(\varphi) =$

$\delta_{\mathbb{X}} E_I|_{\bar\varphi} =$

$$L = -\frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}m^2\varphi^2$$

η^{ab}

E_I $\partial_a\partial_b\, d\varphi^J + \ldots$

$\partial_a\partial_b\varphi^J)$

d eom of $\bar{\varphi}$

"on shell perturbation"

$$E_I(\varphi) = \Box\varphi - m^2\varphi + \lambda\varphi^3$$

$$\delta_X E_I\big|_{\bar{\varphi}} = \Box\,\delta_X\varphi - m^2\delta_X\varphi + 3\lambda\bar{\varphi}^2\delta_X\varphi$$

Thm

Let $\underline{\Omega} := \mathbb{d}\underline{\Theta} \in \Omega^{top-1,2}(M \times F)$ presympl current

then $\mathbb{d}\underline{\Omega} \approx 0$ i.e. $\nabla_a \tilde{\Omega}^a \approx 0$ [on shell]

i.e. the presympl current is conserved on-shell.

Pf: $0 \equiv \mathbb{d}^2 \underline{L} = \mathbb{d}\underline{E} - \underbrace{\mathbb{d}d\underline{\Theta}}_{+d\underline{\Omega}} \approx d\underline{\Omega}$ ∎

[on shell]

shell

$$0 \approx \int_M d\underline{\Omega} = \int_{\Sigma_f} \underline{\Omega} - \int_{\Sigma_i} \underline{\Omega}$$

$$\Rightarrow \boxed{\begin{array}{ll} \omega := \iota^*_{EL} \int_\Sigma \underline{\Omega} & \text{is indep. of chosen Cauchy surface } \Sigma \hookrightarrow M \\ \cap & \\ \Omega^2(\mathcal{F}_{EL}) & \end{array}}$$

$\sim$ a "canonical" $(\mathcal{F}_{EL}, \omega)$ Q: symplectic?

COV. PH. SP.

Side rmk

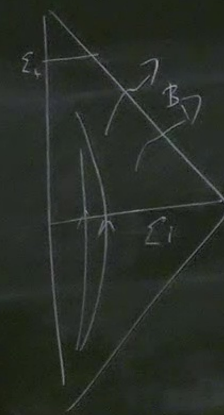
$\partial \Sigma \neq \phi$

$\partial M = \Sigma_f \cup \overline{\Sigma_i} \cup B$ ← timelike

"box"

balance eq

$$\Omega_{\Sigma_i} \approx \Omega_{\Sigma_f} + \underbrace{\int_B \underline{\Omega}}_{\text{"presymplectic flux"}}$$



Ex scalar field (Mink)

$$\tilde{\Theta}^a = -(\nabla^a \varphi)\, \mathbb{d}\varphi$$

$$\tilde{\Omega}^a = \mathbb{d}\varphi \wedge \nabla^a \mathbb{d}\varphi$$

$$\underset{\uparrow \{t=0\}}{\Omega_\Sigma} = -\int_\Sigma \underbrace{n_a \nabla^a \mathbb{d}\varphi}_{\partial_t \mathbb{d}\varphi} \wedge \mathbb{d}\varphi = -\int_\Sigma \mathbb{d}\dot{\varphi} \wedge \mathbb{d}\varphi$$

initial conds $\overset{1\text{ to }1}{\longleftrightarrow}$ sols eom $\sim \mathcal{F}_{EL}$

$$\begin{cases} \pi(\vec{x}) := \dot{\varphi}(t=0, \vec{x}) \\ \phi(\vec{x}) := \varphi(t=0, \vec{x}) \end{cases}$$

$$(\mathcal{F}_{EL}, \omega) \simeq (\text{canonical ph sp})$$

$$\int_\Sigma \mathbb{d}\pi \wedge \mathbb{d}\phi$$

no a "canonical" cov. ph. sp. $(\mathcal{F}_{EL}, \omega)$ Q: symplectic?

scalar field (Mink)

$= -(\nabla^a \varphi)\, d\varphi$

$= d\varphi \wedge \nabla^a d\varphi$

$$= -\int_\Sigma \underbrace{n_a \nabla^a d\varphi}_{\partial_t d\varphi} \wedge d\varphi = -\int_\Sigma d\dot\varphi \wedge d\varphi$$

$\{t=0\}$

initial conds $\overset{1\text{ to }1}{\longleftrightarrow}$ sols eom ~ $\mathcal{F}_{EL}$

$$\begin{cases} \pi(\vec x) := \dot\varphi(t=0, \vec x) \\ \phi(\vec x) := \varphi(t=0, \vec x) \end{cases}$$

$\omega) \simeq$ (canonical ph sp $\int d\pi \wedge d\varphi$)

But $(\mathcal{F}_{EL}, \omega)$ will be degenerate for a gauge theory!