Title: Lecture - Quantum Gravity, PHYS 644

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Collection/Series: Quantum Gravity (Elective), PHYS 644, February 24 - March 28, 2025

Subject: Quantum Gravity

Date: March 03, 2025 - 9:00 AM

URL: https://pirsa.org/25030025

$$i_C : C = \{ \overset{2n}{z^I} : \underset{k}{J_\alpha}(z^I) = 0 \} \hookrightarrow P$$

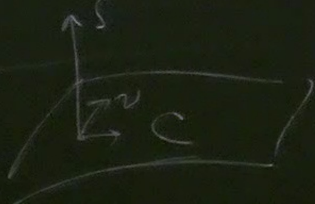
$$\omega = \frac{1}{2}\, \omega_{IJ}\, dz^I \wedge dz^J$$

ω_{IJ}: $2n \times 2n$

non degenerate on P

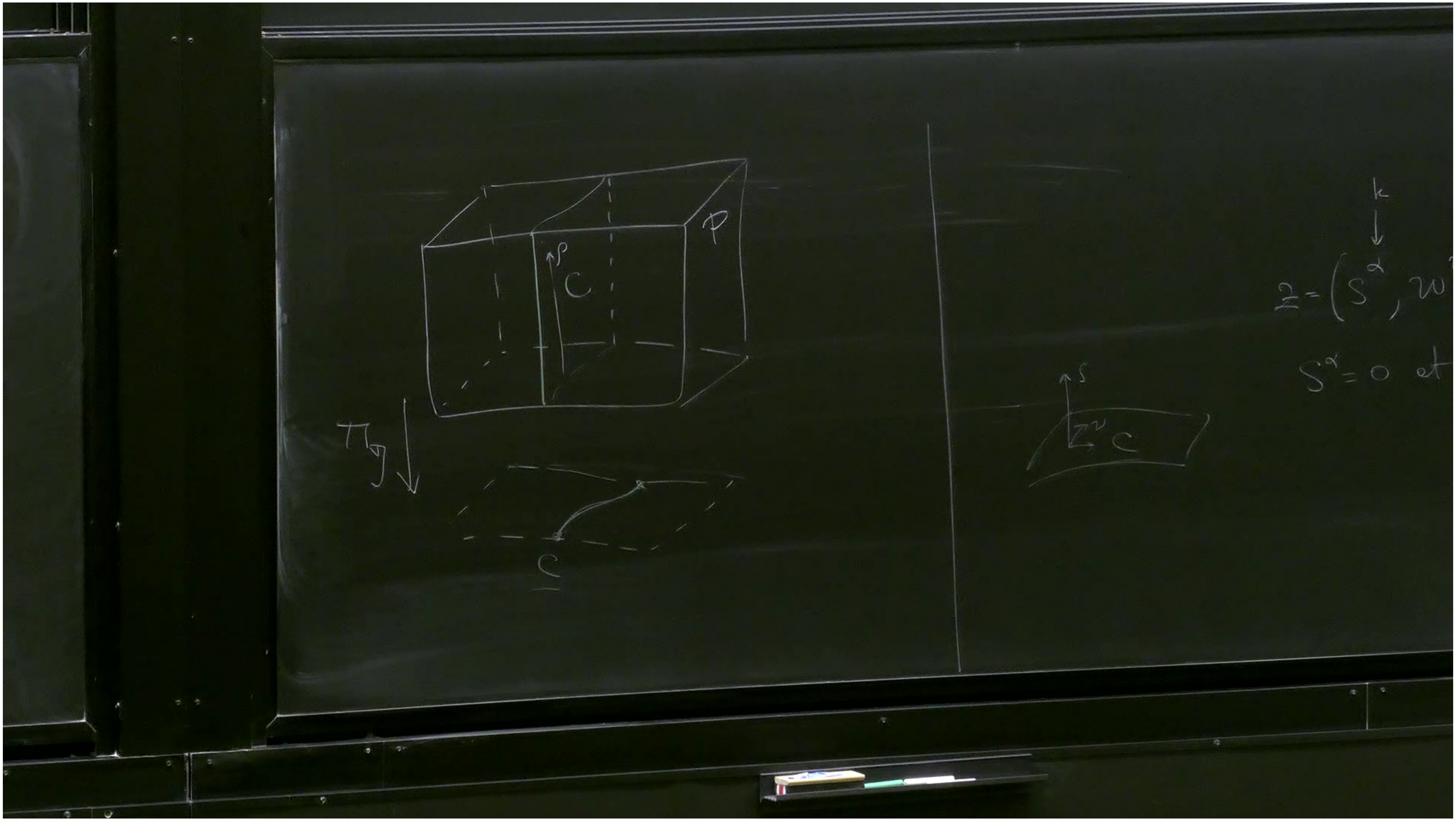
$$z = (\overset{k}{s^\alpha}, w^i)$$

$s^\alpha = 0$ at C



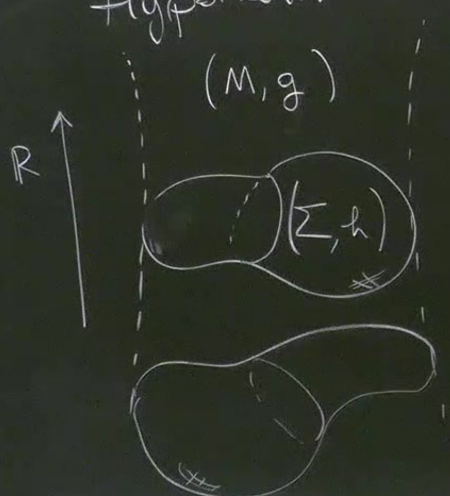
$i_C^*\omega \quad (2n-k)\times(2n-k)$

$\dim(\ker(\quad)) \leq k \qquad \rho(\tau_\alpha)$


P
ρ
C
π_g
C
k
z = (S^α, w
S^α = 0 et
s
z^ν
C

Field Theory

Hypothesis: ① sp.time is globally hyperbolic



$$M \simeq \underbrace{\Sigma}_{\text{Cauchy surface}} \times \underbrace{\mathbb{R}}_{\text{"time"}}$$

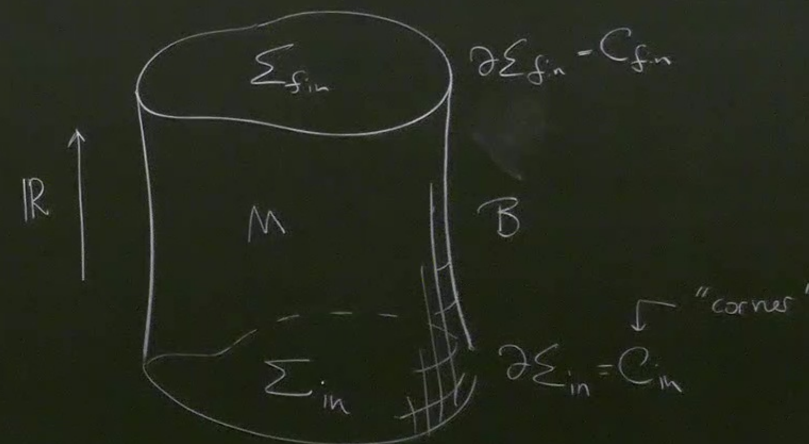
Cauchy surface
"spatial slice"

② The whole universe is closed

$$\partial\Sigma = \phi$$

③ We can still look at subregions

$$[\partial\Sigma \neq \phi]$$

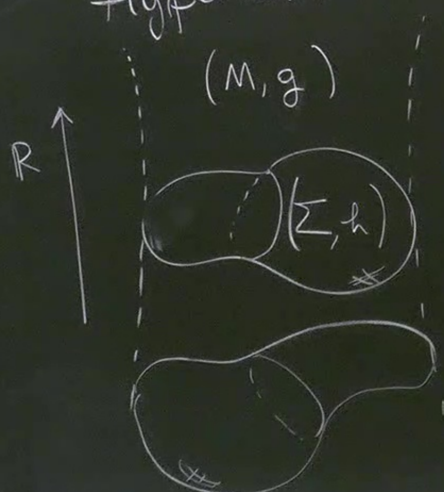


$$\partial M = \Sigma_{in} \cup \Sigma_{fin} \cup B$$

$$B \simeq \partial\Sigma \times [t_{in}, t_{fin}]$$

Field Theory

Hypothesis: ① sp. time is globally hyperbolic



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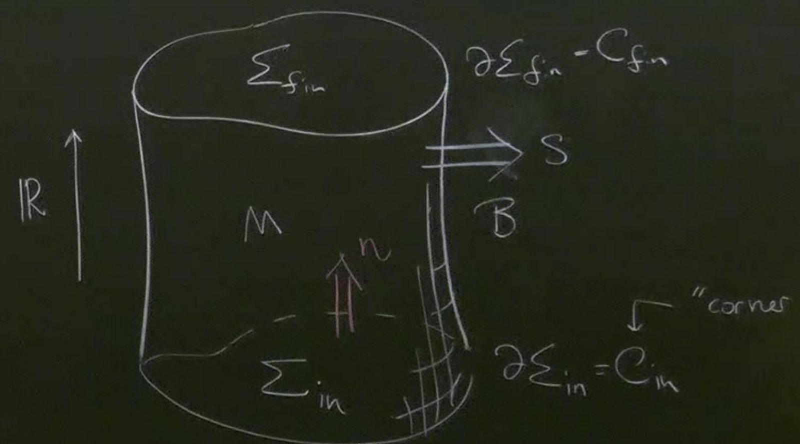

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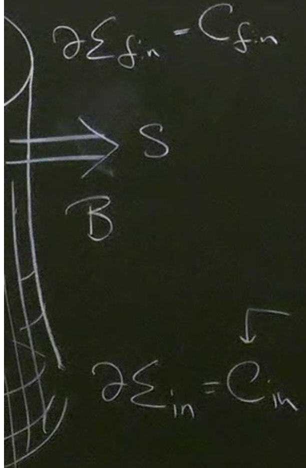


$\cup\, \Sigma_{fin} \cup B$

$\times [t_{in}, t_{fin}]$

- Unit conormals

$n_a \sim$ timelike conormal to $\Sigma \hookrightarrow M$
(future pting)

$s_a \sim$ spacelike conormal to $B \hookrightarrow M$
(outgoing)

$$\varepsilon_{ab} := \frac{n_a s_b - s_a n_b}{\sqrt{1+(s\cdot n)^2}} \sim \text{co-binormal to } C \hookrightarrow M$$

Ex

$M = (\mathbb{R}^4, \eta_{ab})$ ~ Minkowski

$M = (\mathbb{R}^3, \delta_{ab}) \times \mathbb{R}$

Cartesian coords

$\Sigma \simeq$ 3-ball of unit radius $\{r \leq 1\}$

- $\underline{n} = dt \longrightarrow n^a = \partial_t$
- $\underline{s} = dr \longrightarrow s^a = \partial_r$
- $\underline{\varepsilon} = dt \wedge dr$

inkowski

$$(\mathbb{R}^3, \delta_{ab}) \times \mathbb{R}$$

radius $\{r \leq 1\}$

$$n^a = \partial_t$$

$$s^a = \partial_r$$

In adapted coords

where $\Sigma = \{t = cte\}$

$\mathcal{B} = \{r = cte\}$

$\underline{n} \propto dt \longrightarrow n \sim \partial_t + \cdots$

$\underline{s} \propto dr \qquad s \sim \partial_r + \cdots$

$\underline{\varepsilon} \propto dt \wedge dr$

why? X vector tangent to Σ

$$X = (\text{zero})\partial_t + X^i \partial_i$$

$$0 = \underline{n}(X) \rightarrow \underline{n} \propto dt$$

• On sp.t. we can do geometry / diff calculus using Cartan's formalism

$$(d,\ i_X,\ L_X) \qquad X \in \mathfrak{X}^1 \text{ vect field}$$

L_X — Lie derivative along X

i_X — interior diff / inner prod.

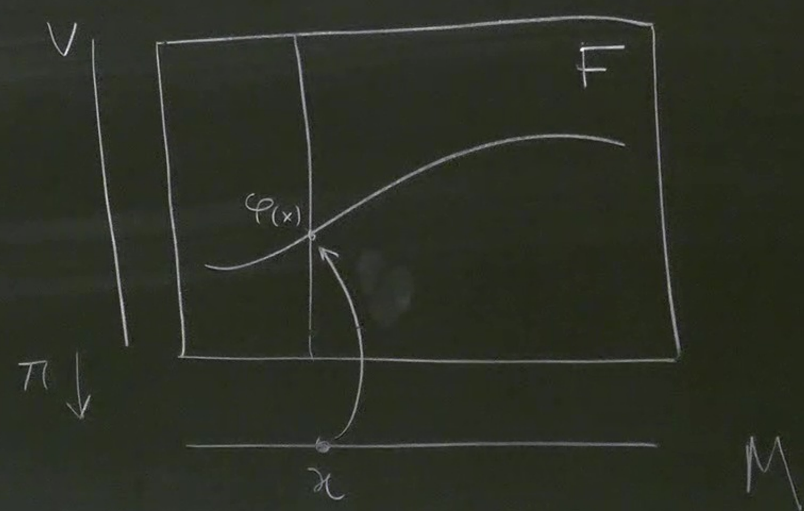
d — exterior diff.

$$L_X = i_X d + d i_X \qquad \text{Cartan's formula}$$

$$\Omega^{top-k}(R) := \Omega^{\dim(R)-k}(R)$$

- But we want to do geometry over field space! (Phase space)

→ Fields are sections of (vector/affine) bundles



$$\pi : F \to M$$

$$V \simeq \pi^{-1}(x)$$

$$\varphi \in \Gamma(M, F)$$

$,F)$

if F is trivial, $F = M \times V$,

$$\varphi \in \Gamma(M,F) \simeq C^\infty(M,V)$$

Examples

- $\mathbb{C}$-scalar field $\quad F = M \times \mathbb{C}$, $\varphi \in C^\infty(M,\mathbb{C})$
- Electromagnetism $\quad A \in \mathcal{F} = \Omega^1(M)$, $\quad F = T^*M$

  (cheating a bit, in reality $F = J^1P/G$)
- YM $\quad A \in \mathcal{F} = \Omega^1(M,\mathfrak{g})$, $\quad F = T^*M \otimes \mathfrak{g}$

$M, \mathbb{C})$

T^*M

$F = J^1P/G)$

$= T^*M \otimes \mathfrak{g}$

- GR $\quad g_{ab} \quad F \sim T^*M \otimes_{sym} T^*M$

- particle in sp time M $\Rightarrow$ 0+1 d field theory

$$F = \underbrace{\mathbb{R}}_{\text{(sp)time}} \times \underbrace{M}_{\text{fibre}}$$

$$\gamma \in \mathcal{F} = C^\infty(\mathbb{R}, M)$$

- $\mathbb{F}$ ∞-dim mfd.

Cartan's geometry on $\mathbb{F}$

$$\left(\mathbb{d},\ \mathbb{i}_{\mathbb{X}},\ \mathbb{L}_{\mathbb{X}} \right) \qquad \mathbb{X} \in \mathfrak{X}^1(\mathbb{F})$$

$$\mathbb{L}_{\mathbb{X}} = \mathbb{i}_{\mathbb{X}}\, \mathbb{d} + \mathbb{d}\, \mathbb{i}_{\mathbb{X}}$$

<u>Rmk</u>: often $\mathbb{d} \to \delta$

here $\delta\varphi$ denotes the component of a vector

dx

here $\delta\varphi$ denotes the component of a vector

	M	$\mathcal{F}$
Coords	$x^a \quad a = 1, \dots, n$	$\varphi^I(x)$ — finite (I), infinite dim (x)
vector	$X = \sum_a X^a(x) \dfrac{\partial}{\partial x^a}$	$\mathbb{X} = \boxed{\int_M \sum_I} (\delta_{\mathbb{X}} \varphi)^I(x) \dfrac{\delta}{\delta\varphi^I(x)}$
differential	$df = \sum_a \dfrac{\partial f}{\partial x^a} dx^a$	If $S = \int_M L(\varphi, \partial\varphi, x)$
		$dS = \int_M \sum_I \underbrace{\left(\dfrac{\partial L}{\partial \varphi^I} + \dfrac{\partial L}{\partial(\partial_a \varphi^I)} \partial_a \right)}_{\delta S / \delta\varphi^I(x)} [d\varphi^I(x)]$ differential operator!

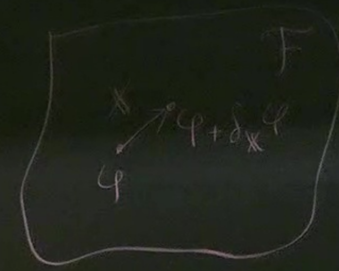
$\mathcal{F}$

, n

$\varphi^{I}(x)$ — I: finite; x: infinite dim

$$\mathbb{X} = \boxed{\int_M \sum_I} (\delta_{\mathbb{X}}\varphi)^I(x) \, \frac{\delta}{\delta\varphi^I(x)}$$

$$d\,\partial_a\varphi^I \equiv \partial_a\, d\varphi^I$$



If $S = \int_M \mathcal{L}(\varphi, \partial\varphi, x)$

$$dS = \int_M \sum_I \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\varphi^I} + \frac{\partial\mathcal{L}}{\partial(\partial_a\varphi^I)}\,\partial_a\right)}_{\delta S/\delta\varphi^I(x)} d\varphi^I(x) = \int_M \sum_I \frac{\delta S}{\delta\varphi^I}\left[d\varphi^I\right]$$

differential operator!