Title: Lecture - Quantum Information, PHYS 635

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Subject: Quantum Information

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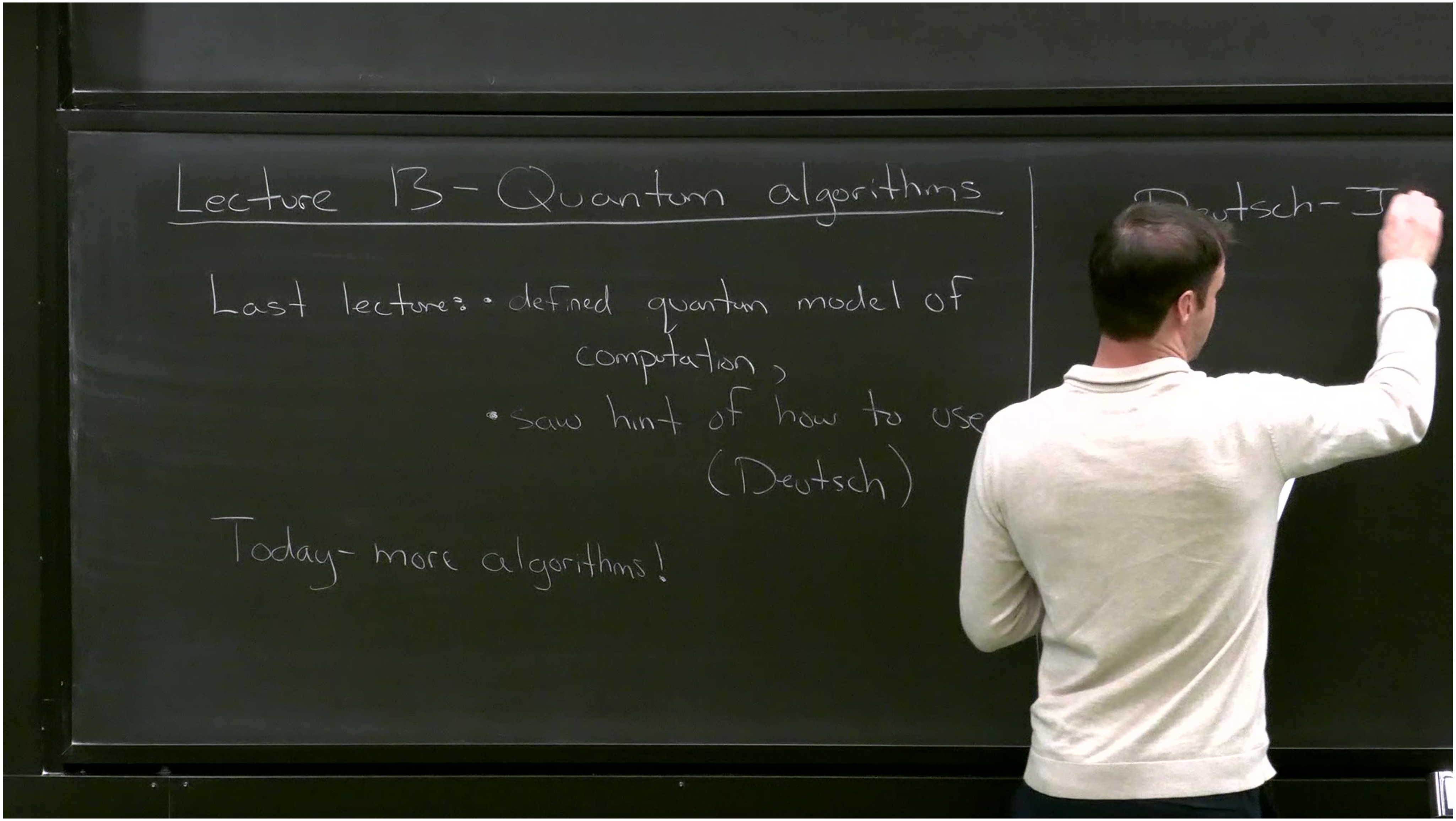
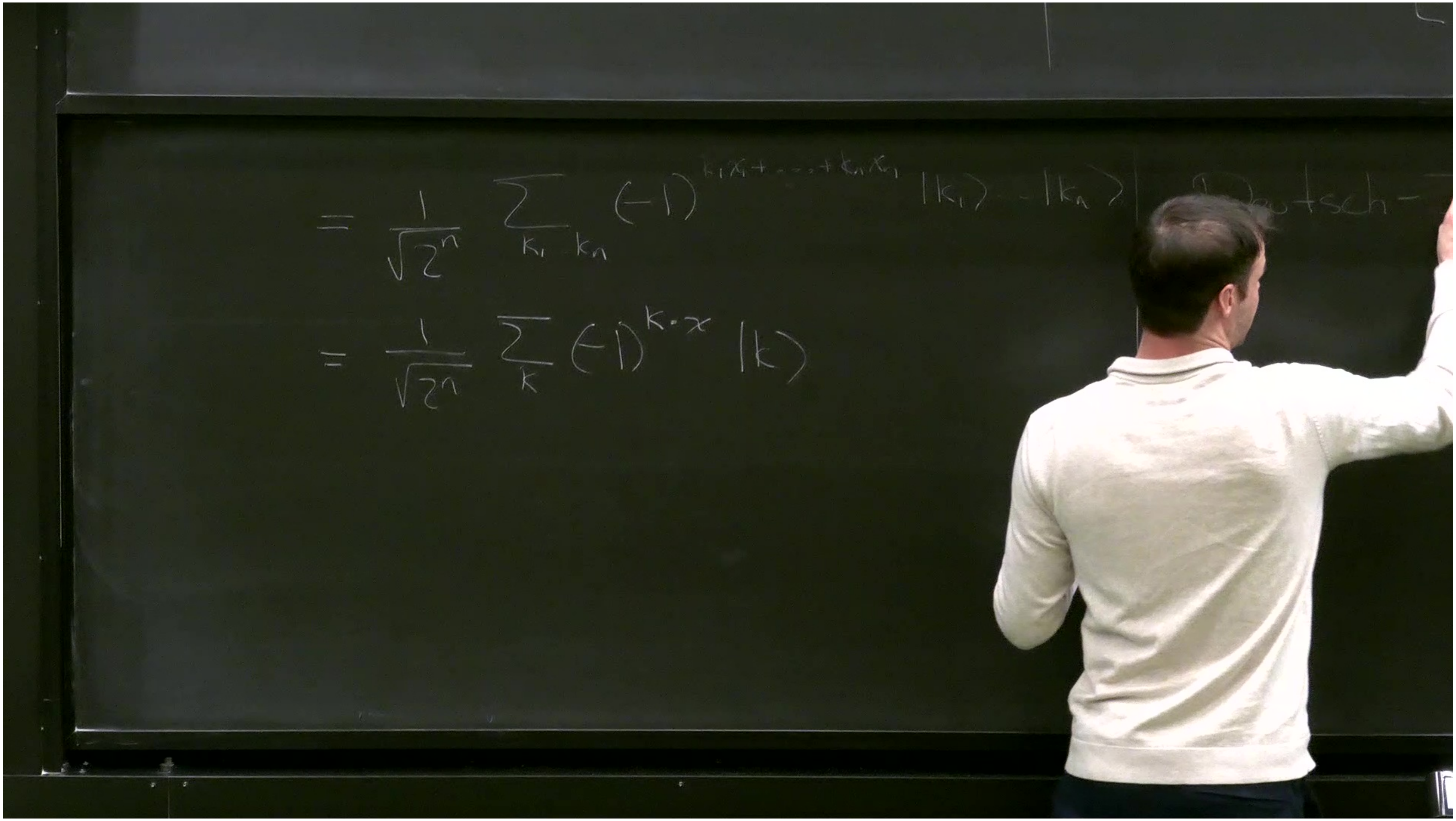

Lecture 13 - Quantum algorithms
Last lecture: • defined quantum model of computation,
• saw hint of how to use
(Deutsch)
Today - more algorithms!
Deutsch-J

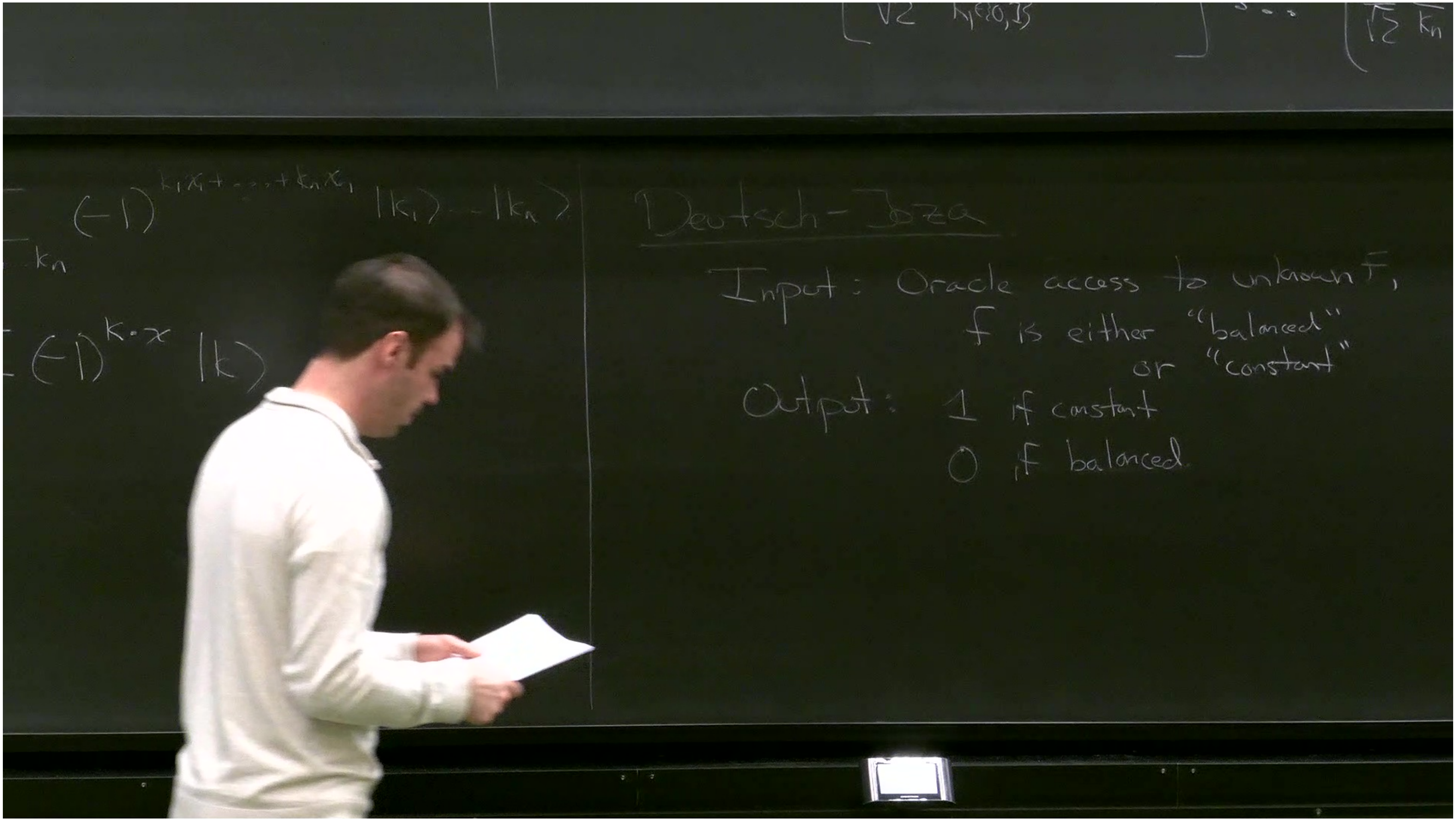
$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_k (-1)^{x\cdot k} |k\rangle \qquad x\cdot k = \sum_{i=1}^{n} x_i k_i$$

$$H^{\otimes n}|x\rangle = H|x_1\rangle H|x_2\rangle \dots H|x_n\rangle$$

$$= \left[\frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{x_1}|1\rangle\right)\right] \dots \left[\frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{x_n}|1\rangle\right)\right]$$

$$= \left[\frac{1}{\sqrt{2}} \sum_{k_1\in\{0,1\}} (-1)^{x_1\cdot k_1} |k\rangle\right] \dots \left[\frac{1}{\sqrt{2}} \sum_{k_n} (-1)^{x_n\cdot k_n} |k_n\rangle\right]$$


$= \frac{1}{\sqrt{2^n}} \sum_{k_1 \ldots k_n} (-1)^{k_1 x_1 + \ldots + k_n x_n} |k_1\rangle \ldots |k_n\rangle$
$= \frac{1}{\sqrt{2^n}} \sum_{k} (-1)^{k \cdot x} |k\rangle$
Deutsch-


Deutsch-Jozsa
Input: Oracle access to unknown f,
f is either "balanced"
or "constant"
Output: 1 if constant
0 if balanced.

0 if balanced

$f(0)\oplus f(1)$

pare $H^{\otimes n}|0\rangle = \frac{1}{\sqrt{2^n}}\sum_x |x\rangle$

Oracle. $\longrightarrow \frac{1}{\sqrt{2^n}}\sum_x (-1)^{f(x)}|x\rangle$

pply $H^{\otimes n}$ $\longrightarrow \frac{1}{2^n}\sum_x (-1)^{f(x)}\sum_k (-1)^{x\cdot k}|k\rangle$

$$= \frac{1}{2^n}\sum_{x,k}(-1)^{f(x)+x\cdot k}|k\rangle$$

$$O_f|z\rangle|x\rangle = |z+f(x)\rangle|x\rangle$$

$$O_f|-\rangle|x\rangle = (-1)^{f(x)}|-\rangle|x\rangle$$

$$\hookrightarrow O_f|x\rangle = (-1)^{f(x)}|x\rangle$$

Output: 1 if constant

0 if balanced

$f(0) \oplus f(1)$

prepare $\quad H^{\otimes n}|0\rangle = \frac{1}{\sqrt{2^n}} \sum_x |x\rangle$

apply Oracle

$$\rightarrow \frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} |x\rangle$$

apply $H^{\otimes n}$

$$\rightarrow \frac{1}{2^n} \sum_x (-1)^{f(x)} \sum_k (-1)^{x\cdot k} |k\rangle$$

$$|\psi_f\rangle = \frac{1}{2^n} \sum_{x,k} (-1)^{f(x)+x\cdot k} |k\rangle$$

$$P_0 = |\langle 0|\psi_f\rangle|^2$$

$$= \left| \frac{1}{2^n} \sum_{x,k} (-1)^{f(x)+x\cdot k} \underbrace{\langle 0|k\rangle}_{\delta_{0k}} \right|^2$$

$$= \frac{1}{2^{2n}} \left| \sum_x (-1)^{f(x)} \right|^2$$

$$= \begin{cases} 0 & f = \text{balanced.} \\ 1 & f = \text{constant.} \end{cases}$$

$$\frac{1}{\sqrt{2^n}} \sum_k (-1)^{\dots} |k\rangle$$

or "constant"

Output: 1 if constant

0 if balanced

$f(x) \oplus f(1)$

prepare $H^{\otimes n}|0\rangle = \frac{1}{\sqrt{2^n}} \sum_x |x\rangle$

apply Oracle $\longrightarrow \frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} |x\rangle$

apply $H^{\otimes n}$ $\longrightarrow \frac{1}{2^n} \sum_x (-1)^{f(x)} \sum_k (-1)^{x \cdot k} |k\rangle$

$$|\psi_f\rangle = \frac{1}{2^n} \sum_{x,k} (-1)^{f(x) + x \cdot k} |k\rangle$$

$$P_0 = |\langle 0|\psi_f\rangle|^2$$

$$= \left| \frac{1}{2^n} \sum_{x,k} (-1)^{f(x)+x\cdot k} \underbrace{\langle 0|k\rangle}_{\delta_{0k}} \right|^2$$

$$= \frac{1}{2^n} \left| \sum_x (-1)^{f(x)} \right|^2$$

$$= \begin{cases} 0 & f = \text{balanced} \\ 1 & f = \text{constant} \end{cases}$$

$q \to 1$ oracle calls

$c \to \Theta(n)$ calls

Grover search

$$\mathcal{O}_A|z\rangle|x\rangle = \begin{cases} |z\oplus 1\rangle|x\rangle & \text{if } x = x^* \\ |z\rangle|x\rangle & \text{otherwise} \end{cases}$$

$x \in \{1, \dots, N\}$

Claim: classically: $2/3\, N$

quantum $O(\sqrt{N})$

(Sketch) Algorithm

1) Prepare $|u\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle$

2) Apply oracle: $\mathcal{O} = \sum_x (-1)^{\delta_{xx^*}} |x\rangle\langle x|$

$$= \mathbb{I} - 2|x^*\rangle\langle x^*|$$

3) Apply $R_u = 2|u\rangle\langle u| - \mathbb{I}$

Claim | R_u, O preserve $V = \text{span}\{|u\rangle, |x^*\rangle\}$

$R_u |u\rangle$

Claim R_u, O preserve $V = \mathrm{span}\{|u\rangle, |x^*\rangle\}$

$$R_u|u\rangle = (2|u\rangle\langle u| - \mathbb{I})|u\rangle$$
$$= 2|u\rangle - |u\rangle = |u\rangle.$$

$$O|u\rangle = (\mathbb{I} - 2|x^*\rangle\langle x^*|)|u\rangle = |u\rangle - 2\langle x^*|u\rangle|x^*\rangle$$

$$O|x^*\rangle = -|x^*\rangle.$$

$$V = \mathrm{span}\left\{ |x^*\rangle ,\ |v\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq x^*} |x\rangle \right\}$$

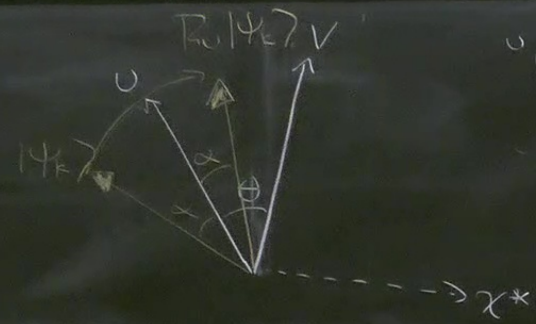
$$\mathcal{O} = \mathbb{I} - 2|x^*\rangle\langle x^*|$$

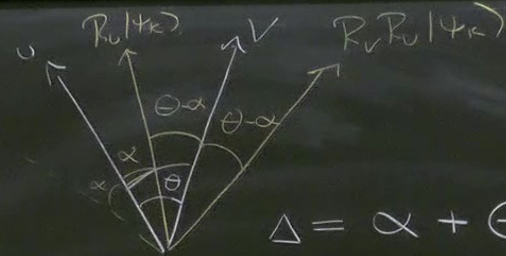
$$\mathcal{O}_V = \mathbb{I}_V - 2|x^*\rangle\langle x^*|$$

$$= \left(|v\rangle\langle v| + |x^*\rangle\langle x^*| \right) - 2|x^*\rangle\langle x^*|$$

$$= |v\rangle\langle v| - |x^*\rangle\langle x^*| = 2|v\rangle\langle v| - \mathbb{I}_V = R_V$$

$(o) \oplus f(1)$





$$\Delta = \alpha + \Theta + (\Theta - \alpha)$$

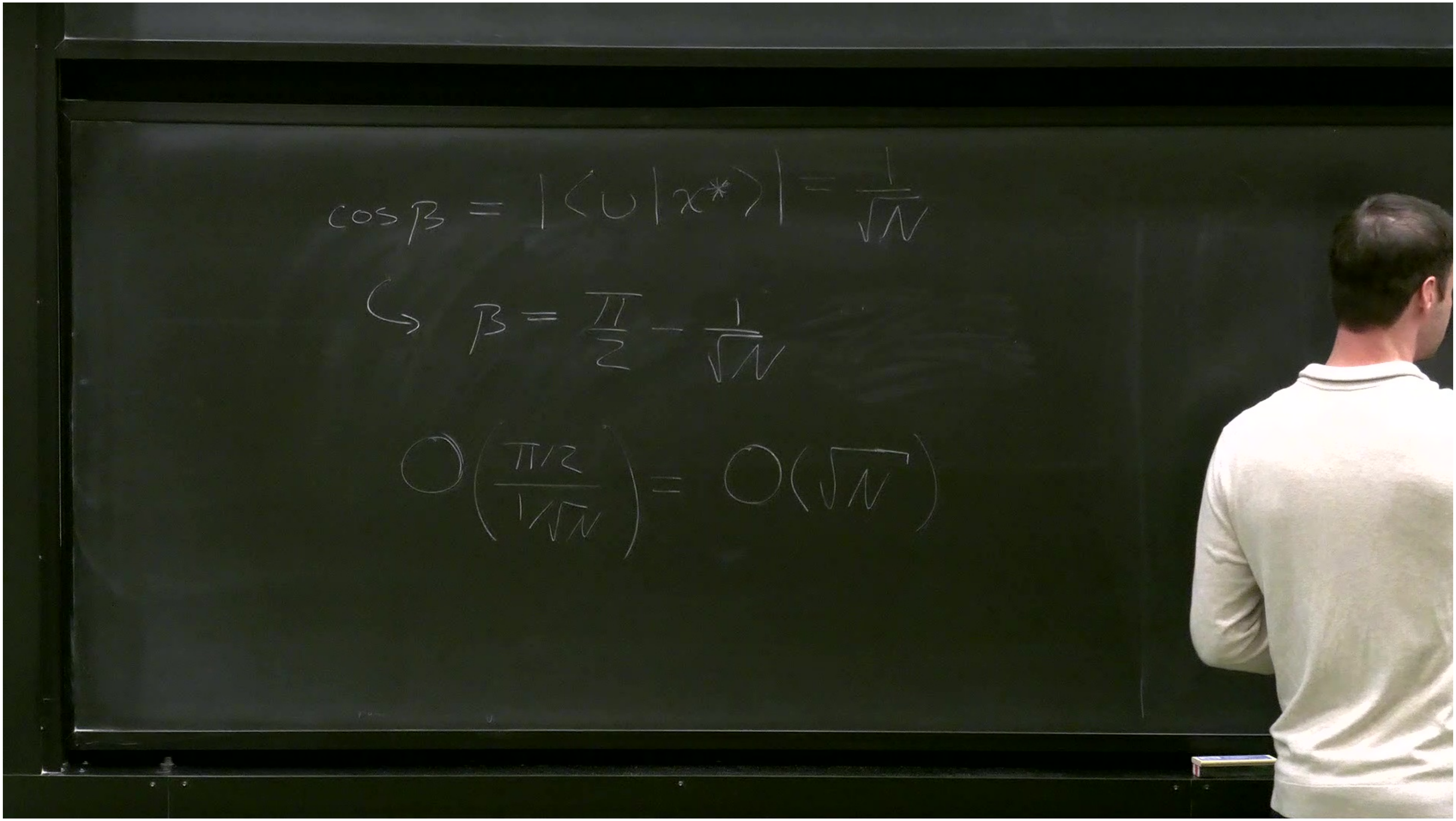
$$\Delta = 2\Theta$$

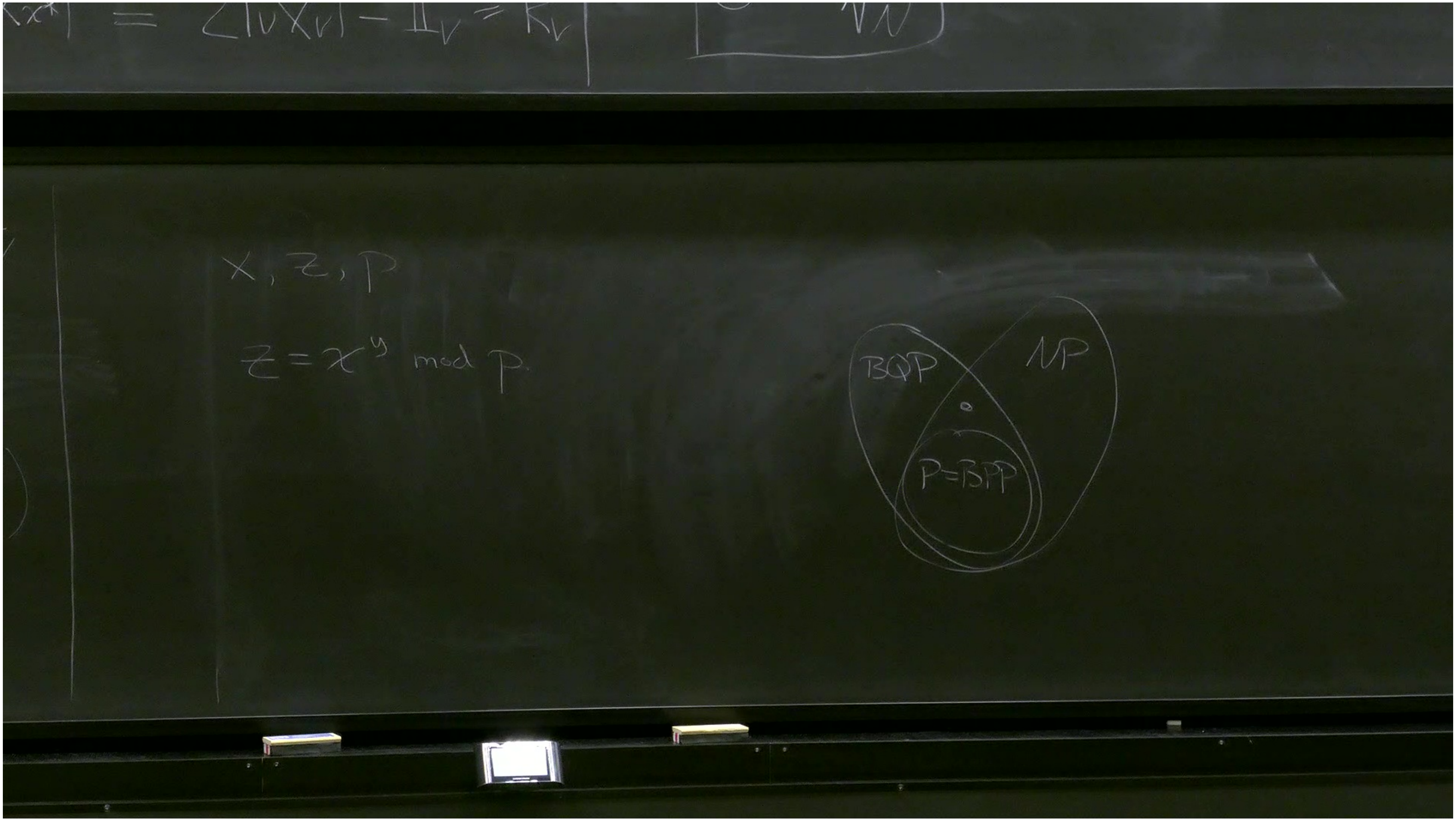
$$|u\rangle \longrightarrow |x^*\rangle$$

$$\cos\Theta = |\langle u|v\rangle| = \sqrt{1-\frac{1}{N}}$$

$$\Theta \simeq 1/\sqrt{N}$$

R_v


$\cos\beta = |\langle u | x^* \rangle| = \frac{1}{\sqrt{N}}$
$\hookrightarrow \beta = \frac{\pi}{2} - \frac{1}{\sqrt{N}}$
$O\left(\frac{\pi/2}{1/\sqrt{N}}\right) = O(\sqrt{N})$


X, Z, P
Z = x^y mod p
BQP
NP
P=BPP