Title: Lecture - Quantum Information, PHYS 635

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Subject: Quantum Information

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Shor's code (again)

Def[s]

$$|P\rangle \equiv \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

$$|M\rangle \equiv \frac{1}{\sqrt{2}}\left(|000\rangle - |111\rangle\right)$$

$$\boxed{\begin{aligned}
\Pi_0 &= |000\rangle\langle 000| + |111\rangle\langle 111| \\
\Pi_1 &= |100\rangle\langle 100| + |011\rangle\langle 011| \\
\Pi_2 &= |010\rangle\langle 010| + |101\rangle\langle 101| \\
\Pi_3 &= |001\rangle\langle 001| + |110\rangle\langle 110|
\end{aligned}}$$

$$\boxed{\begin{aligned}
\Pi_0' &= |PPP\rangle\langle PPP| + |MMM\rangle\langle MMM| \\
\Pi_1' &= |MPP\rangle\langle MPP| + |PMM\rangle\langle PMM| \\
\Pi_2' &= |PMP\rangle\langle PMP| + |MPM\rangle\langle MPM| \\
\Pi_3' &= |PPM\rangle\langle PPM| + |MMP\rangle\langle MMP|
\end{aligned}}$$

Encod

$|0\rangle \rightarrow$

$|1\rangle \rightarrow$

De

Encode:

$|0\rangle \to |PPP\rangle$

$|1\rangle \to |MMM\rangle$

Consider error $X_1 Z_1$

$$|\psi\rangle = \alpha|PPP\rangle + \beta|MMM\rangle$$

$$\xrightarrow{X_1 Z_1} \alpha\, X_1 Z_1\big(|000\rangle + |111\rangle\big)|PP\rangle + \beta\, X_1 Z_1\big(|000\rangle - |111\rangle\big)|MM\rangle$$

$$= \alpha\big(|100\rangle - |011\rangle\big)|PP\rangle + \beta\big(|100\rangle + |011\rangle\big)|MM\rangle$$

Decode

Step 1

measure $\{\Pi_i \otimes \Pi_j \otimes \Pi_k\}$

$\hookrightarrow$ obtain $\Pi_1 \otimes \Pi_0 \otimes \Pi_0 \to$ apply X_1

obtain state

$$\alpha|MPP\rangle + \beta|PMM\rangle$$

step 2

measure $\{\Pi_i'\}$, get $\Pi_1' \to$ apply Z_1

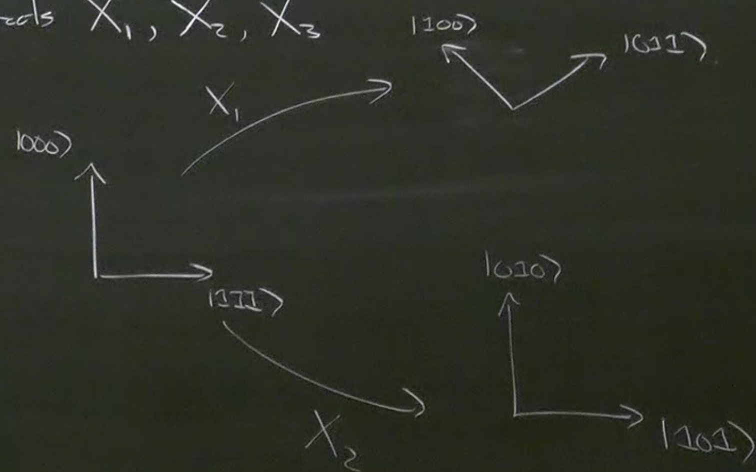
$$\alpha|PPP\rangle + \beta|MMM\rangle$$

encoding: $|0\rangle \to |000\rangle$

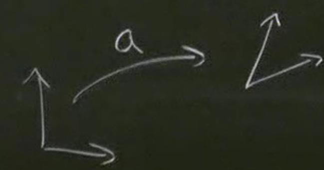
$|1\rangle \to |111\rangle$

$\mathrm{span}\{|000\rangle, |111\rangle\} = \mathrm{Im}\, V =$ "codespace" $\subseteq \mathcal{H}_P$

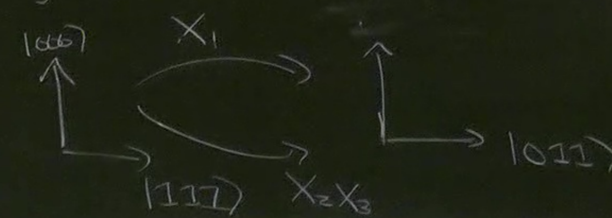
corrects X_1, X_2, X_3



1) don't change inner product ✗



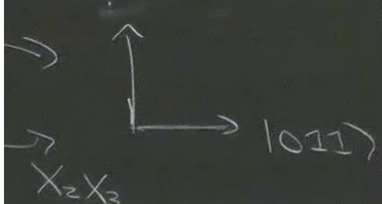
2) Don't let images under error a overlap.



$\mathbb{Z}/p$

X

...or a



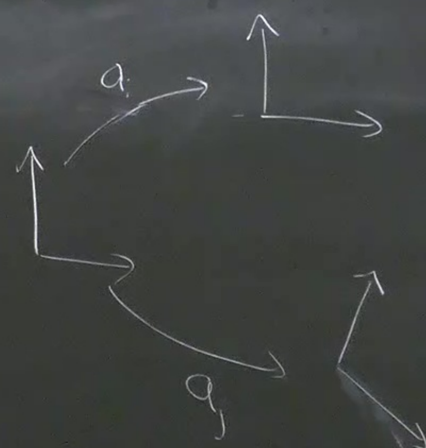
$|\psi\rangle, |\varphi\rangle \in C \quad \{a_i\}$ s.t.

$$(\langle\psi|a_i^\dagger)(a_j|\varphi\rangle) = \delta_{ij}\langle\psi|\varphi\rangle$$

can also correct

$(a_i + a_j)$

$\alpha a_i + \beta a_j$



can correct: $b_\ell = \sum_i M_{\ell i} a_i$

$$\langle\psi| b_\ell^\dagger b_k |\varphi\rangle = \sum_{i,j} M^*_{\ell j} M_{ki} \overbrace{\langle\psi| a_j^\dagger a_i |\varphi\rangle}^{\delta_{ij}\langle\psi|\varphi\rangle}$$

$$= \left(\sum_i M^*_{\ell i} M_{ki}\right)\langle\psi|\varphi\rangle$$

$$\boxed{\langle\psi| b_\ell^\dagger b_k |\varphi\rangle = \alpha_{\ell k}\langle\psi|\varphi\rangle}$$

Hermitian

"Knill-LaFlamme"

Hermitian

Lecture 8 - Stabilizer formalism

Def

$|\psi\rangle \in \mathcal{H}$ with n qubits

$|\psi\rangle = \sum_i \alpha_i |i\rangle$

2^n α_i's.

$|00\ldots0\rangle$, $(\alpha_1|0\rangle + \beta_1|1\rangle) \otimes \ldots \otimes (\alpha_n|0\rangle + \beta_n|1\rangle)$

$g|\psi\rangle = |\psi\rangle$

"g stabilizes $|\psi\rangle$"

Def Pauli group, G_n, on n qubits.

$$G_n = \left\{ \begin{array}{c} \text{all products of Paulis,} \\ \text{with } \pm 1, \pm i \end{array} \right\}$$

$$G_1 = \langle X, Z \rangle = \{ \pm \mathbb{1}, \pm i\mathbb{1}, \pm X, \pm iX, \ldots \}$$

$$G_2 = \langle \mathbb{1}X, X\mathbb{1}, \mathbb{1}Z, Z\mathbb{1} \rangle$$

specify $|\psi\rangle$.

$$|\psi\rangle \longrightarrow S = \{g_1, g_2, \dots\}$$

$$g_i \in G_n$$

$$\forall g_i \in S,\ g_i|\psi\rangle = |\psi\rangle.$$

$$|\psi\rangle = |0\rangle \longrightarrow S = \{Z\}$$

$$|+\rangle \longrightarrow S = \{X\}$$

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \longrightarrow S = \{XX, ZZ\}$$

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + \right.$$

specify $|\psi\rangle$.

$$|\psi\rangle \longrightarrow S = \{g_1, g_2, \dots\}$$

$$g_i \in G_n$$

$$\forall g_i \in S, g_i|\psi\rangle = |\psi\rangle.$$

$$|\psi\rangle = |0\rangle \longrightarrow S = \{Z\}$$

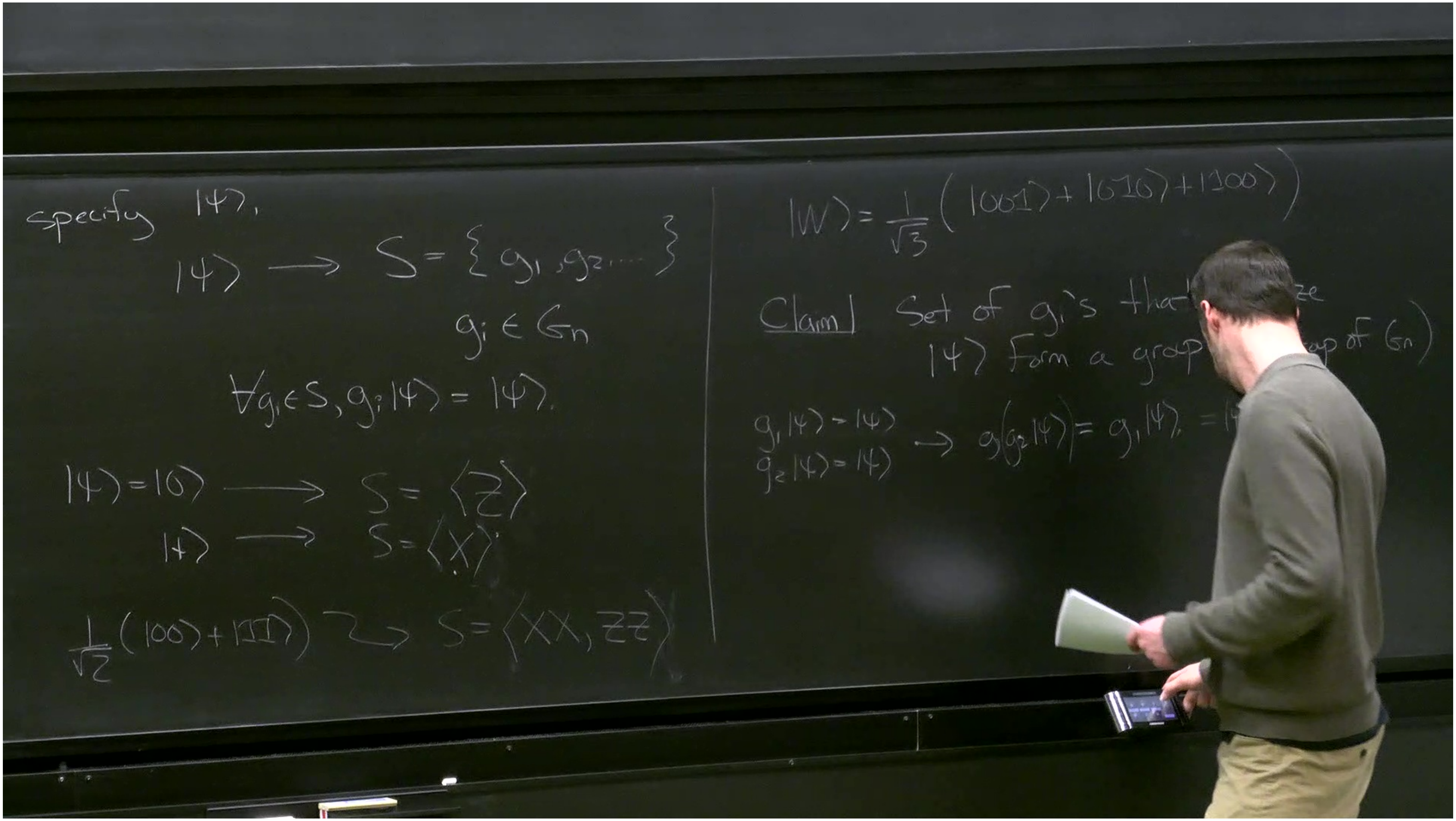
$$|+\rangle \longrightarrow S = \{X\}$$

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \longrightarrow S = \{XX, ZZ\}$$

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$$

Claim: Set of g_i's that stabilize $|\psi\rangle$ form a group (subgroup of G_n)

$$\begin{matrix} g_1|\psi\rangle = |\psi\rangle \\ g_2|\psi\rangle = |\psi\rangle \end{matrix} \rightarrow g_1(g_2|\psi\rangle) = g_1|\psi\rangle = |\psi\rangle.$$


specify |ψ⟩.
|ψ⟩ → S = {g1, g2, ...}
gi ∈ Gn
∀gi ∈ S, gi|ψ⟩ = |ψ⟩.
|ψ⟩ = |0⟩ → S = ⟨Z⟩
|+⟩ → S = ⟨X⟩
1/√2 (|00⟩ + |11⟩) → S = ⟨XX, ZZ⟩
|W⟩ = 1/√3 (|001⟩ + |010⟩ + |100⟩)
Claim
Set of gi's that
|ψ⟩ form a group
g1|ψ⟩ = |ψ⟩
g2|ψ⟩ = |ψ⟩
g1(g2|ψ⟩) = g1|ψ⟩

Not all subgroups of G_n are "good" stabilizer subgroups,

$\times \quad -\mathbb{I} \in S \qquad -\mathbb{I}|\psi\rangle = -|\psi\rangle$

$\times \quad S = \langle g_1, g_2, \ldots \rangle \qquad |\psi\rangle = g_1 g_2 |\psi\rangle = -g_2 g_1 |\psi\rangle = -|\psi\rangle$

$g_1 g_2 = -g_2 g_1$

Lemma $S = \langle g_1, \dots, g_{n-k} \rangle \subseteq G_n$

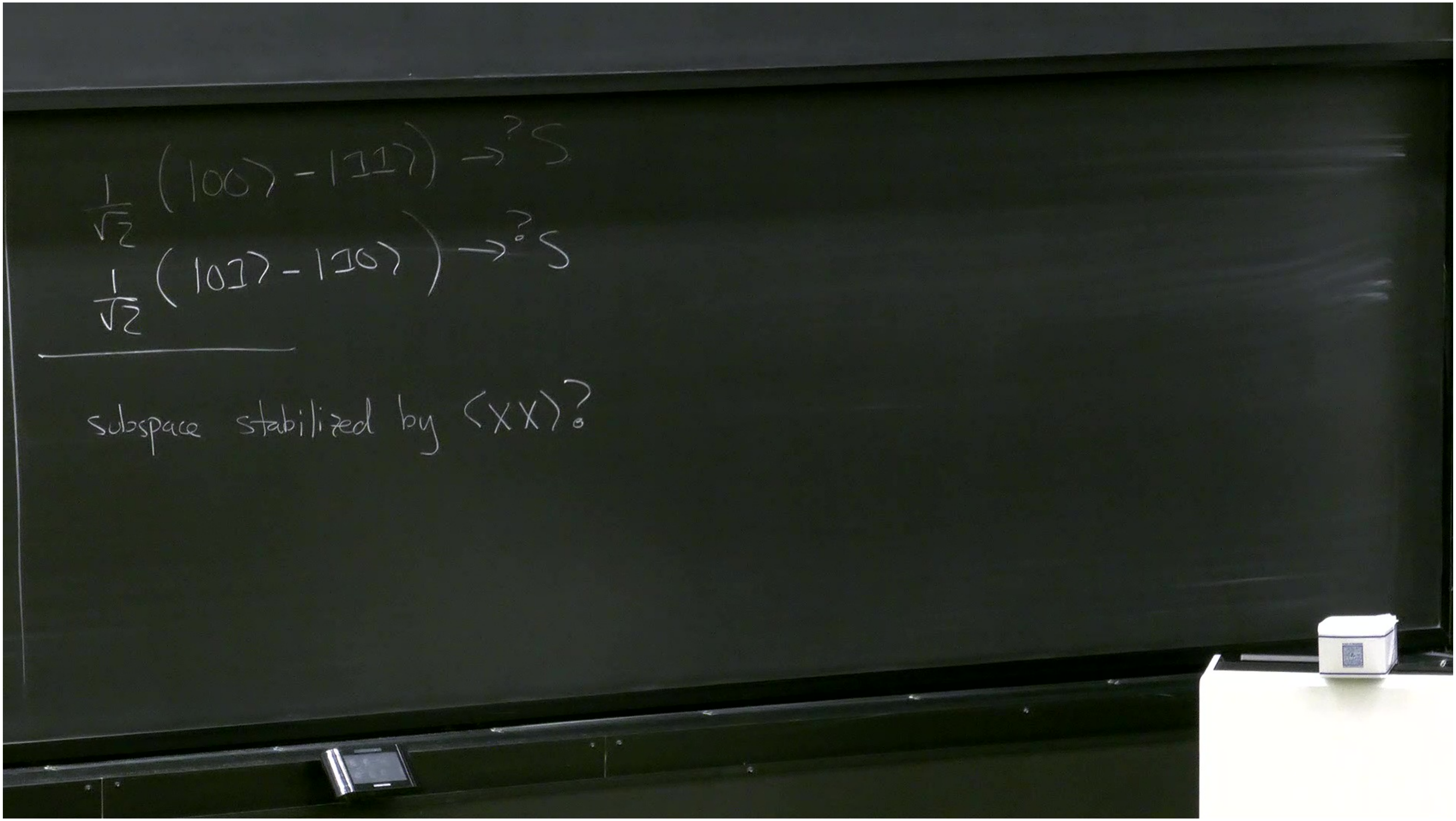
- g_i are independent, commute.
- $-\mathbb{1} \notin S$

Then S stabilizes a 2^k dimensional subspace

$$2^n / 2^{n-k} = 2^k$$

$$\frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$


$\frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \xrightarrow{?} S$
$\frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \xrightarrow{?} S$
subspace stabilized by $\langle XX \rangle$?

$$\langle ZZ, -XX \rangle$$

$$\langle -ZZ, -XX \rangle$$

$$\langle XX \rangle \longrightarrow \text{span}\left\{ \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big), \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big) \right\}$$

$$S = \langle g_1, g_2, \dots \rangle \longrightarrow S' = \langle U g_1 U^\dagger, U g_2 U^\dagger, \dots \rangle$$

$$\updownarrow$$

$$|\psi\rangle \longrightarrow U|\psi\rangle \qquad (U g_1 U^\dagger)(U|\psi\rangle) = U g_1 |\psi\rangle$$

$$= U|\psi\rangle$$

restrict U s.t. $U g_1 U^\dagger \in G_n$

$\ldots \rangle$

$)(U|\psi\rangle = U g_1 |\psi\rangle$

$= U|\psi\rangle$

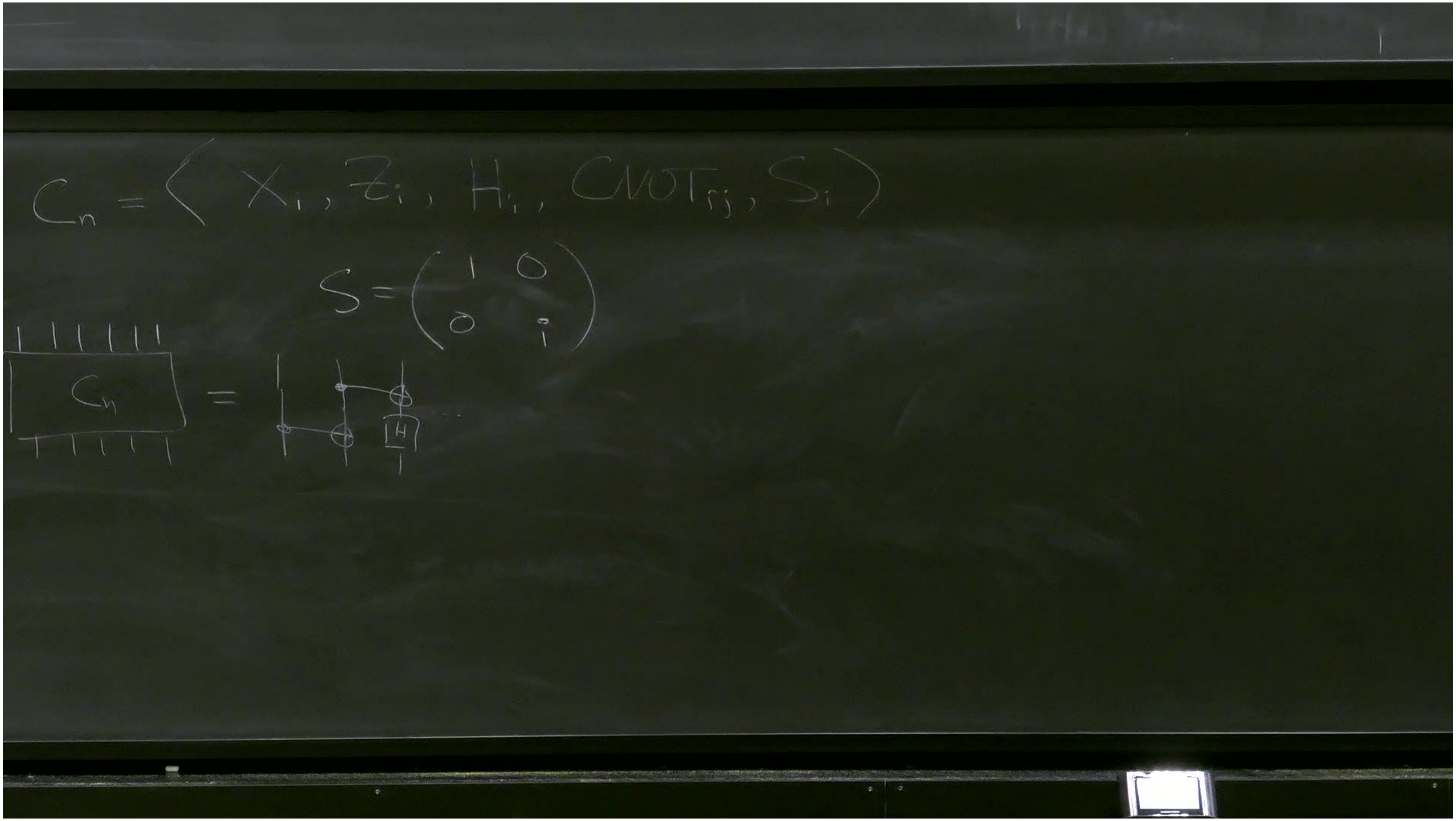
Def Clifford group $\mathcal{C}_n$ (on n qubits)

is all unitaries C s.t.

$$\forall g \in G_n, \quad C g C^\dagger \in G_n$$

$C_1, C_2 \in \mathcal{C}_n,$

$$C_1 (C_2 g C_2^\dagger) C_1^\dagger = C_1 g' C_1^\dagger = g'' \in G_n$$


$C_n = \langle X_i, Z_i, H_i, CNOT_{ij}, S_i \rangle$
$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
C_n
$=$
H