**Title:** Lecture - Quantum Information, PHYS 635

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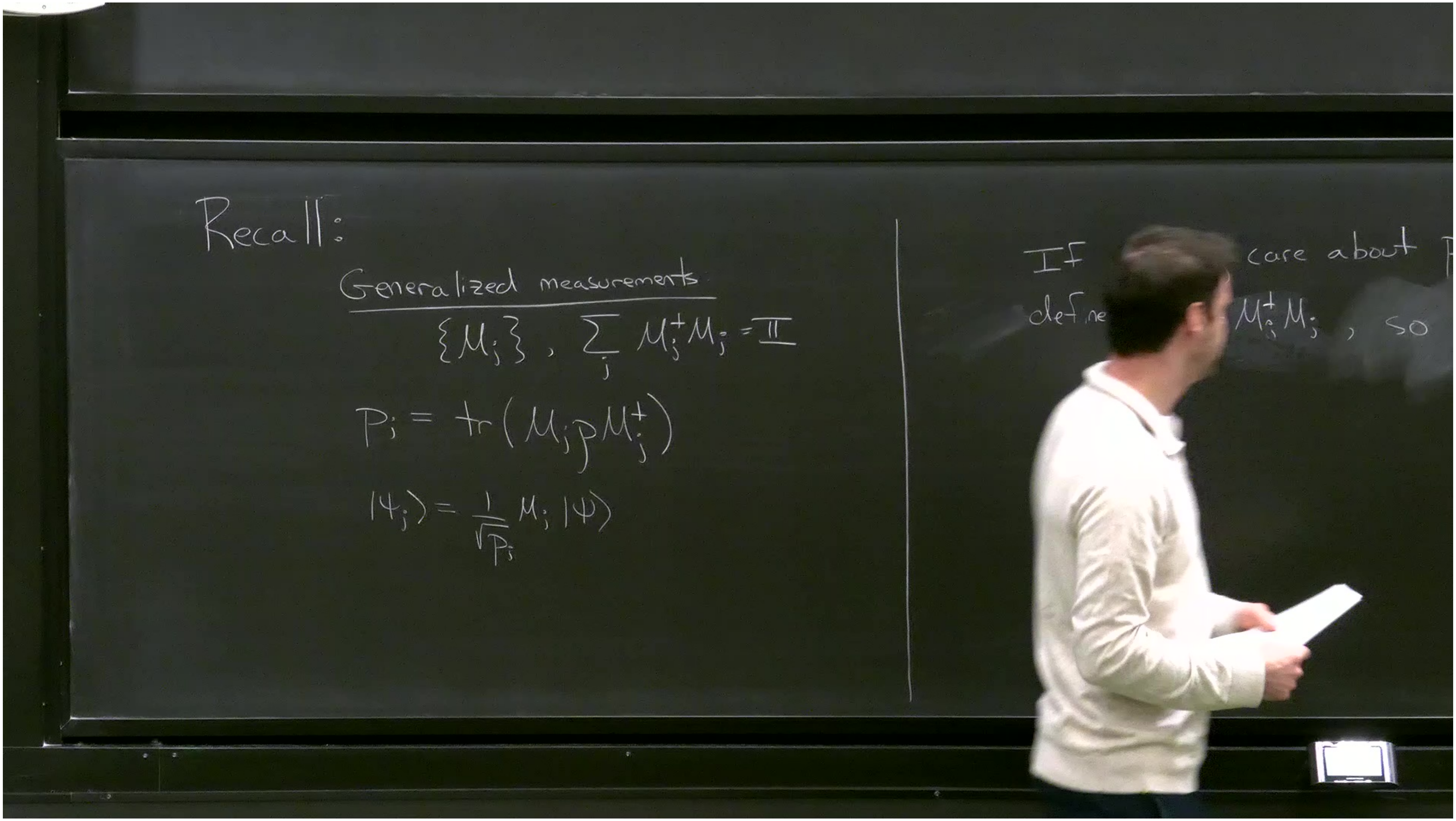

Recall:
Generalized measurements
$\{M_j\}$, $\sum_j M_j^\dagger M_j = \mathbb{1}$
$p_j = \mathrm{tr}(M_j \rho M_j^\dagger)$
$|\psi_j\rangle = \frac{1}{\sqrt{p_j}} M_j |\psi\rangle$
If
care about
define
$M_j^\dagger M_j$ , so

ed measurements

$\{ M_j \}$, $\sum_j M_j^\dagger M_j = \mathbb{I}$

$(M_j \rho M_j^\dagger)$

$\frac{1}{\sqrt{p_j}} M_j |\psi\rangle$

If we only care about $p_j$,

define $\Lambda_j = M_j^\dagger M_j$, so $p_j = \mathrm{tr}(\Lambda_j \rho)$

Def POVM $\{\Lambda_j\}$, $\sum_j \Lambda_j = \mathbb{I}$, $\Lambda_j \geq 0$

$p_j = \mathrm{tr}(\Lambda_j \rho)$

Herm

bout $p_j$,

, so $p_j = \mathrm{tr}(\Lambda_j \rho)$

, $\sum_j \Lambda_j = \mathbb{I}$, $\Lambda_j \geq 0$

Hermitian $B, A$, $\quad A \geq 0$

$$A \geq B \iff A - B \geq 0$$

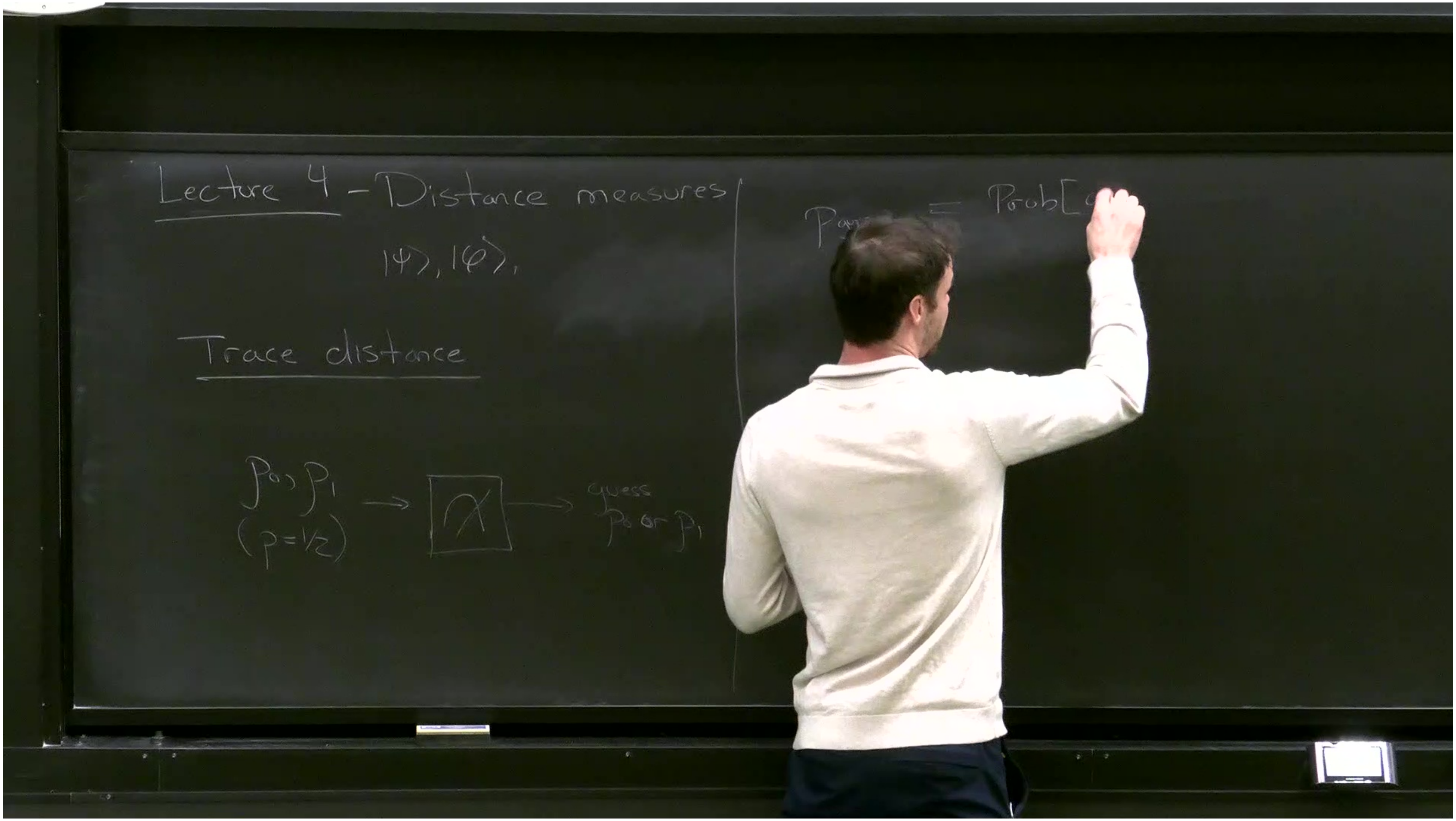
Fact: $A$ Herm., $0 \leq A \leq \mathbb{I}$

$$\mathrm{tr}(AC) \leq \mathrm{tr}\, C$$

$$\mathrm{tr}(AC) = \sum_i a_i \,\mathrm{tr}(|i\rangle\langle i| C)$$

$$= \sum_i a_i \langle i|C|i\rangle$$

$$\leq \sum_i \langle i|C|i\rangle = \mathrm{tr}\, C$$


Lecture 4 - Distance measures
|ψ⟩, |φ⟩,
Trace distance
ρ0, ρ1
(p=1/2)
guess
ρ0 or ρ1
Prob[

measures

$\rightarrow$ guess $\rho_0$ or $\rho_1$

$$P_{guess} = \text{Prob}[\text{given } \rho_0]\, \text{Prob}[\text{guess } \rho_0 | \rho_0] + \text{Prob}[\text{given } \rho_1]\, \text{Prob}[\text{guess } \rho_1 | \rho_1]$$

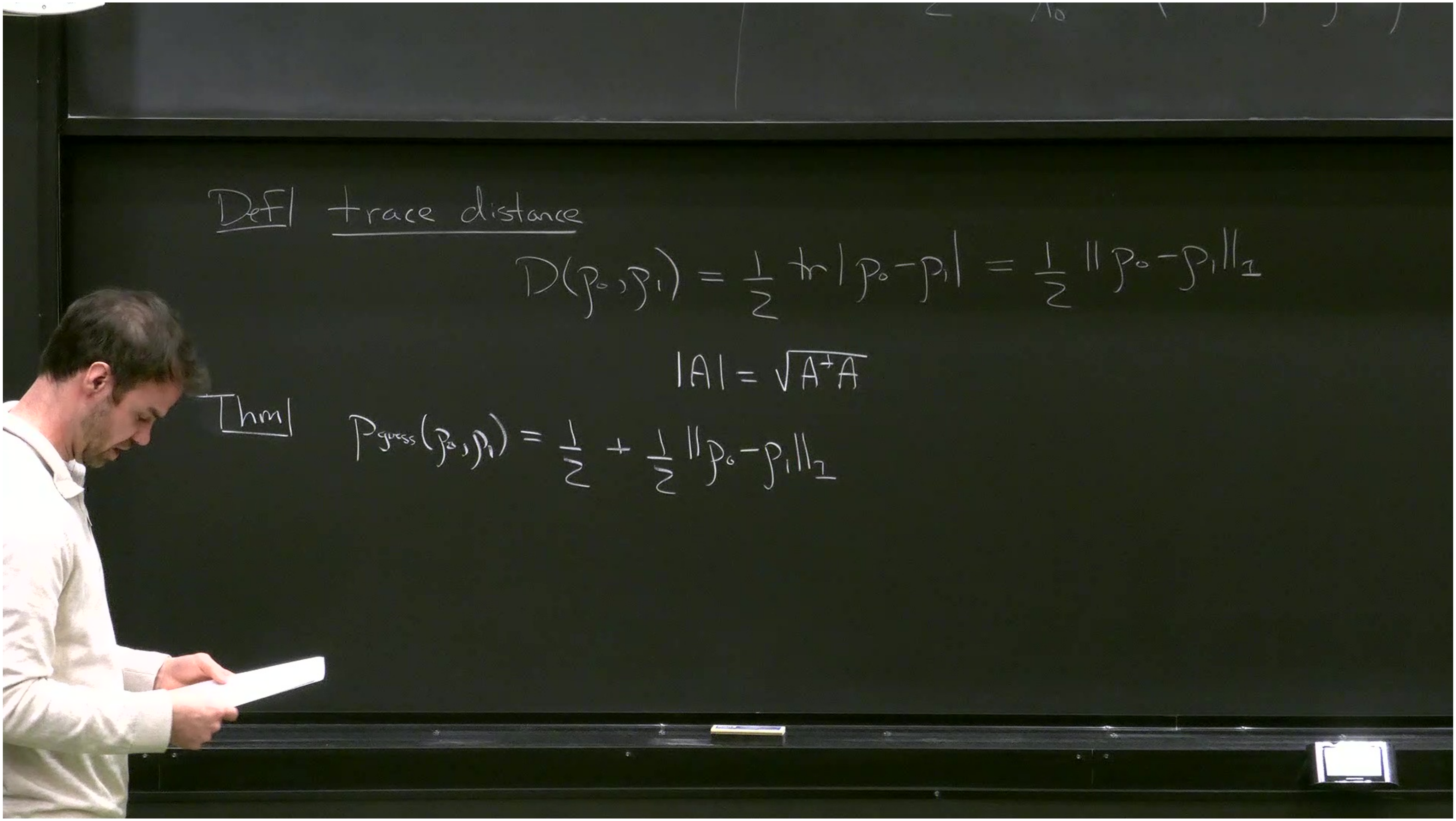
$\{\Lambda_0, \Lambda_1\}$

$$P_{guess} = \max_{\Lambda_0, \Lambda_1} \left[ \frac{1}{2} \mathrm{tr}(\Lambda_0 \rho_0) + \frac{1}{2} \mathrm{tr}(\Lambda_1 \rho_1) \right]$$

$\Lambda_1 = \mathbb{I} - \Lambda_0$

$$= \max_{\Lambda_0} \frac{1}{2} \mathrm{tr}(\Lambda_0 \rho_0) + \frac{1}{2} \mathrm{tr}(\mathbb{I} \rho_1) - \frac{1}{2} \mathrm{tr}(\Lambda_0 \rho_1)$$

$$P_{guess} = \frac{1}{2} + \max_{\Lambda_0} \mathrm{tr}\left( \Lambda_0 (\rho_0 - \rho_1) \right)$$


Def| trace distance
$D(\rho_0,\rho_1) = \frac{1}{2}\,\mathrm{tr}|\rho_0-\rho_1| = \frac{1}{2}\|\rho_0-\rho_1\|_1$
$|A| = \sqrt{A^\dagger A}$
Thm| $p_{guess}(\rho_0,\rho_1) = \frac{1}{2} + \frac{1}{2}\|\rho_0-\rho_1\|_1$

$(\rho_0 - \rho_1))$

Proof

$$\rho_0 - \rho_1 = \sum_i \lambda_i |i\rangle\langle i| \quad (\lambda_i \text{ +ve or -ve})$$

$$P = \sum_{\lambda_i \geq 0} \lambda_i |i\rangle\langle i| \qquad \Pi_P = \sum_{\lambda_i \geq 0} |i\rangle\langle i|$$

$$Q = \sum_{\lambda_i < 0} |\lambda_i| \, |i\rangle\langle i| \qquad \Pi_Q = \sum_{\lambda_i < 0} |i\rangle\langle i|$$

$$\mathrm{tr}\,|\rho_0 - \rho_1| = \mathrm{tr}\,|P - Q|$$
$$= \mathrm{tr}\,P + \mathrm{tr}\,Q = 2\,\mathrm{tr}\,P$$

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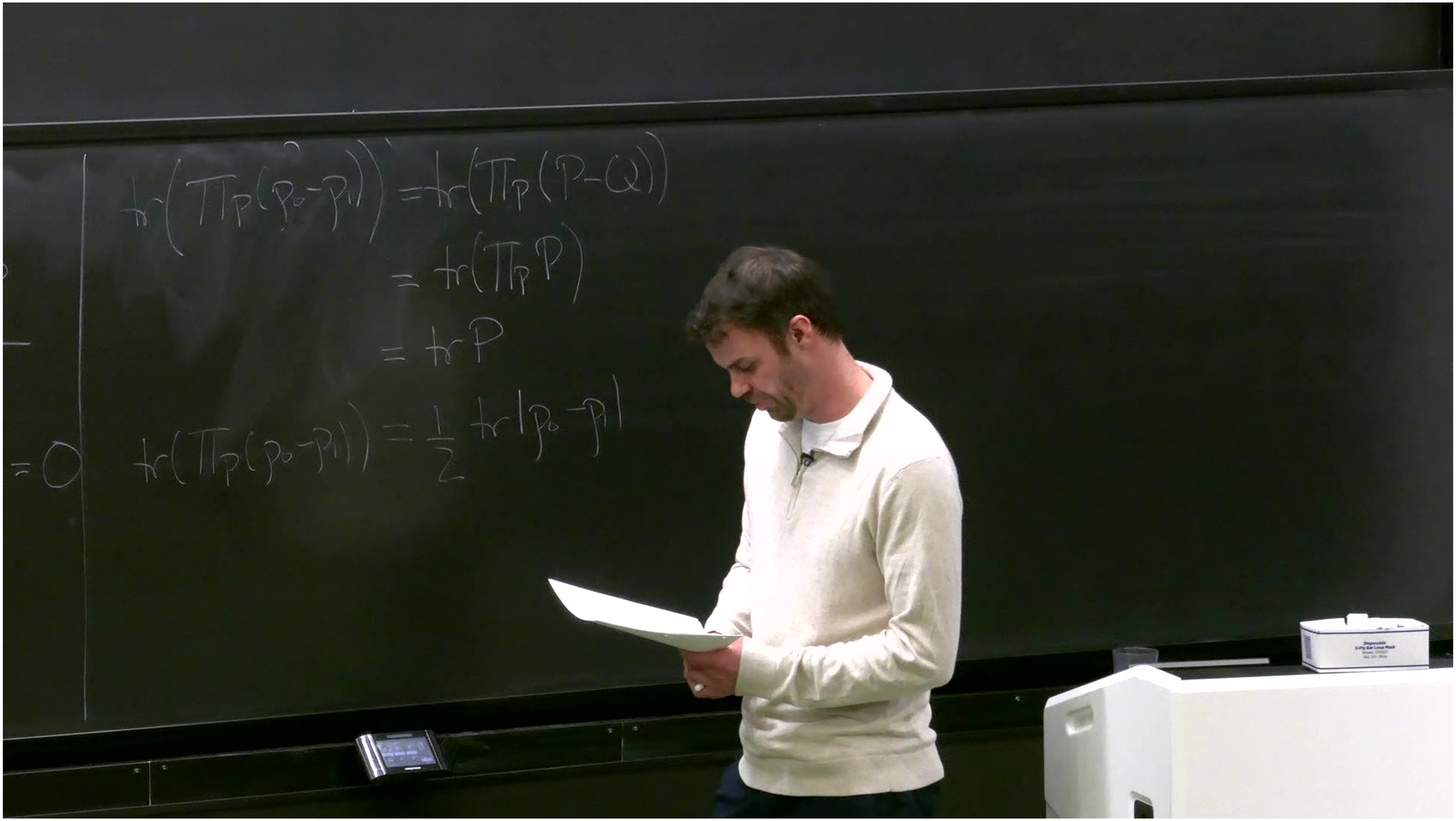
$$\mathrm{tr}\,P - \mathrm{tr}\,Q = \mathrm{tr}(P - Q)$$
$$= \mathrm{tr}(\rho_0 - \rho_1) = \mathrm{tr}\,\rho_0 - \mathrm{tr}\,\rho_1 = 0$$



$\mathrm{tr}(\Pi_P(\rho_0-\rho_1)) = \mathrm{tr}(\Pi_P(P-Q))$
$= \mathrm{tr}(\Pi_P P)$
$= \mathrm{tr}\, P$
$= 0$
$\mathrm{tr}(\Pi_P(\rho_0-\rho_1)) = \frac{1}{2}\mathrm{tr}|\rho_0-\rho_1|$

$$
\begin{aligned}
\operatorname{tr}\left(\Lambda_0(\rho_0-\rho_1)\right) &= \operatorname{tr}\left(\Lambda_0(P-Q)\right)\\
&\leq \operatorname{tr}(\Lambda_0 P)\\
&\leq \operatorname{tr} P = \frac{1}{2}\operatorname{tr}|\rho_0-\rho_1|
\end{aligned}
$$

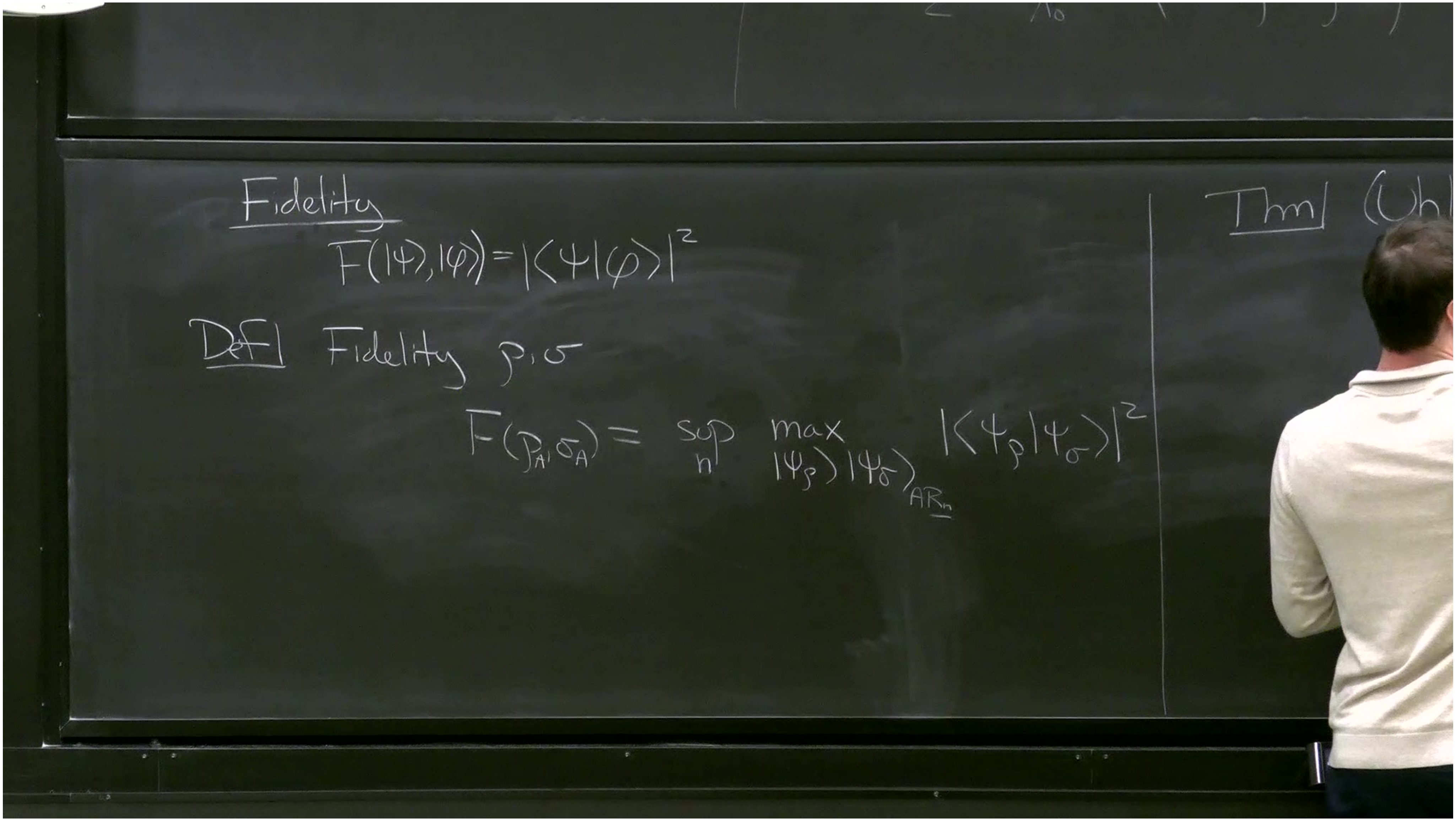
$$\mathrm{tr}|M| = \|M\|_1$$

$$\mathrm{tr}(|cM|) = |c|\,\mathrm{tr}|M|$$

$$\mathrm{tr}(|M+N|) \leq \mathrm{tr}|M| + \mathrm{tr}|N|$$

$$D(\rho,\rho) = 0 \qquad D(\rho,\sigma) = D(\sigma,\rho)$$

$$D(\rho,\sigma) \geq 0 \qquad D(\rho,\sigma) \leq D(\rho,\gamma) + D(\gamma,\sigma)$$


Fidelity
$F(|\psi\rangle,|\varphi\rangle) = |\langle\psi|\varphi\rangle|^2$
Def Fidelity $\rho, \sigma$
$F(\rho_A, \sigma_A) = \sup_n \max_{|\psi_\rho\rangle |\psi_\sigma\rangle_{AR_n}} |\langle\psi_\rho|\psi_\sigma\rangle|^2$
Thm (Uhl

$$\operatorname{tr}\left(\Lambda_0(\rho_0-\rho_1)\right)$$

$\rangle|^2$

**Thm** (Uhlmanns theorem)

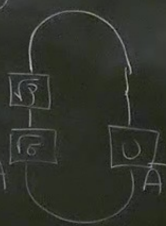
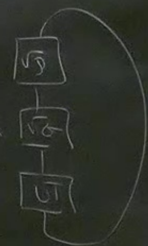
$$F(\rho_A,\sigma_A) = \left\|\sqrt{\rho_A}\sqrt{\sigma_A}\right\|_1 = \operatorname{tr}\left(\sqrt{\sqrt{\sigma}\,\rho\,\sqrt{\sigma}}\right)$$

$$|\psi_\rho\rangle = V_{\bar{A}\to R_n}\left(\sqrt{\rho_A}\otimes \mathbb{I}_{\bar{A}}\right)|\Phi^+\rangle_{A\bar{A}}$$

$$|\psi_\sigma\rangle = W_{\bar{A}\to R_n}\left(\sqrt{\sigma_A}\otimes \mathbb{I}_{\bar{A}}\right)|\Phi^+\rangle_{A\bar{A}}$$

$$\sqrt{F} = \sup_n \max_{V,W} \langle\Phi^+|\left(\sqrt{\rho_A}\otimes\mathbb{I}_{\bar{A}}\right)\underbrace{V^\dagger_{\bar{A}\to R_n}W_{\bar{A}\to R_n}}_{U_{\bar{A}}}\left(\sqrt{\sigma_A}\otimes\mathbb{I}_{\bar{A}}\right)|\Phi^+\rangle$$

$$\sqrt{F} = \max_U \langle \Psi^+ | \sqrt{\rho_A}\sqrt{\sigma_A} \otimes U_{\bar{A}} | \Psi^+ \rangle = \dots = \| \sqrt{\rho_A}\sqrt{\sigma_A} \|_1$$

$= \max_{U, A}$  $= \max_U$  $= \max_U \mathrm{tr}(\sqrt{\rho}\sqrt{\sigma}\, U)$

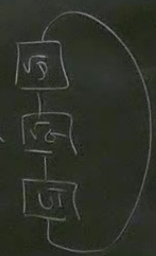
$= \| \sqrt{\rho}\sqrt{\sigma} \|_1$

$$\max_U \left( \langle \Psi^+ | \sqrt{\rho}_A \otimes \mathbb{I}_{\bar{A}} \right.$$

$$U_{\bar{A}}$$

$$\ldots\sqrt{\rho_A}\sqrt{\sigma_A}\otimes U_{\bar{A}}\,|\Phi^+\rangle = \ldots = \|\sqrt{\rho_A}\sqrt{\sigma_A}\|_1$$

$$= \max_U$$



$$= \max_U \mathrm{tr}\left(\sqrt{\rho}\sqrt{\sigma}\,U^T\right) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$$

$$\max_U \underbrace{\left(\langle\Phi^+|\sqrt{\rho}_A\otimes \mathbb{I}_{\bar{A}}\right)}_{|\psi_\rho\rangle}\underbrace{\left(\sqrt{\sigma_A}\otimes U_{\bar{A}}\,|\Phi^+\rangle_{A\bar{A}}\right)}$$

$$F(\rho,\sigma) = \max_{|\psi_\sigma\rangle} |\langle\psi_\rho|\psi_\sigma\rangle|^2$$

$\|M\|_1$

$\mathrm{tr}(|cM|) = |c|\,\mathrm{tr}|M|$

$\mathrm{tr}(|M+N|) \le \mathrm{tr}|M| + \mathrm{tr}|N|$

$D(\rho,\sigma) = D(\sigma,\rho)$

$D(\rho,\sigma) \le D(\rho,\gamma) + D(\gamma,\sigma)$

$$\underbrace{\|M\|_1 = \max_U \mathrm{tr}(UM)}$$

$D(\rho,\sigma) \le \epsilon$

$p_{guess} = \frac{1}{2} + \frac{D(\rho,\sigma)}{2} = 1$

$p_{guess} \le \frac{1}{2} + \epsilon/2$

$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\frac{1}{2}\mathrm{tr}|\rho-\sigma| = \frac{1}{2}\mathrm{tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$