Title: Lecture - Quantum Field Theory III - PHYS 777

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Subject: Quantum Fields and Strings

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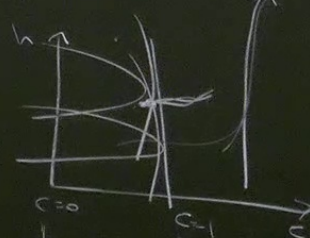
Minimal models & Ising CFT

Recap Unitarity of CFTs: unitary if $\nexists\ |\chi\rangle \neq 0$ $\langle\chi|\chi\rangle \leq 0$.

→ $c<0$ or $\exists\ h_i<0 \Rightarrow$ CFT is not unitary.

→ $c>1,\ h>0$ - CFTs are unitary $\qquad \det S^{(\ell)}(h,c)$



→ $0<c<1,\ h>0$ - most are not unitary. Special ones are unitary $\mathcal{M}(m+1,m)$

zeroes of Kac determinant

→ Unitary minimal models $\mathcal{M}(m+1,m)$

$$c = 1 - \frac{6}{m(m+1)}, \quad h_{r,s}(m) = \frac{\left[(m+1)r - ms\right]^2 - 1}{4m(m+1)} \qquad \begin{matrix} 1 \leq r < m+1 \\ 1 \leq s < m \end{matrix}$$

Ising CFT $\quad m=3 \quad : \mathcal{M}(4,3) \qquad c=1/2$

$\mathbb{1} \qquad \sigma_i \rightsquigarrow \sigma \qquad \sigma_i\sigma_{i+1} \rightsquigarrow \varepsilon$

$(h,\bar h)=(0,0) \qquad (h,\bar h)_\sigma=\left(\frac{1}{16},\frac{1}{16}\right) \qquad (h,\bar h)_\varepsilon=\left(\frac{1}{2},\frac{1}{2}\right)$

OPEs: $\sigma\times\sigma=\mathbb{1}+\varepsilon \quad , \quad \sigma\times\varepsilon=\sigma \quad , \quad \varepsilon\times\varepsilon=\mathbb{1}$ (these are consistent with $\mathbb{Z}_2$)

- $\Delta_\sigma=h_\sigma+\bar h_\sigma=\frac{1}{8} \qquad \Delta_\varepsilon=h_\varepsilon+\bar h_\varepsilon=1$

i.e. $\langle\varepsilon(z)\varepsilon(0)\rangle=\frac{1}{|z|^{2\Delta_\varepsilon}}$

$\langle\sigma(z)\sigma(0)\rangle=\frac{1}{|z|^{2\Delta_\sigma}}$

$G(r)\propto r^{-2(d-y_\sigma)}$

$$\boxed{\alpha=0,\ \beta=\frac{1}{8} \quad \gamma=\frac{7}{4} \quad \delta=15 \quad \nu=1 \quad \eta=1/4}$$

$$\boxed{y_\sigma=\frac{15}{8} \quad y_\varepsilon=1} \quad \Leftarrow$$

Singular vectors

Sing. vect. $|\chi\rangle$: $\quad L_n|\chi\rangle = 0 \quad n>0 \quad |\chi\rangle \in V_{h,c}$

0	$(\lvert h\rangle)$
1	$L_{-1}\lvert h\rangle$
2	$(L_{-1})^2\lvert h\rangle \quad L_{-2}\lvert h\rangle$
ℓ	$\lvert\chi\rangle$

→ Singular vectors are orthogonal to all vectors in $V_{h,c}$:

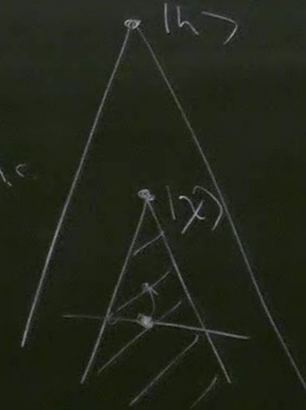
$$\langle\lambda,h|\chi\rangle = (\langle h|L_{\lambda_1}\ldots L_{\lambda_n}|\chi\rangle) = 0 \qquad \lambda_i > 0$$

⇒ Kac determinant $\det S^{(\ell)}$ have zero at such h. that $|\chi\rangle$ exists in $V_{h,c}$

$\Rightarrow$ Each s.v. gives rise to Verma submodule $V_{\tilde{h},c} \subset V_{h,c}$

where $\tilde{h}$: $L_0 |\chi\rangle = \tilde{h} |\chi\rangle$, $\tilde{h} = h + \ell$

$$V_{\tilde{h},c} = \{ L_{-\lambda_n} \dots L_{-\lambda_1} |\chi\rangle \}$$



$\rightarrow$ Every vector in $V_{\tilde{h},c}$ is orthogonal to whole $V_{h,c}$

$$\langle h | L_{\mu_1} \dots L_{\mu_m} \cdot L_{-\lambda_n} \dots L_{-\lambda_1} |\chi\rangle$$

$$[L_n, L_m] = (n-m) L_{n+m} + c \ \dots \ \delta_{n+m,0}$$

$$\sum \mu_i > \sum \lambda_i \Rightarrow \langle h | \dots L_k |\chi\rangle = 0$$

$k > 0$

that $|\chi\rangle$ exists in $V_{h,c}$

We can understand structure of Kac determinant better:

$$\det S^{(\ell)} = A_\ell \prod_{\substack{r,s\geq 1 \\ r\cdot s\leq \ell}} \left[h - h_{r,s}(c)\right]^{p(\ell - r\cdot s)}$$

$$h_{r,s}(c) = h_0 + \frac{(r\alpha_+ + s\alpha_-)^2}{4}$$

$$h_0 = \frac{1}{24}(c-1) = -\frac{1}{24}(1-c)$$

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}$$

→ The conformal dim. $h = h_{r,s}$ are such that $V_{h_{r,s},c}$ has singular vector on the level $r\cdot s$.

→ $\det S^{(\ell)}$ on levels $\ell \geq r\cdot s$ has zero of multiplicity $p(\ell - rs)$ at $h = h_{r,s}$ because of descendants of $|\chi\rangle$

$$L_{-\lambda_n} \dots L_{-\lambda_1} |\chi\rangle \in V_{\tilde{h},c} \subset V_{h_{r,s},c}, \quad \sum \lambda_i = \ell - r\cdot s$$

- One Verma modul can contain several singular vector.

$$M(h,c) = V(h,c) \Big/ \left(\begin{matrix}\text{all singular} \\ \text{submodules}\end{matrix}\right)$$

The resulting scalar product doesn't have null vectors.

$\mathcal{M}(p', p)$ - minimal models are not unitary in general

Selection rules for degenerate operators

Let's start with $h_{2,1}$: $|\chi\rangle = \left(L_{-2} - \frac{3}{2(2h+1)}(L_{-1})^2\right)|h\rangle$ - singular if $h = h_{2,1}$

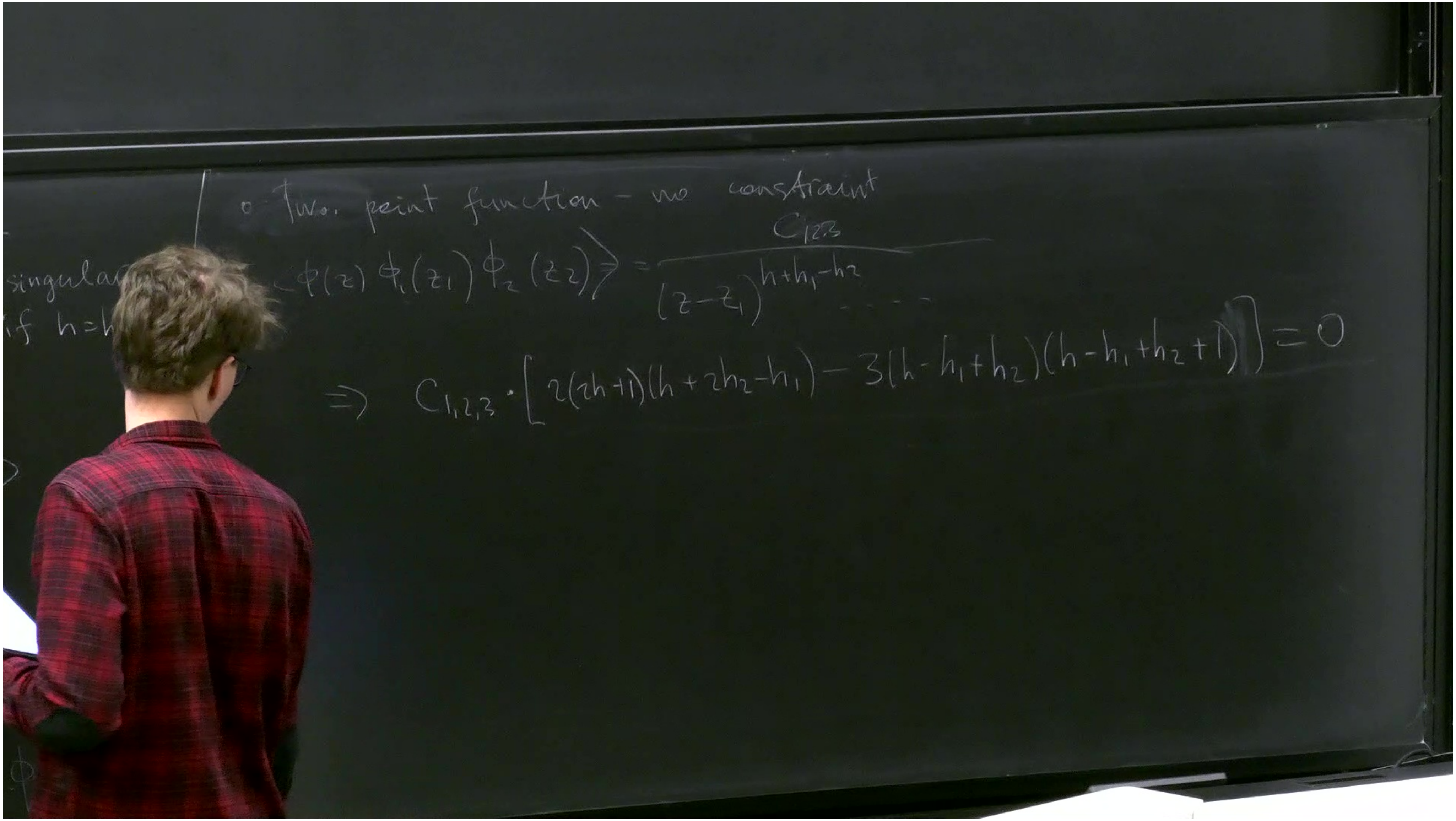
$$h = h_{2,1} = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right)$$

$$\chi(z) = (L_{-2}\phi)(z) - \frac{3}{2(2h+1)} \frac{\partial^2 \phi(z)}{\partial z^2}, \quad |h\rangle = \lim_{z \to 0} \phi(z)|0\rangle$$

- Decoupling condition: $\langle \chi(z) X \rangle = 0$

$$\Rightarrow \left\{ \sum_{i=1}^{n} \left(\frac{1}{z - z_i} \frac{\partial}{\partial z_i} + \frac{h_i}{(z - z_i)^2} \right) - \frac{3}{2(2h+1)} \frac{\partial^2}{\partial z^2} \right\} \langle \phi(z) X \rangle = 0$$

$X = \phi_1(z_1) \ldots \phi_n(z_n)$



∘ two point function – no constraint
$\langle \phi(z)\, \phi_1(z_1)\, \phi_2(z_2) \rangle = \frac{C_{123}}{(z-z_1)^{h+h_1-h_2}} \cdots$
$\Rightarrow C_{1,2,3} \cdot \left[2(2h+1)(h+2h_2-h_1) - 3(h-h_1+h_2)(h-h_1+h_2+1) \right] = 0$

singular if $h = h_{2,1}$

$) = 0$

$\phi_n(z_n))$

$\langle \phi(z)\, \phi_1(z_1)\, \phi_2(z_2) \rangle = \dfrac{\quad}{(z-z_1)^{h+h_1-h_2}} \cdots$

$$\Rightarrow \quad C_{1,2,3} \cdot \left[2(2h+1)(h+2h_2-h_1) - 3(h-h_1+h_2)(h-h_1+h_2+1) \right] = 0$$

$$h(\alpha) = \frac{1}{24}(c-1) + \frac{1}{4}\alpha^2$$

$h_1 = h(\alpha_1) \qquad h_2 = h(\alpha_2)$

$\alpha_2 = \alpha_1 \pm \alpha_+ \quad (\text{if } h = h_{2,1})$

$\alpha_2 = \alpha_1 \pm \alpha_- \quad (\text{if } h = h_{1,2})$

$$\phi_{(2,1)} \times \phi_{(\alpha)} = \phi_{(\alpha-\alpha_+)} + \phi_{(\alpha+\alpha_+)}$$

$$\phi_{(1,2)} \times \phi_{(\alpha)} = \phi_{(\alpha-\alpha_-)} + \phi_{(\alpha+\alpha_-)}$$

Commutativity of fushion rules gives extra constraints

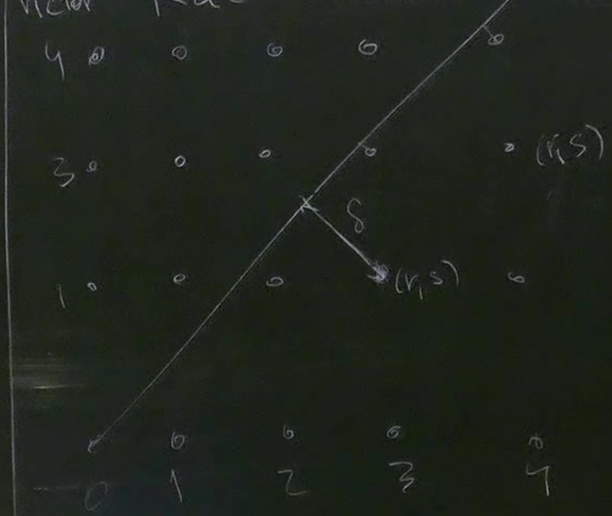
$$\Rightarrow \quad \phi_{(r_1;s_1)} \cdot \phi_{(r_2;s_2)} = \sum_{\substack{k=1+|r_1-r_2| \\ k+r_1+r_2=1 \bmod}}^{k=r_1+r_2-1} \; \sum_{\substack{\ell=1+|s_1-s_2| \\ \ell+s_1+s_2=1 \bmod 2}}^{\ell=s_1+s_2-1} \phi_{(k,\ell)}$$

$$\phi_{(r,s)} = \phi_{\substack{h=h_{r,s} \\ \alpha = r\alpha_+ + s\alpha_-}}$$

$\Rightarrow$ Degenerate fields form closed operator algebra.

It is generated by $\phi_{(1,2)}$ and $\phi_{(2,1)}$.

Victor Kac table

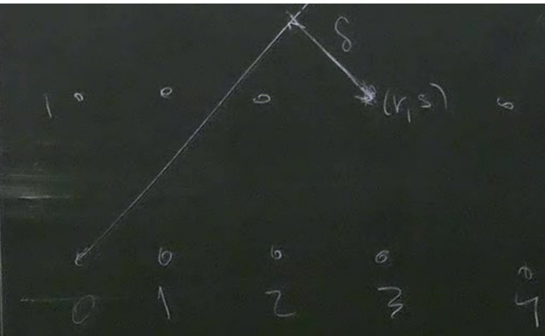


Mimal models

$$\tan\theta = -\frac{\alpha_+}{\alpha_-}, \quad h_{r,s} = h_0 + \frac{1}{4}\delta^2(\alpha_+^2 + \alpha_-^2)$$

$\tan\theta \notin \mathbb{Q}$ $\quad \delta \sim 0, h_0 < 0 \Rightarrow h_{r,s} < 0$

$\tan\theta \in \mathbb{Q} \Rightarrow \tan\theta = \frac{p}{p'}$ – minimal models $M(p', p)$.



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$\tan\theta \in \mathbb{Q} \Rightarrow \tan\theta = \frac{p}{p'}$ – minimal models

$\mathcal{M}(p', p)$.

$$c = 1 - 6\,\frac{(p-p')^2}{p\,p'}\,, \quad h_{r,s} = \frac{(pr-p's)^2-(p-p')^2}{4pp'}$$

(r,s) — $h = h_{r,s}$ ... $+ r_1 + r_2 = 1 \mod$ ... $+ s_1 + s_2 = 1 \mod 2$

$\alpha = r\alpha_+ + s\alpha_-$

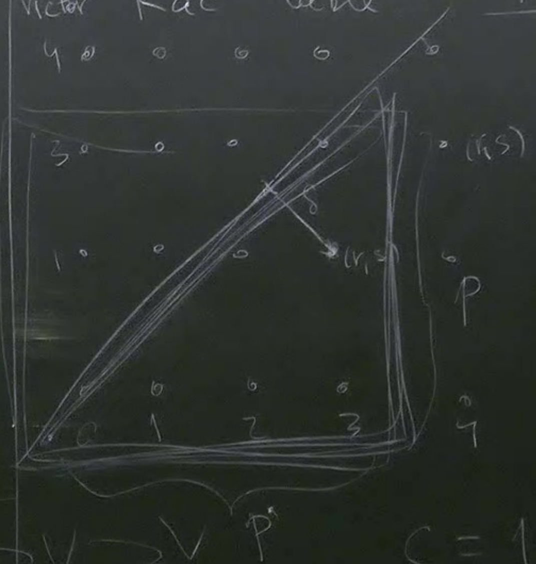
$\Rightarrow$ Degenerate fields form closed operator algebra.

It is generated by $\phi_{(1,2)}$ and $\phi_{(2,1)}$.

- $h_{r,s} = h_{r+p', s+p}$
- $h_{r,s} + rs = h_{p'+r, p-s} = h_{p'-r, p+s}$
- $h_{r,s} + (p'-r)(p-s) = h_{r, 2p-s} = h_{2p'-r, s}$

$V \supset V$

Victor Kac table



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$$c = 1 - 6\frac{(p-p')^2}{p\,p'}, \quad h_{r,s} = \frac{(pr - p's)^2 - (p-p')^2}{4pp'}$$

$$\boxed{\text{Ising: } \mathbb{1} = \phi_{(1,1)} \quad \sigma = \phi_{(2,2)} \quad \varepsilon = \phi_{(2,1)}}$$