Title: Lecture - Quantum Field Theory III - PHYS 777

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Subject: Quantum Fields and Strings

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Recap Last time we talked about conformal blocks (holomorphic building blocks for 4 pt fns)

$$\phi_{h_1}(z,\bar z)\,\phi_{h_2}(w,\bar w) = \sum_{h_3}\ \sum_{\lambda,\mu\text{-Young}} \underbrace{C^{h_3}_{h_1,h_2}}_{\text{3-pt fns.}}\ \beta^{\lambda}\,\bar\beta^{\mu}\,(z-w)^{h_3-h_1-h_2+|\lambda|}\,(\bar z-\bar w)^{\boxed{\bar h_3-\bar h_1-\bar h_2+|\mu|}} \cdot \left(L_{-\lambda}\bar L_{-\mu}\phi_{h_3}\right)(w)$$

$$\mathcal{H} = \bigoplus_{(h,\bar h)} M_{h,\bar h}$$

$$\langle h_1\bar h_1|\,\phi_2(1)\,\phi_3(x)\,|h_4,\bar h_4\rangle = \sum_{(h,\bar h)}\ \sum_{\lambda,\bar\lambda}\ \sum_{\mu,\bar\mu} \langle h_1,\bar h_1|\,\phi_2(1)\,|\lambda,\bar\lambda;h,\bar h\rangle\, Q_{\lambda,\mu}(h,c)\, Q_{\bar\lambda,\bar\mu}(\bar h,c)\,\langle \mu,\bar\mu;h,\bar h|\,\phi_3(x)\,\boxed{|h_4,\bar h_4\rangle}$$

$$= \sum C_{h_1,h_2,h}\, C_{h,h_3,h_4}\, \mathcal{F}^{21}_{3,4}(x)\, \bar{\mathcal{F}}^{21}_{3,4}(\bar x)$$

- $Q\phi$ ...
- $[$ ...

$)$ $\boxed{h_2 + |\mu|}$

$$\mathcal{F}^{21}_{34}(x) = \sum_h \sum_{\lambda,\mu} \frac{\langle h_1|\phi_2(1) L_{-\lambda}|h\rangle}{\langle h_1|\phi_2(1)|h\rangle} \cdot Q_{\lambda,\mu}(h,c) \cdot \frac{\langle h|L_\mu \phi_3(x)|h_4\rangle}{\langle h|\phi_3(x)|h_4\rangle}$$

- $Q_{\phi,\phi} = 1$, $Q_{\square,\square} = \left(\langle h|L_1 L_{-1}|h\rangle\right)^{-1} = (2h)^{-1}$
- $[L_n, \phi(w,\bar{w})] = h(n+1)\, w^n \phi(w,\bar{w}) + w^{n+1}\partial \phi(w,\bar{w})$

$\boxed{|h_4, \bar{h}_4\rangle}$

$$\langle h_1|\phi_{h_2}(1) L_{-1}|h_3\rangle = C_{123} \lim_{x\to 1}\left(-\frac{\partial}{\partial x}\right)\frac{1}{x^{h_2+h_3-h_1}} = C_{123}(h_2+h_3-h_1)$$

$$\langle h_1|L_1 \phi_{h_2}(x)|h_3\rangle = C_{123}\left(2h_2 x + x^2\frac{\partial}{\partial x}\right)\frac{1}{x^{h_2+h_3-h_1}} = C_{123}\frac{h_2+h_1-h_3}{x^{h_2+h_3-h_1}}\, x$$

$$\mathcal{F}^{21}_{34}(x) = 1 + x\frac{(h+h_2-h_1)(h+h_3-h_4)}{2h} + O(x^2)$$

Remark

$\mathcal{F}(x)$ is an analytical function with finite radius of convergence ($|x|<1$)

Question: How much can we constraint which conformal families appear in 2d CFT?

The most natural constraint - unitarity: $\forall\, |\chi\rangle \in M_{(h,\bar{h})}$ $\langle\chi|\chi\rangle > 0$

$|\chi\rangle = 0 \Leftrightarrow \langle\chi|\chi\rangle = 0$

$$\mathcal{H} = \bigoplus_{(h,\bar{h})} M'_{h,c} \otimes \overline{M}'_{\bar{h},c}$$

The scalar products are encoded $S_{\lambda,\mu}(h,c) = \langle h | L_{\lambda} L_{-\mu} | h\rangle$

$S_{\lambda,\mu}(h,c) = 0$ if $|\lambda| \neq |\mu|$

$S^{(\ell)}$ - K.-S. form at the level ℓ; $|\lambda|$ - level of $|\lambda, h\rangle$.

$$S^{(0)} = 1, \quad S^{(1)} = 2h, \quad S^{(2)} = \begin{pmatrix} 4h + c/2 & 6h \\ 6h & 4h(2h+1) \end{pmatrix}$$

• The existence of negative norm states $\Leftrightarrow$ existence of negative eigenvalues of $Q^{(\ell)}$ for some ℓ

$$S^{(\ell)} = U^{\dagger} \Lambda U, \quad \Lambda = \mathrm{diag}(\Lambda_1, \dots, \Lambda_{p(\ell)})$$

$p(\ell)$ – number of Young diagrams with ℓ-boxes.

$$\Lambda_1 < 0 \rightsquigarrow \chi : \quad S^{(\ell)} \chi = -|\Lambda_1| \chi$$

$$\chi^T S^{(\ell)} \chi = -|\Lambda_1|$$

χ – is of negative norm

$$\chi = \sum c_i \psi_i, \quad S^{(\ell)} \psi_i = \Lambda_i \psi_i$$

$$\Downarrow$$

$$0 > \chi^T Q^{(\ell)} \chi = \sum \Lambda_i |c_i|^2$$

• The existence of negative norm states $\Longleftrightarrow$ existence of negative eigenvalues of $Q^{(\ell)}$ for some ℓ.

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$$0 > \chi^T Q^{(\ell)} \chi = \sum \Lambda_i |c_i|^2$$

This is important for us, because by keeping track of signs of eigenvalues, we can keep track of unitarity.

We will treat $\det Q^{(\ell)}$ as analytic function (polynomial) in (c, h) – plane

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$S^{(\ell)}$ – K.-S. form at the level $\ell = |\lambda|$ – level of $|\lambda, h\rangle$.

$$S^{(0)} = 1, \quad S^{(1)} = 2h, \quad S^{(2)} = \begin{pmatrix} 4h + c/2 & 6h \\ 6h & 4h(2h+1) \end{pmatrix}$$

$$\det S^{(2)} = \left(4h + \frac{c}{2}\right) 4h(2h+1) - 36h^2 = h\left(32h^2 + 4h(c-5) + 2c\right)$$

Eigenvalues of $S^{(\ell)}$ can change a sign $\Leftrightarrow$ $\det(S^{(\ell)})$ is vanishing.

• Fact: the $\det S^{(\ell)}$ can be computed: $h_{1,1}(c)=0$.

$$h_{1,2}(c)=\frac{1}{16}\left(5-c-\sqrt{(1-c)(25-c)}\right)$$

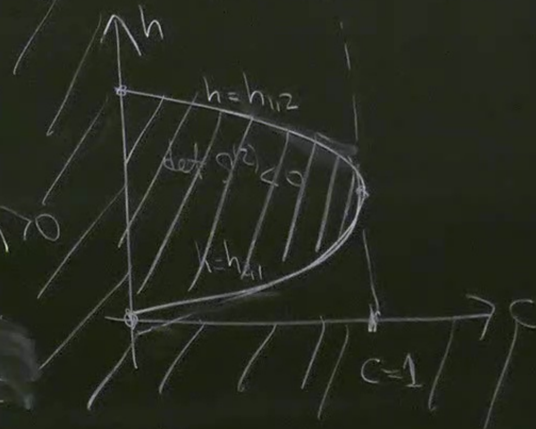
$$\det S^{(2)} = 32\,h(h-h_{1,2})(h-h_{2,1})$$

$$h_{2,1}(c)=\frac{1}{16}\left(5-c+\sqrt{(1-c)(25-c)}\right)$$

$h>0$

$c>0.$

$$\langle h|L_1L_{-1}|h\rangle = 2h>0$$
$\quad S^{(1)}$



$$\det S^{(\ell)} = A_\ell \prod_{\substack{r,s\geq 1\\ r\cdot s\leq \ell}} \left[h-h_{r,s}(c)\right]^{p(\ell-r\cdot s)}$$

$$A_\ell = \prod_{\substack{r,s\geq 1\\ rs\leq \ell}} \left[(2r)^s\, s!\right]^{p(\ell-rs)-p(\ell-r(s+1))}$$

$$\langle h|L_nL_{-n}|h\rangle = $$
$$= 2nh + \frac{c}{12}(n^2-1)n > 0 \ \forall n.$$
$$\Rightarrow c>0$$

$$h_{r,s} = h_0 + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2 = \qquad h_0 = \frac{1}{24}(c-1)$$

$$= h_0 + \frac{\left((r+s)\sqrt{1-c} + (r-s)\sqrt{25-c}\right)^2}{96}. \qquad \alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}$$

Answer for $c>1$ - all the modules are unitary.

for $0<c<1$ - almost every module is non-unitary

there is discrete set of unitary ones

for very particular values of

$$m \in \mathbb{Z}_{>0} \quad c = 1 - \frac{6}{m(m+1)}$$

unitary minimal models

$$h_{r,s}(m) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}$$

$\underline{c>1}$ • $h_{r,s}(c) = \frac{1-c}{96}\left\{\left[(r+s)+(r-s)\sqrt{\frac{25-c}{1-c}}\right]^2 - 4\right\}$ $\boxed{1<c<25}$ ✓

- real iff $r=s$, $h_{r,r} = \frac{1-c}{96}\{4r^2-4\} < 0$

• ✓ $\boxed{c>25}$ $\quad 0 < \sqrt{\frac{c-25}{c-1}} = k < 1 \Rightarrow \left[(r+s)+(r-s)\sqrt{\frac{25-c}{1-c}}\right]^2 =$

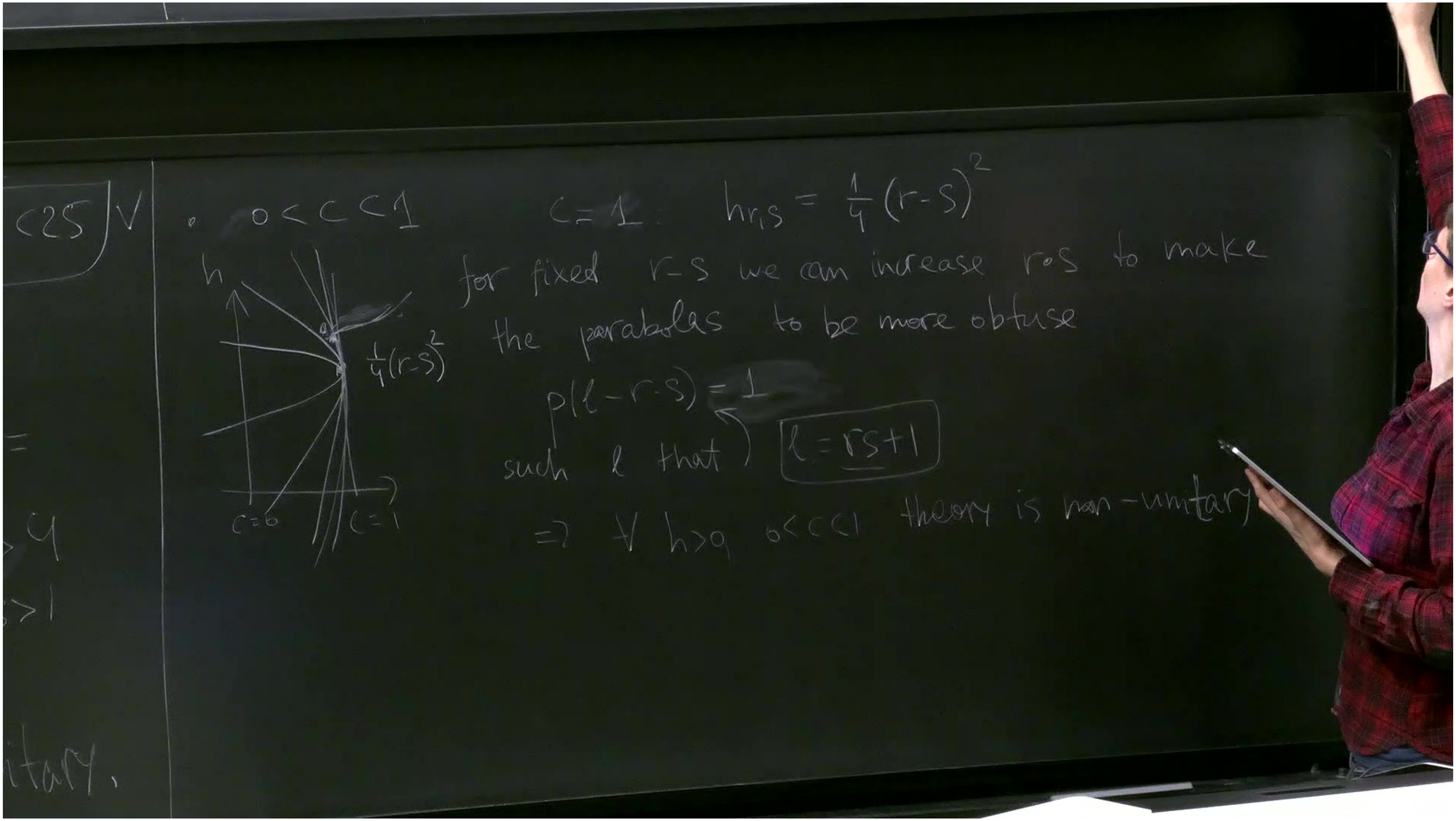
$= \left[(1+k)r + (1-k)s\right]^2 > 4$

$r,s>1$

$\Rightarrow h_{r,s}(c) < 0$

• If $c=\text{const}$, $h\to\infty$

$\Rightarrow S^{(1)}$ are positive defined



0 < c < 1
c = 1   h_{r,s} = 1/4 (r-s)^2
for fixed r-s we can increase rs to make
the parabolas to be more obtuse
p(l - r·s) = 1
such l that
l = rs+1
=> ∀ h>0, 0<c<1 theory is non-unitary
h
1/4 (r-s)^2
c=0
c=1

> 4

> 1

itary.

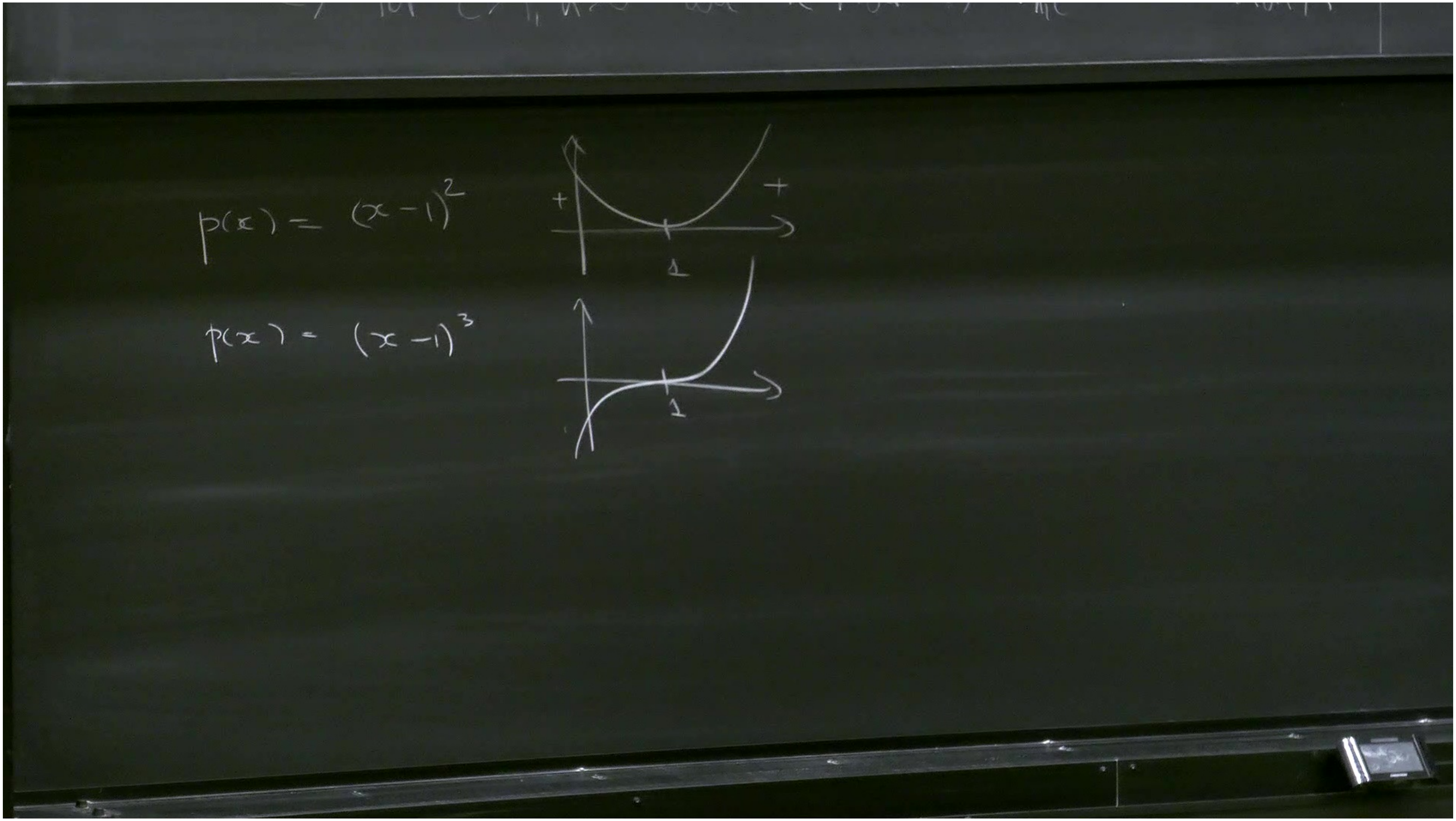
$c=0$ $c=1$

such ℓ that) $\boxed{\ell = \underline{rs}+1}$

$\Rightarrow \forall\ h>0,\ 0<c<1$ theory is non-unitary

• Minimal models are coming from the cases when we can "decouple" non-unitary vectors.

$$M(h,c) = V_{h,c}/(\cdots\cdots)$$


$p(x) = (x-1)^2$
+
+
1
$p(x) = (x-1)^3$
1