Title: Lecture - Quantum Field Theory III - PHYS 777

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Recap Last lecture: OPE & Ward identities in Path Integral $\rightsquigarrow$ contour integrals (in operator formalism) commutators (i.e. algebra) & representations

· Rep

· $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \Rightarrow [L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^2-1)n\delta_{n+m,0}$

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

also similar for $[\bar{L}_n, \bar{L}_m]$, $[L_n, \bar{L}_m] = 0$

c - central charge

· Action on Vir on primary fields: $\phi(z) = \sum_{k\in\mathbb{Z}} z^{-h-k}\phi_k$

$$\Rightarrow [L_n, \phi_m] = (n(h-1)-m)\phi_{n+m} \Leftarrow T(z)\phi(w) \sim \frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}$$

· <u>Representations</u>: $Vir \oplus \overline{Vir}$ acts on Verma modules:

· $|0\rangle$, $L_n|0\rangle = 0$, $n \geq -1$, $L_n|0\rangle \neq 0$, $n < -1$

· $|h\rangle = \lim_{z \to 0} \phi(z)|0\rangle = \phi_{-h}|0\rangle$, $h \in \mathbb{Z}$

$L_0|h\rangle = h|h\rangle$, $L_n|h\rangle = 0$, $n > 0$, $L_n|h\rangle \neq 0$, $n < 0$.

vector of highest weight h

· $|(\lambda,\mu),(h,\bar{h})\rangle = L_{-\lambda_n} \dots L_{-\lambda_1} \bar{L}_{-\mu_m} \dots \bar{L}_{-\mu_1} |(h,\bar{h})\rangle$

$\lambda_1 \geq \lambda_2 \dots \geq \lambda_1 > 0$ $\quad \mu_m \geq \dots \geq \mu_1 > 0$

$V_{(h,\bar{h}),c} = \{ |(\lambda,\mu),(h,\bar{h})\rangle \}$ — Verma module (conformal family, in general)

$|0\rangle$

$L_{-1}|0\rangle$

$(L_{-1})^2|0\rangle$ $\quad L_{-2}|0\rangle$

$|h\rangle$

$L_{-1}|h\rangle$

$(L_{-1})^2|h\rangle$ $\quad L_{-2}|h\rangle$

$\vdots$

Remark In general there exist CFTs wher the action of L_0 is non-diagonalizable (i.e. Jordan blocks). This results of log in OPEs

e.g. $L_0 |(\lambda,\mu),(h,\bar{h})\rangle = \left(h + \sum_{i=1}^{\ell(\lambda)} \lambda_i\right) |(\lambda,\mu),(h,\bar{h})\rangle$

for scalar $\phi(z)\phi(w) = \log(z-w) + \log(\bar{z}-\bar{w}) + \text{reg.}$

- From algebraic perspective $V_{(h,\bar{h}),c} = V_{h,c} \otimes V_{\bar{h},c}$, with Vir acting on $V_{h,c}$ and $\overline{Vir}$ on $V_{\bar{h},c}$

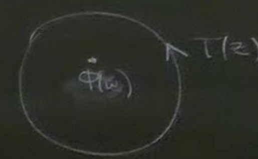
In general in CFTs: $\mathcal{H} = \bigoplus_{(h,\bar{h})} M_{h,c} \otimes M_{\bar{h},c}$

in particular might be V.

s
OPEs

· So far we when constructing states out of operators.

Let's go opposite direction:



$$(L_{-n}\phi)(w) = \oint_{w} \frac{dz}{2\pi i}(z-w)^{-n+1}\, T(z)\,\phi(w)$$

$$\Updownarrow$$

$$L_{-n}|\phi\rangle$$

$$\Rightarrow \quad \cdot\,\phi(w) \qquad (L_{-1}\phi)(w)\cdot \qquad (L_{-2}\phi)(w)\cdot \qquad (L_{-1}^2\phi)(w)\cdot$$

$$|\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i$$

· Now for OPEs:

$$\phi_{h_1,\bar h_1}(z,\bar z)\,\phi_{h_2,\bar h_2}(w,\bar w) = \sum_{h_3,\bar h_3}\sum_{\lambda,\mu} C^{h_3,\lambda,\mu}_{h_1,h_2}\,(z-w)^{h_3-h_1-h_2+|\lambda|}\cdot(\bar z-\bar w)^{\bar h_3-\bar h_1-\bar h_2+|\mu|}\cdot(L_{-\lambda}\bar L_{-\mu}\phi_{h_3,\bar h_3})(w)$$

$$C^{h_3,\lambda,\mu}_{h_1,h_2} = \boxed{C^{h_3}_{h_1,h_3}}\,\beta^{\lambda}\,\bar\beta^{\mu}$$

structural constants

$\beta^{\lambda}, \bar\beta^{\mu}$ depends only on (h_i, c)

- Assume that we know $C_{h_1,h_2}^{h_3}$, $(h,\bar{h})$, c. Let's compute 4-pt function.
- Resolution of identity (holomorphic part). $\mathbb{1} = \sum_h \sum_{N=0}^{+\infty} \sum_{|\lambda|=N} \sum_{|\mu|=N} Q_{\lambda,\mu}(h,c)\, |\lambda,h\rangle\langle\mu,h|$

$\rightsquigarrow$ Vectors from different $V_{h,c}$ are orthogonal.
This is equivalent to choice of basis in primary fields s.t. $\langle \phi_1(z_1)\phi_2(z_2)\rangle = \dfrac{\delta_{12}}{z_{12}^{2h}\,\bar{z}_{12}^{2\bar{h}}}$

$\rightsquigarrow$ Inside of $V_{h,c}$ vectors with different eigenvalues of L_0 are orthogonal ion.

$h|$

al

- $\left(h + \sum_{i=1}^{\ell(\lambda)} \lambda_i\right) \langle\ \rangle = \langle \lambda, h | L_0 | \mu, h \rangle = \left(h + \sum_i \mu_i\right) \langle \lambda, h | \mu, h \rangle$

$(|\lambda, h\rangle)^\dagger = \langle h | L_\lambda$ "-1"

- $L_n^\dagger = L_{-n}$

- Matrix $Q_{\lambda,\mu}(h,c) = \left(\langle \mu, h | \lambda, h \rangle\right)^{-1}$ is call Kac-Shapavalar form

- $\underline{\text{example}}$ $Q_{\phi,\phi} = \langle h | h \rangle = 1$

- $Q_{\square,\square} = \left(\left(\langle h | L_1\right)\left(L_{-1} | h\rangle\right)\right)^{-1} = \left(\langle h | \left(\cancel{L_{-1} L_1} + 2L_0\right) | h \rangle\right)^{-1} = \frac{1}{2h}$

- $Q_{\square\square} \Rightarrow \langle h | L_2 L_{-2} | h \rangle = \langle h | \left(\cancel{L_{-2} L_2} + 4L_0 + \frac{c}{12} 2(4-1)\right) | h \rangle = 4h + \frac{c}{2}$

$$\langle \phi_1(\infty)\,\phi_2(1)\,\phi_3(x)\,\phi_4(0)\rangle \sim \langle h_1;\bar{h}_1|\,\phi_2(1)\,\hat{\mathbb{1}}\,\phi_3(x)\,|h_4,\bar{h}_4\rangle =$$

$$= \sum_{(h,\bar{h})}\sum_{\lambda,\bar{\lambda}}\sum_{\mu,\bar{\mu}} \langle h_1,\bar{h}_1|\,\phi_2(1)\,|\lambda,\bar{\lambda};(h,\bar{h})\rangle\, Q_{\lambda,\mu}(h)\, Q_{\bar{\lambda},\bar{\mu}}(\bar{h})\,\langle(\mu,\bar{\mu});(h,\bar{h})|\,\phi_3(x)\,|h_4,\bar{h}_4\rangle$$

$$= \sum_{(h,\bar{h})} C_{h_1,h_2,h}\, C_{h,h_3,h_4}\, \mathcal{F}^{12}_{34}(x)\, \bar{\mathcal{F}}^{12}_{34}(\bar{x}),$$

$\mathcal{F}$ – conformal block.

$$\mathcal{F}^{21}_{34}(x) = \sum_{h}\sum_{\lambda,\mu} \frac{\langle h_1|\,\phi_2(1)\,L_{-\lambda}\,|h\rangle}{\langle h_1|\,\phi_2(1)\,|h\rangle}\cdot Q_{\lambda,\mu}(h,c)\, \frac{\langle h|\,L_{\mu}\,\phi_3(x)\,|h_4\rangle}{\langle h|\,\phi_3(x)\,|h_4\rangle}$$

• Let's compute β^λ using algebra.

$$[L_n, \phi(w,\bar{w})] = h(n+1) w^n \phi(w,\bar{w}) + w^{n+1} \partial \phi(w,\bar{w})$$

so: $\langle h_1 | \phi_{h_2}(1) L_{-n} L_{-\lambda'} | h_3 \rangle = \langle h_1 | \cancel{L_{-n}} \phi_{h_2}(1) L_{-\lambda'} | h_3 \rangle +$

$$+ \lim_{x \to 1} \left(-x^{-n} h(-n+1) - x^{-n+1} \frac{\partial}{\partial x} \right) \left(\langle h_1 | \phi_{h_2}(x) L_{-\lambda'} | h_3 \rangle \right)$$

• $\langle h_1 | \phi_{h_2}(1) L_{-1} | h_3 \rangle = \lim_{x \to 1} \left(-\frac{\partial}{\partial x} \right) \left(\frac{C_{123}}{x^{h_2+h_3-h_1}} \right) = \underbrace{C_{123} (h_2 + h_3 - h_1)}_{\beta^{\square}}$

$\langle h_1 | \phi_{h_2}(x) | h_3 \rangle$

• $\langle h_1 | L_1 \phi_{h_2}(1) | h_3 \rangle = \lim_{x \to 1} (2h_2 x + x^2 \partial_x) \langle h_1 | \phi_{h_2}(x) | h_3 \rangle = C_{123} (h_2 + h_1 - h_3)$

Using also that $Q_{\square,\square} = \frac{1}{2h}$ we find:

$$\mathcal{F}^{21}_{34}(x) = 1 + x\,\frac{(h+h_2-h_1)(h+h_3-h_4)}{2h} + O(x^2)$$

- At this point we've computed all corr. functions for 2d CFT
  - → We know $c_i(h,\bar{h})$, C_{123}. (conformal bootstrap)
  - → We assumed that $\langle\lambda, h|\mu, h\rangle$ is non-degenerate (decouple zero-norm states)

Next

- Minimal models – there is only fin. num. of conf. fam.