Title: Lecture - Quantum Field Theory III - PHYS 777

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Subject: Quantum Fields and Strings

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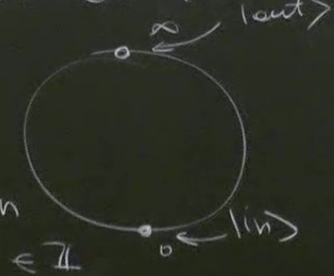
<u>Recap</u> Last time we started operator formalism in CFTs.

Radial quantization: $z = e^{\frac{2\pi}{L}(t+ix)}$ $x \sim x + L$



- Operator ⇄ state correspondence. Let's ϕ - field with some conf. dimension $(h,\bar{h})$, $h,\bar{h} \in \mathbb{Z}$

" ... Vertex algebras ... "

$$\Rightarrow \quad |\phi\rangle = \lim_{z,\bar{z}\to 0} \phi(z)|0\rangle \qquad Y: \mathcal{H} \to \mathrm{End}(\mathcal{H}) \otimes \mathbb{C}[[z,\bar{z}]]$$

$$\langle\phi| = \lim_{z,\bar{z}\to\infty} z^{2h}\bar{z}^{2\bar{h}} \langle 0|\phi(z) \leftarrow I: z \mapsto \frac{z}{|z|^2} = \frac{1}{\bar{z}}$$

- We also had mode expansion: $\phi(z,\bar{z}) = \sum_{n,m\in\mathbb{Z}} z^{-n-h}\bar{z}^{-m-\bar{h}} \phi_{n,m}$, $\Rightarrow (\phi_{n,m})^{\dagger} = \phi_{-n,-m}$

$$\phi_{n,m} = \oint \frac{dz}{2\pi i} \oint \frac{d\bar{z}}{2\pi i} \phi(z,\bar{z})\, z^{n+h-1} \bar{z}^{m+\bar{h}-1}$$

- $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n$ ← generators of conf. symmetry.

- ϕ, of conf. dim. $(h, \bar{h})$

$$\lim_{z \to 0} \phi(z, \bar{z})|0\rangle = \lim_{z \to 0} \sum \phi_{n,m} z^{-n-h} \bar{z}^{-m-\bar{h}} |0\rangle = \phi_{-h,-\bar{h}}|0\rangle$$

$$\phi_{n,m}|0\rangle = 0 \quad \text{if} \quad n > -h \quad \text{or} \quad m > -\bar{h}.$$

Q) What about products (or commutators?) of operators?

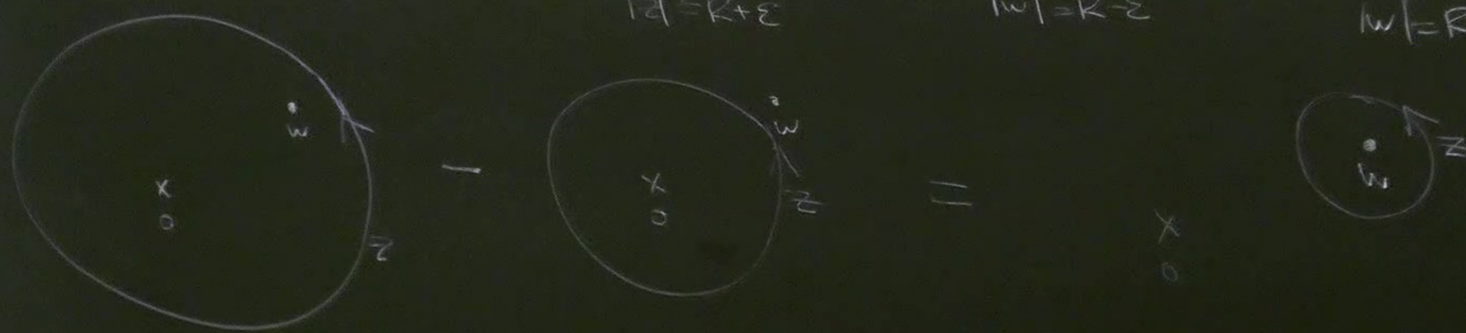
- In usual QFT when we match path. int and operators formalism, our correlators are always time ordered.

In radial quantization: we have radial ordering: instead

$$R\,\phi_1(z)\phi_2(w) = \begin{cases} \phi_1(z)\phi_2(w), & \text{if } |z| > |w| \\ \phi_2(w)\phi_1(z), & \text{if } |z| < |w| \end{cases}$$

Let's compute now commutators for modes of $T(z)$:

$$[L_n, L_m] \; = \oint_{|z|=R+\varepsilon} \frac{dz}{2\pi i} z^{n+1} T(z) \oint_{|w|=R-\varepsilon} \frac{dw}{2\pi i} w^{m+1} T(w) - \oint_{|w|=R+\varepsilon} \frac{dw}{2\pi i} w^{m+1} T(w) \oint_{|z|=R-\varepsilon} \frac{dz}{2\pi i} z^{n+1} T(z)$$



$$= \oint_{|w|=R} \frac{dw}{2\pi i} w^{m+1} \oint_{w} \frac{dz}{2\pi i} z^{n+1} T(z) T(w) = \left|\begin{matrix}\text{using}\\ \text{OPE}\end{matrix}\right| = \oint_{|w|=R} \frac{dw}{2\pi i} w^{m+1} \oint_{w} \frac{dz}{2\pi i} z^{n+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \right)$$

$$= \oint_{|w|=R} \frac{dw}{2\pi i} w^{m+1} \left(\frac{c}{2}\frac{1}{3!}(n+1)n(n-1)\, w^{n-2} + 2(n+1)\, w^{n}\, T(w) + w^{n+1}\,\partial T(w) \right) =$$

$$\oint \frac{dz}{2\pi i} z^{n} = \delta_{n,-1}$$

$$= \frac{c}{12}(n+1)n(n-1)\,\delta_{n+m,0} + 2(n+1)L_{n+m} - (n+m+2)L_{n+m}$$

$$\boxed{[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^2-1)n\,\delta_{n+m,0}}$$

– Virassoro algebra

- Notice that if $c=0$ this algebra becomes Witt algebra $l_n = -z^{n+1}\frac{\partial}{\partial z}$
- The term with c is called central extension of the Witt algebra.
- c is called central charge and measures quantum corrections to conformal theory.

$$= \oint_{|w|=R} \frac{dw}{2\pi i} w^{m+1} \left(\frac{c}{2} \frac{1}{3!} (n+1)n(n-1) w^{n-2} + 2(n+1) w^{n} T(w) + w^{n+1} \partial T(w) \right) =$$

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$$\boxed{[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n^2-1) n \, \delta_{n+m,0}}$$

$\langle T_\mu{}^\mu(x) \rangle = \frac{c}{12} R(x)$ if you have background metric?

– Virassoro algebra.

- Notice that if $c=0$ this algebra becomes Witt algebra $l_n = -z^{n+1}\frac{\partial}{\partial z}$
- The term with c is called central extension of the Witt algebra.
- c is called central charge and measures quantum corrections to conformal theory.

- We also have anti-holomorphic generators $\bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{L}_n \bar{z}^{-n-2}$

  They satisfy same relations (with potentially different c)

  $[L_n, \bar{L}_m] = 0$

- There is also sl_2-subalgebra L_{-1}, L_0, L_1 which is not affected by central ext.

$$[L_0, L_{-1}] = L_{-1}$$
$$[L_{-1}, L_{+1}] = 2L_0$$
$$[L_0, L_1] = -L_1$$

- How can we act by modes on operators?

$$\partial_\mu \langle j^\mu(y)\, X \rangle = \sum_a \delta(y - x_a)\, i G_a \langle X \rangle$$

• c is called <u>central charge</u> and measures quantum corrections to conformal theory.

Using Stokes theorem: $[Q, \phi_a(x_a)] = i(G_a \phi_a)(x_a)$

$$Q = \oint_{r=\text{const}} dn_\mu j^\mu$$

For the conformal symmetry: $Q = \oint dz\, \epsilon(z) T(z) = \sum_{k\in\mathbb{Z}} \epsilon_k L_k$

$\epsilon(z) = \sum \frac{\epsilon_k}{2\pi i} z^{k+1}$ $\qquad \epsilon(z)\frac{\partial}{\partial z}$

$\epsilon(z) = 1, z, z^2$

• Let's take closer look $L_0, \bar{L}_0$:

On a classical level $l_0 = -z\frac{\partial}{\partial z}$, $\bar{l}_0 = -\bar{z}\frac{\partial}{\partial \bar{z}}$

$z = e^{\frac{2\pi}{L}(t+ix)}$ $\qquad \bar{z} = e^{\frac{2\pi}{L}(t-ix)}$

$l_0 + \bar{l}_0 \sim \frac{\partial}{\partial t}$

$l_0 - \bar{l}_0 \sim \frac{\partial}{\partial x}$

$\Rightarrow$ $L_0 + \bar{L}_0$ is "Hamiltonian"

$L_0 - \bar{L}_0$ is "Momentum"

Q] How do we act by L_n on primary fields?

$$[L_n, \phi(w,\bar{w})] = \oint_{|z|>|w|} \frac{dz}{2\pi i} z^{n+1} T(z)\phi(w) - \oint_{|z|<|w|} \frac{dz}{2\pi i} z^{n+1} \phi(w) T(z) =$$

$$= \oint_{w} \frac{dz}{2\pi i} z^{n+1} \left(\frac{h\,\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w} \right) = h(n+1)w^{n}\phi(w,\bar{w}) + w^{n+1}\partial\phi(w,\bar{w})$$

or in modes: $[L_n, \phi_m] = (n(h-1) - m)\,\phi_{n+m}$.

In particular: • $[L_0, \phi_m] = -m\,\phi_m$

• $[L_{-1}, \phi(w)] = \partial\phi(w)$

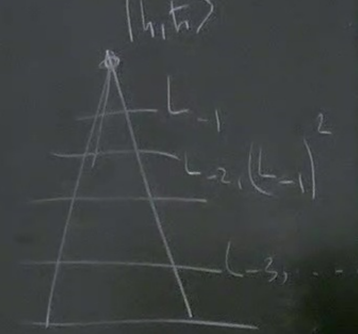
• We had assymptotic state $|0\rangle$. How does L_n, L_m act on $\lim_{z\to 0} \phi(z)|0\rangle$?
$L_n, \bar{L}_n$

$$|h,\bar h\rangle = \lim_{z\to 0} \phi(z,\bar z)|0\rangle = \phi_{-h,\bar h}|0\rangle$$ for primary $\phi(z,\bar z)$ of conf. dim $(h,\bar h)$.

Using commutators we can see:

$$L_0|h,\bar h\rangle = h|h,\bar h\rangle, \quad \bar L_0|h,\bar h\rangle = \bar h|h,\bar h\rangle$$

$$L_n|h,\bar h\rangle = 0 \quad \text{if} \quad n>0$$



• The descendant states can be obtained as

$$|\lambda; h,\bar h\rangle = L_{-\lambda_n} L_{-\lambda_{n-1}} \cdots L_{-\lambda_1}|h,\bar h\rangle, \quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n \geq 0$$

- All the fields $\{ |\lambda; h, \bar{h}\rangle \}$ form <u>conformal family</u> $V_{h, \bar{h}, c}$

- There is a natural "grading" by the eigenvalues of L_0:

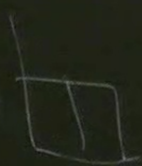
$$L_0 |\lambda; h, \bar{h}\rangle = \left(h + \sum_{i=1}^{n} \lambda_i\right) |\lambda; h \bar{h}\rangle$$

- In mathematical literature such representations of algebras are called Verma modules.

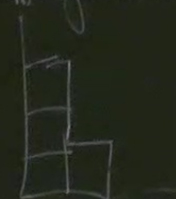
- Such sequences λ are encoded by Young diagrams:

$\lambda = \{1\}$ 

$L_{-1} |h, \bar{h}\rangle$

$\lambda = \{1, 1\}$ 

$(L_{-1})^2 |h, \bar{h}\rangle$

$\lambda = \{3, 1\}$ 

$L_{-3} L_{-1} |h \bar{h}\rangle$

- There is a very nice formula for the counting of states in $V_{h,c}$:

$$Z = \mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H} \rightsquigarrow \mathrm{Tr}_{V_{h\bar{h},c}} \left(q^{L_0} \bar{q}^{\bar{L}_0} \right) = \qquad q \cdot \bar{q} = e^{-\beta}$$

$$= \sum_{\lambda,\mu} \left(q^{h + \sum_{i=1}^{\ell(\lambda)} \lambda_i} \bar{q}^{\bar{h} + \sum_{j=1}^{\ell(\mu)} \mu_i} \right) = \frac{q^{h} \cdot \bar{q}^{\bar{h}}}{\prod_{i=1}^{\infty}(1-q^i) \prod_{i=1}^{\infty}(1-\bar{q}^i)}$$

$$p(q) = \sum_{\lambda} q^{|\lambda|} = \prod_{k=1}^{\infty} \frac{1}{1-q^k}$$

Two tricky points

- To actually compute Z we need to know all the primaries in the theory.
- Not every module generated by primary would be Verma module

$) |\lambda, h_F\rangle$

representations of algebras are called Verma modules

y Young diagrams:

$\lambda = \{3,1\}$

$L_{-3} L_{-1} |h_F\rangle$

$= \sum_{\lambda,\mu} \Big(q \quad \overline{q} \quad \prod_{i=1} (1-q^i) \prod_{i=1}$

$$p(q) = \sum_{\lambda} q^{|\lambda|} = \prod_{k=1}^{\infty} \frac{1}{1-q^k}$$

Two tricky points

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- Not every module generated by primary w

In Minkowski QFT: $Z = \int D\phi \, e^{\frac{i}{\hbar}S[\phi]} = \lim_{T\to\infty} \langle 0| e^{-iHT/\hbar} |0\rangle \qquad T \longrightarrow -i\beta$

$\phi : \mathbb{R}^n \to \mathbb{R}$

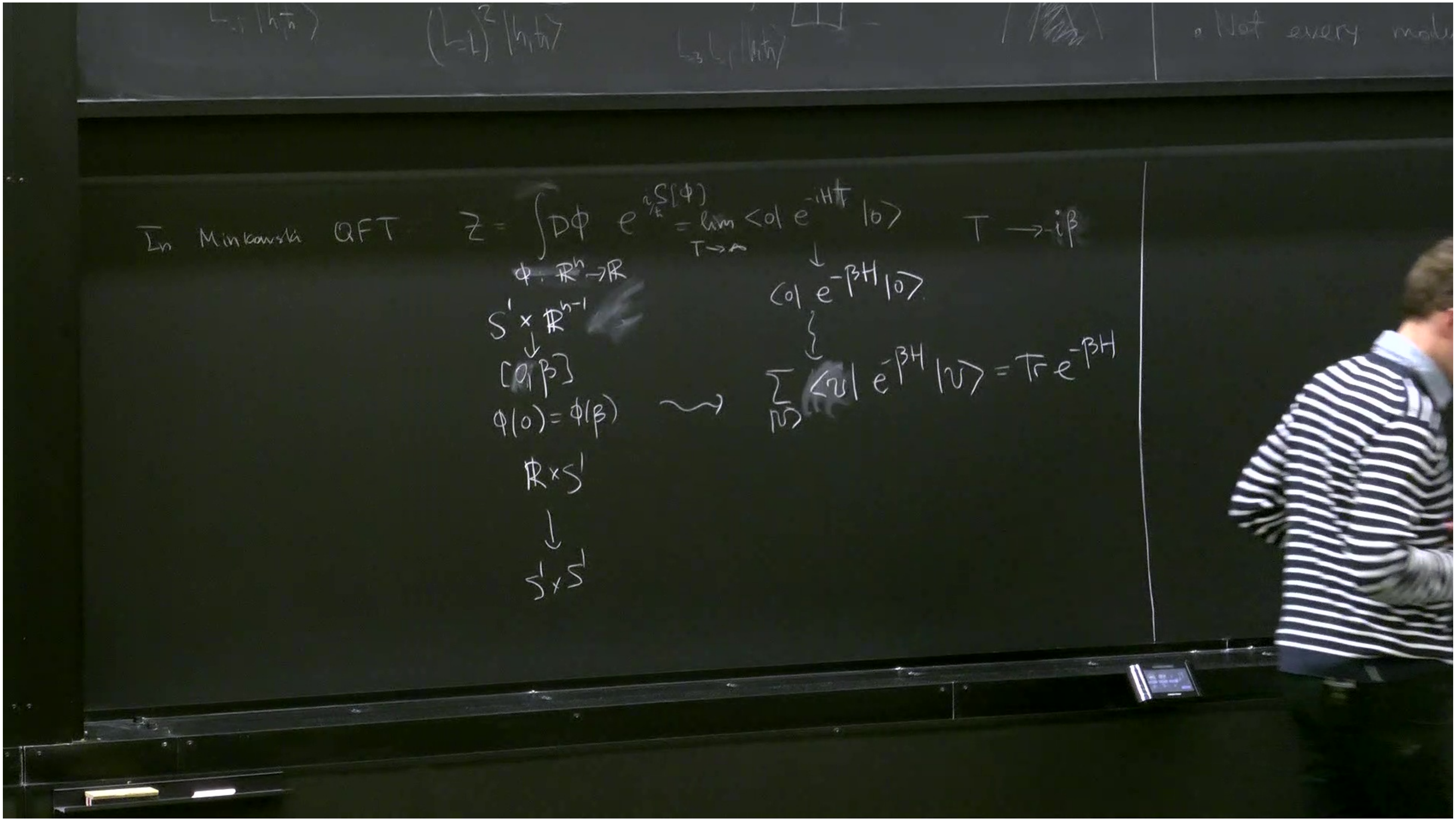
$$\langle 0| e^{-\beta H} |0\rangle$$

$$S^1 \times \mathbb{R}^{n-1}$$

$$\downarrow$$

$$[0,\beta]$$

$$\phi(0) = \phi(\beta) \quad \rightsquigarrow \quad \sum_{|v\rangle} \langle v| e^{-\beta H} |v\rangle$$


In Minkowski QFT
Z = ∫Dφ e^{iS[φ]/ħ} = lim_{T→∞} <0| e^{-iHT} |0>
T → -iβ
φ: ℝ^n → ℝ
<0| e^{-βH} |0>
S^1 × ℝ^{n-1}
[0, β]
φ(0) = φ(β)
Σ_{|v>} <v| e^{-βH} |v> = Tr e^{-βH}
ℝ × S^1
S^1 × S^1