Title: Lecture - Quantum Field Theory III - PHYS 777

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Subject: Quantum Fields and Strings

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Recap Ward identities.

Noether thm. 1 param. family of symm. $\longrightarrow$ conserved (on-shell) current j^μ.

$$S[\phi - i\underset{\mathbb{R}}{\omega} G\phi] - S[\phi] = O(\omega^2) \quad \text{(off-shell)}$$

$$\tilde{\phi}(x) - \phi(x) = -i\omega G\phi \qquad \Longrightarrow \quad \partial_\mu j^\mu = 0.$$

Ward identity $X = \phi_1(x_1)\dots\phi_n(x_n)$, $[D\tilde{\phi}] = [D\phi]$

$$\Rightarrow \frac{\partial}{\partial x^\mu}\langle j^\mu X\rangle = -i\sum_{i=1}^{n} \delta(x - x_i)\, G_i\langle X\rangle$$

$$G_i\langle X\rangle = \langle \dots (G_i\phi_i)(x_i)\dots\rangle$$

For the conf. symm in 2d $j^\mu = T^{\mu\nu}\epsilon_\nu$, $\partial_{\bar z}\epsilon^z = 0$, $\partial_z \epsilon^{\bar z} = 0$

$\partial_{\bar z}\epsilon_{\bar z} = 0$, $\partial_z \epsilon_z = 0$

$$\delta_{\epsilon,\bar\epsilon}\langle X\rangle = -\frac{1}{2\pi i}\oint_C \epsilon(z)\langle T(z) X\rangle dz + \frac{1}{2\pi i}\oint_C \bar\epsilon(\bar z)\langle \bar T(\bar z) X\rangle d\bar z$$

$$T(z) = -2\pi T_{zz}(z), \quad \bar T(\bar z) = -2\pi T_{\bar z\bar z}(\bar z), \quad \epsilon = \epsilon^z, \quad \bar\epsilon = \epsilon^{\bar z}$$

$$\langle T_{z\bar z}, T_{\bar z z} X\rangle \sim \delta$$

$\rightsquigarrow Q^{x_1}$, Q^{x_2}, Q^{x_4}, Q^{x_3}

$$\langle \ldots \delta_{\epsilon,\bar\epsilon}\phi_i(z_i)\ldots\rangle = \Big\langle \ldots\Big(-\oint_{z_i}\epsilon(z)T(z)\phi_i(z_i)\frac{dz}{2\pi i} + \oint_{z_i}\bar\epsilon(\bar z)\bar T(\bar z)\phi_i(\bar z)\frac{d\bar z}{2\pi i}\Big)\ldots\Big\rangle$$

$$\delta_{\epsilon,\bar\epsilon}\phi_i = -(h_i\partial\epsilon + \epsilon\partial)\phi_i + \text{a.h.} \Rightarrow T(z)\phi_i(z_i) = \frac{h_i\phi_i(z_i)}{(z-z_i)^2} + \frac{\partial\phi_i(z_i)}{z-z_i} + \text{reg}(z-z_i)$$

+ similar formula for $\bar T(\bar z)\cdot\phi(z_i) = \ldots$

OPE

What about stress-energy tensor?

T has $\Delta = 2$, $s = 0$, $h = 2$, $\bar{h} = 0$

$\bar{T}$: $\Delta = 2$, $s = 2$, $h = 0$, $\bar{h} = 2$.

$$T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots$$

~~$\frac{A(z)}{(z-w)^3}$~~

c - central charge

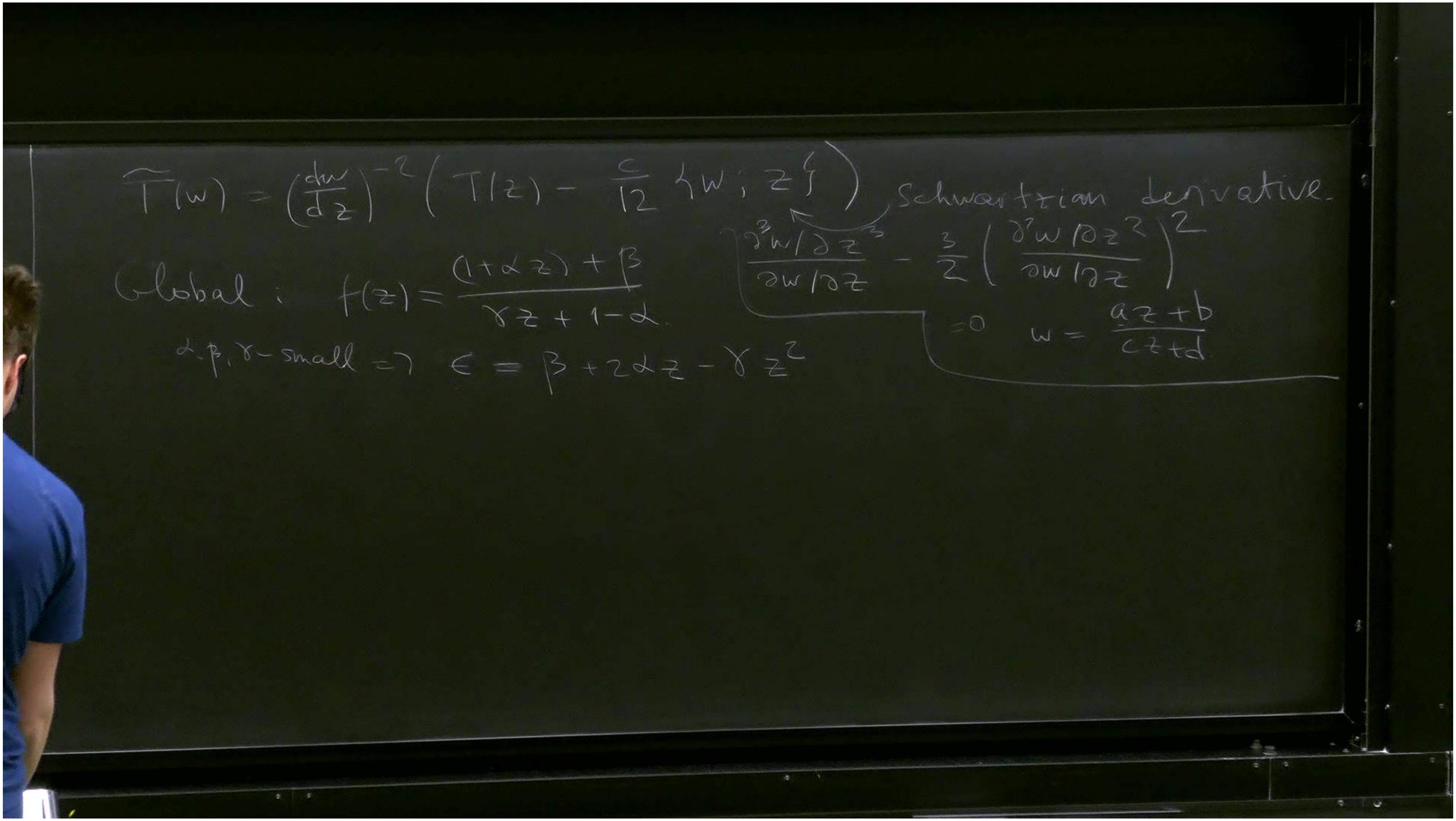
$$T(z)\,\bar{T}(\bar{w}) = \text{reg.}$$

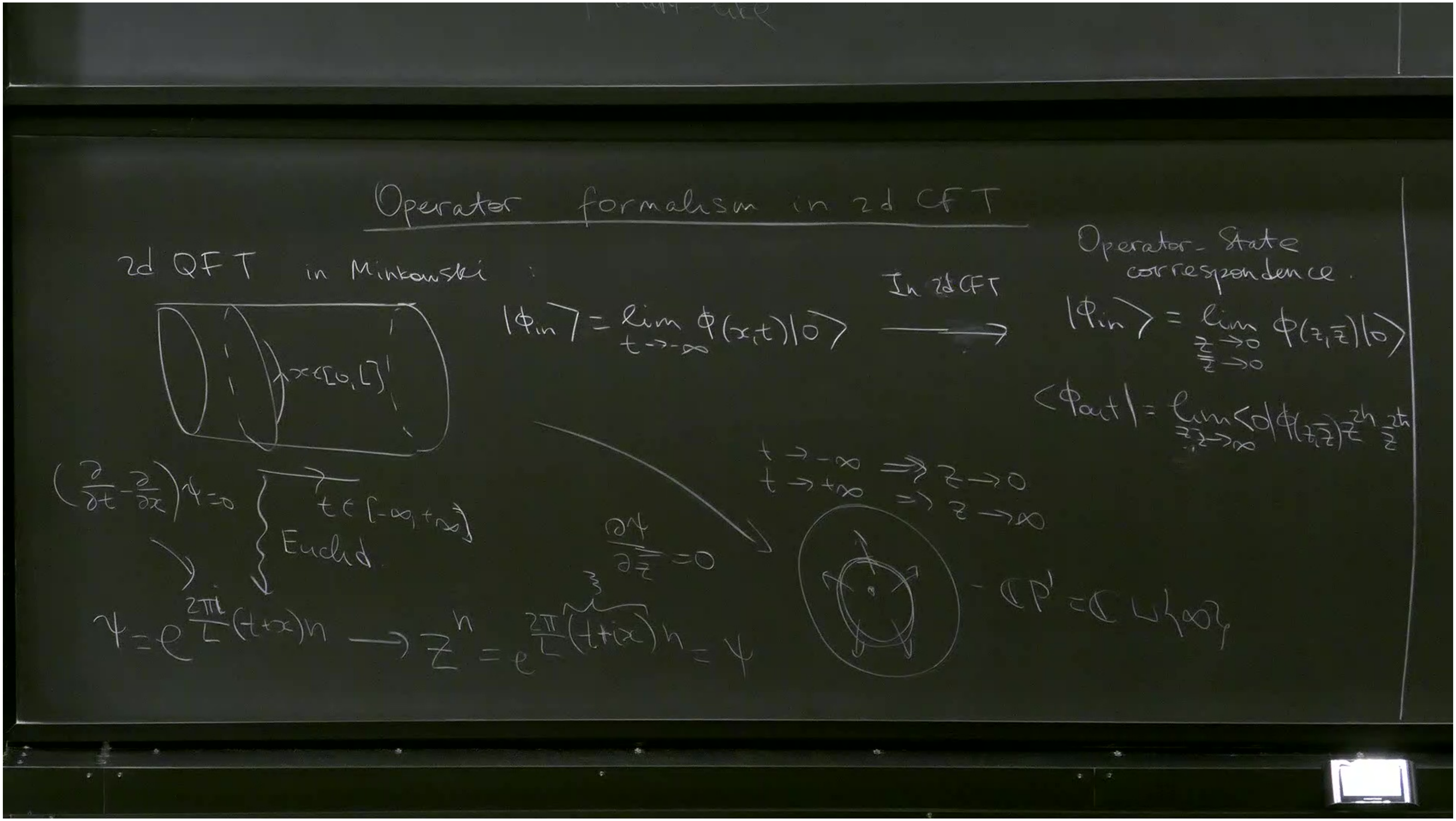
$$\bar{T}(\bar{z})\,\bar{T}(\bar{w}) = \dots$$

$$\delta_\epsilon T(w) = -\frac{1}{2\pi i}\oint_w dz\, \epsilon(z)\,T(z)\,T(w) = -\frac{1}{2\pi i}\oint_w dz\, \epsilon(z)\left(\frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\right)$$

$$= -\frac{c}{12}\,\epsilon'''(w) - 2\,\epsilon'(w)\,T(w) - \epsilon(w)\,\partial T(w)$$

primary-like


$\tilde{T}(w) = \left(\frac{dw}{dz}\right)^{-2}\left(T(z) - \frac{c}{12}\{w;z\}\right)$ Schwartzian derivative.
$\frac{\partial^3 w/\partial z^3}{\partial w/\partial z} - \frac{3}{2}\left(\frac{\partial^2 w/\partial z^2}{\partial w/\partial z}\right)^2$
$=0 \quad w = \frac{az+b}{cz+d}$
Global: $f(z) = \frac{(1+\alpha z) + \beta}{\gamma z + 1 - \alpha}$
α, β, γ - small $\Rightarrow \epsilon = \beta + 2\alpha z - \gamma z^2$


Operator formalism in 2d CFT
2d QFT in Minkowski :
x ∈ [0, L]
$|\phi_{in}\rangle = \lim_{t \to -\infty} \phi(x,t)|0\rangle$
In 2d CFT
Operator - State correspondence.
$|\phi_{in}\rangle = \lim_{z \to 0, \bar{z} \to 0} \phi(z,\bar{z})|0\rangle$
$\langle\phi_{out}| = \lim_{z,\bar{z} \to \infty} \langle 0|\phi(z,\bar{z}) z^{2h} \bar{z}^{2\bar{h}}$
$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\psi = 0$
t ∈ [-∞, +∞]
Euclid.
$t \to -\infty \Rightarrow z \to 0$
$t \to +\infty \Rightarrow z \to \infty$
$\frac{\partial\psi}{\partial\bar{z}} = 0$
$\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$
$\psi = e^{\frac{2\pi i}{L}(t+x)n} \longrightarrow z^n = e^{\frac{2\pi}{L}(t+ix)n} = \psi$

ce.

$z,\bar{z})|0\rangle$

$\bar{z})z^{2h}\bar{z}^{2\bar{h}}$

Why extra factor for $\langle \phi_{out}|$?

- $\langle \phi_{out} | \phi_{in} \rangle = \lim_{w\to\infty} \lim_{z\to 0} \left(\langle 0 | \phi(w,\bar{w})\, w^{2h}\, \bar{w}^{2\bar{h}} \right)\left(\phi(z,\bar{z}) |0\rangle \right) =$

$$= \lim_{w\to\infty} \lim_{z\to 0} \frac{C}{(w-z)^{2h}(\bar{w}-\bar{z})^{2\bar{h}}} \cdot w^{2h}\,\bar{w}^{2\bar{h}} = C$$

$h = \bar{h}$, $s = 0$: $\frac{C}{|z-w|^{2\Delta}}$

- In 2d CFTs we have inversion : $z \mapsto \frac{z}{|z|^2} = \frac{1}{\bar{z}}$

$$\tilde{\phi}(w,\bar{w}) = (-\bar{w}^2)^{-h}(-w^2)^{-\bar{h}}\,\phi\left(\frac{1}{\bar{w}}, \frac{1}{w}\right)$$

We can set $\langle \phi_{out}| = \lim_{w \to 0} \langle 0| \phi(w, \bar{w}) \underset{w = \frac{1}{\bar{z}}}{\overset{\text{up to } (-1)^{h+\bar{h}}}{=\!=}} \lim_{z \to 0} \langle 0| z^{2h} \bar{z}^{2\bar{h}} \phi(z, \bar{z})$

• Hermitian conjugation for fields: $\langle \phi_{out}| = \left(|\phi_{in}\rangle \right)^{\dagger}$

$$\Rightarrow (*) \quad \left(\phi(z, \bar{z}) \right)^{\dagger} = \phi\left(\frac{1}{\bar{z}}, \frac{1}{z} \right) \bar{z}^{-2h} z^{-2\bar{h}}$$

• Mode expansion $(**)$ $\phi(z, \bar{z}) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} z^{-m-h} \bar{z}^{-n-\bar{h}} \phi_{m,n}$

$$(*) \Rightarrow \left(\phi(z, \bar{z}) \right)^{\dagger} = \sum_{n,m} (\bar{z})^{m+h} (z)^{n+\bar{h}} \phi_{m,n} \bar{z}^{-2h} z^{-2\bar{h}} = \sum_{\substack{n \to -n \\ m \to -m}} \bar{z}^{-m-h} z^{-n-\bar{h}} \phi_{-m,-n} \quad \checkmark$$

$$(**) \Rightarrow \left(\phi(z, \bar{z}) \right)^{\dagger} = \sum_{m,n} \bar{z}^{-m-h} z^{-n-\bar{h}} \left(\phi_{m,n} \right)^{\dagger} \Rightarrow \left(\phi_{m,n} \right)^{\dagger} = \phi_{-m,-n}$$

$(z)^{*} = \bar{z}$

$h \; \bar{z}^{2\bar{h}} \phi(z,\bar{z})$

$\left(\frac{1}{z}\right) \bar{z}^{-2h} \; \bar{z}^{-2\bar{h}}$

...mary fields.

$-h \; \phi_{-m,-h}$

Q) Operator - state corresp in modes ?

- $\phi_{m,n} |0\rangle = 0, \quad m > -h, \quad n > -\bar{h}$

$\left(\lim_{z \to 0} z^{-1} \phi_{..} |0\rangle \to \infty \text{ otherwise} \right)$

- $\lim_{z \to 0} T(z) |0\rangle = \lim_{z \to 0} \sum_{n \in \mathbb{Z}} L_n z^{-n-2} |0\rangle = L_{-2} |0\rangle$, in particular

$L_n |0\rangle = 0, \; n \geq -1$,

$\ell_{-1,0,1} \to L_{-1,0,1} |0\rangle = 0$

↗ invararinace of $|0\rangle$ under global conf. transformations.

- Remark this is not always the case

e.g. Whittaker vectors: $L_1 |\text{Wit.}\rangle = \Lambda |\text{Wit}\rangle$

$[L_{-1,0,1}, \phi(z)] = \partial \phi$