Title: Lecture - Quantum Field Theory III - PHYS 777

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Collection/Series: Quantum Field Theory III, PHYS 777-, February 24 - March 28, 2025

Subject: Quantum Fields and Strings

Date: March 10, 2025 - 10:15 AM

URL: https://pirsa.org/25030008

Recap Last time: phase transitions (of second order)

Free energy: $f = -\frac{T}{N} \log Z$

$\sigma_i = \pm 1$

(conn.) correlation func: $G(r) = \langle \sigma_r \sigma_0 \rangle - \langle \sigma_r \rangle \langle \sigma_0 \rangle$

$G(r) \propto e^{-r/\xi}$, $r \to \infty$

ξ – correlation length

$H = h \sum \sigma_i + K \sum \sigma_i \sigma_j$

Second order p.t.: $\chi = \frac{\partial^2 f}{\partial h^2}$, $C = \frac{\partial^2 f}{\partial T^2}$, $\xi \to \infty$

$t = \frac{T - T_c}{T_c}$

- $C\big|_{h=0} \propto |t|^{-\alpha}$
- $\chi\big|_{h=0} \propto |t|^{-\gamma}$
- $\xi\big|_{h=0} \propto |t|^{-\nu}$
- $G\big|_{t=h=0} \propto \frac{1}{r^{d-2+\eta}}$
- $M\big|_{h\to 0^-} \propto (-t)^{\beta}$
- $M\big|_{t=0} \propto |h|^{1/\delta}$

$\alpha, \beta, \gamma, \delta, \nu, \eta$ – critical exponents.

It can be justified by RG picture.

$$Z_{\Sigma_i}(K') = Z_{s_i}(K) \Rightarrow K' = RG(K)$$

critical points ($\xi \to \infty, 0$): $K^* = RG(K^*)$.

$$u_i \mapsto b^{y_i} u_i, \quad u_i = \sum_a e^i_a (K_a - K^*_a)$$

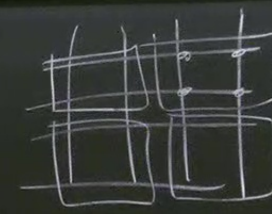
y_i - scaling dimensions, $y_i > 0$ - relevant.

$$\Rightarrow f(u_t, u_h) = (t/t_0)^{d/y_t}\, \Phi\left(\frac{h/h_0}{(t/t_0)^{y_h/y_t}}\right)$$

$$G(r, u_t) = (t/t_0)^{\frac{2(d-y_h)}{y_t}}\, \Psi\left(r/(t/t_0)^{-1/y_t}\right)$$

$$\Rightarrow \quad \alpha = 2 - \frac{d}{y_t}, \quad \beta = \frac{d-y_h}{y_t}, \quad \gamma = \frac{2y_h - d}{y_t}$$

$$\delta = \frac{y_h}{d - y_h}, \quad \nu = 1/y_t, \quad \eta = d + 2 - 2y_h$$



$s_i = \pm 1$

$\Sigma_i = \pm 1$

$N' = \frac{N}{b^d}$

$b = 2$

$\xi' = \frac{\xi}{b}$

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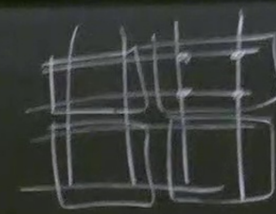

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$\Rightarrow$

$$\alpha = 2 - \frac{d}{y_t} \qquad \delta = \frac{y_h}{d - y_h}$$

$$\beta = \frac{d - y_h}{y_t} \qquad \nu = 1/y_t$$

$$\gamma = \frac{2y_h - d}{y_t} \qquad \eta = d + 2 - 2y_h$$

$y_t = 1$, $y_h = \frac{15}{8}$

2d conformal field theories, conf alg in 2d

Conf. transf.: $\frac{\partial x'^{\lambda}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \eta_{\lambda\sigma} = e^{\omega} \eta_{\mu\nu}$.

infinitesimally: $x'^{\mu} = x^{\mu} + \epsilon^{\mu}$, $e^{\omega} \simeq 1 + \omega$

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \omega \eta_{\mu\nu}$$

$$\partial_\mu \epsilon^\mu = \omega$$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

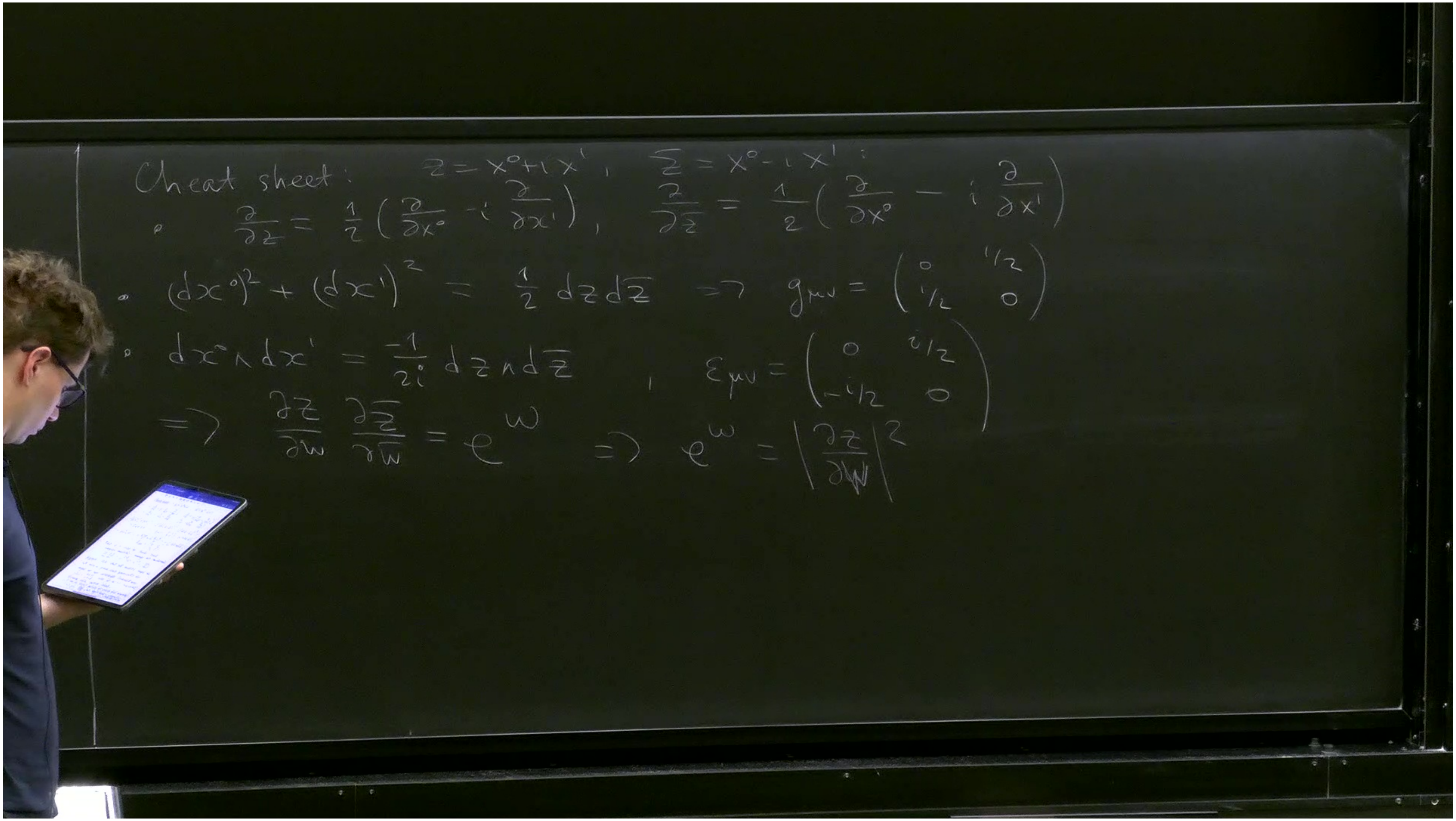
$\mu,\nu = (0,0)$: $\cancel{\partial_0 \epsilon_0} + \partial_0 \epsilon_0 = \cancel{\partial_0 \epsilon_0} + \partial_1 \epsilon_1$

$\mu,\nu = (0,1)$: $\partial_0 \epsilon_1 + \partial_1 \epsilon_0 = 0$

$\epsilon = \epsilon^0 + i \epsilon^1$

$z = x^0 + i x^1$

$\Leftrightarrow \frac{\partial \epsilon}{\partial \bar{z}} = 0$, $\epsilon = \epsilon(z)$


Cheat sheet: $z = x^0 + i x^1$, $\bar{z} = x^0 - i x^1$:
• $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} - i \frac{\partial}{\partial x^1}\right)$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} - i \frac{\partial}{\partial x^1}\right)$
• $(dx^0)^2 + (dx^1)^2 = \frac{1}{2} dz d\bar{z}$ ⇒ $g_{\mu\nu} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$
• $dx^0 \wedge dx^1 = \frac{-1}{2i} dz \wedge d\bar{z}$, $\varepsilon_{\mu\nu} = \begin{pmatrix} 0 & i/2 \\ -i/2 & 0 \end{pmatrix}$
⇒ $\frac{\partial z}{\partial w} \frac{\partial \bar{z}}{\partial \bar{w}} = e^{\omega}$ ⇒ $e^{\omega} = \left|\frac{\partial z}{\partial w}\right|^2$

Cheat sheet: $z = x^0 + i x^1$, $\bar{z} = x^0 - i x^1$;

- $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x^0} - i \dfrac{\partial}{\partial x^1}\right)$, $\dfrac{\partial}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x^0} + i \dfrac{\partial}{\partial x^1}\right)$
- $(dx^0)^2 + (dx^1)^2 = \frac{1}{2} dz\, d\bar{z} \Rightarrow g_{\mu\nu} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$
- $dx^0 \wedge dx^1 = \dfrac{-1}{2i} dz \wedge d\bar{z}$, $\varepsilon_{\mu\nu} = \begin{pmatrix} 0 & i/2 \\ -i/2 & 0 \end{pmatrix}$

$$\Rightarrow \frac{\partial z}{\partial w} \frac{\partial \bar{z}}{\partial \bar{w}} = e^{\omega} \Rightarrow e^{\omega} = \left|\frac{\partial z}{\partial w}\right|^2$$

Remark: In higher dimensions conf. trans. were forming groups. $SO(d,2)$

$w = f(z)$, but f is usually non-invertible

e.g. $f = z^2$ two-to-one

$f^{-1} = \sqrt{z}$

- f should not have branch cuts & e.s.p. $\Rightarrow f = \dfrac{P(z)}{Q(z)}$
- P, Q should be of deg $= 1 \Rightarrow f = \dfrac{az+b}{cz+d}$

$\Rightarrow SL(2,\mathbb{C}) \cong SO(3,1)$ $ad - bc = 1$

Remark Sometimes we will treat $z, \bar{z}$ as independent $\mathbb{C}$-variables

Real hyperplane : $(z)^* = \bar{z}$

Algebra of conformal transformations: $l_n = -z^{n+1}\frac{\partial}{\partial z}$, $\bar{l}_n = -\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}$, $n \in \mathbb{Z}$

$[l_n, l_m] = (n-m)\, l_{n+m}$, $[\bar{l}_n, l_m] = 0$, $[\bar{l}_n, \bar{l}_m] = (n-m)\, \bar{l}_{n+m}$

$\mathrm{Witt} \oplus \mathrm{Witt}$

$\cup$ $sl_2 \oplus sl_2$.

$\mathrm{Witt} \supset sl_2 \longrightarrow$

$l_{-1} = -\partial_z$ – translations

$l_0 = -z\partial_z$ – dilatations & rot $\left(l_0 + \bar{l}_0,\ i(l_0 - \bar{l}_0)\right)$

$l_1 = -z^2\partial_z$ – SCT

Remark actual conf. transf. should preserve real hyperplane ex

z

$= -\bar{z}^{-n+1} \frac{\partial}{\partial \bar{z}}$

$n) \bar{l}_{n+m}$

$l_0) \Big)$

Transformation properties of the fields

Rotations: $SO(2)$ $\quad z \mapsto e^{i\varphi} z, \ \bar{z} \mapsto e^{-i\varphi} \bar{z}$, $\quad \tilde{\phi}(e^{i\varphi} z, e^{-i\varphi} \bar{z}) = e^{-i\varphi s} \phi(z, \bar{z})$

Dilatations: $\mathbb{R}^*_{>0}$ $\quad z \mapsto e^{\sigma} z, \ \bar{z} \mapsto e^{\sigma} \bar{z}$ $\quad \tilde{\phi}(e^{\sigma} z, e^{\sigma} \bar{z}) = e^{-\sigma \cdot \Delta} \phi(z, \bar{z})$

- fields which transform like this are called <u>quasi-primary</u>.

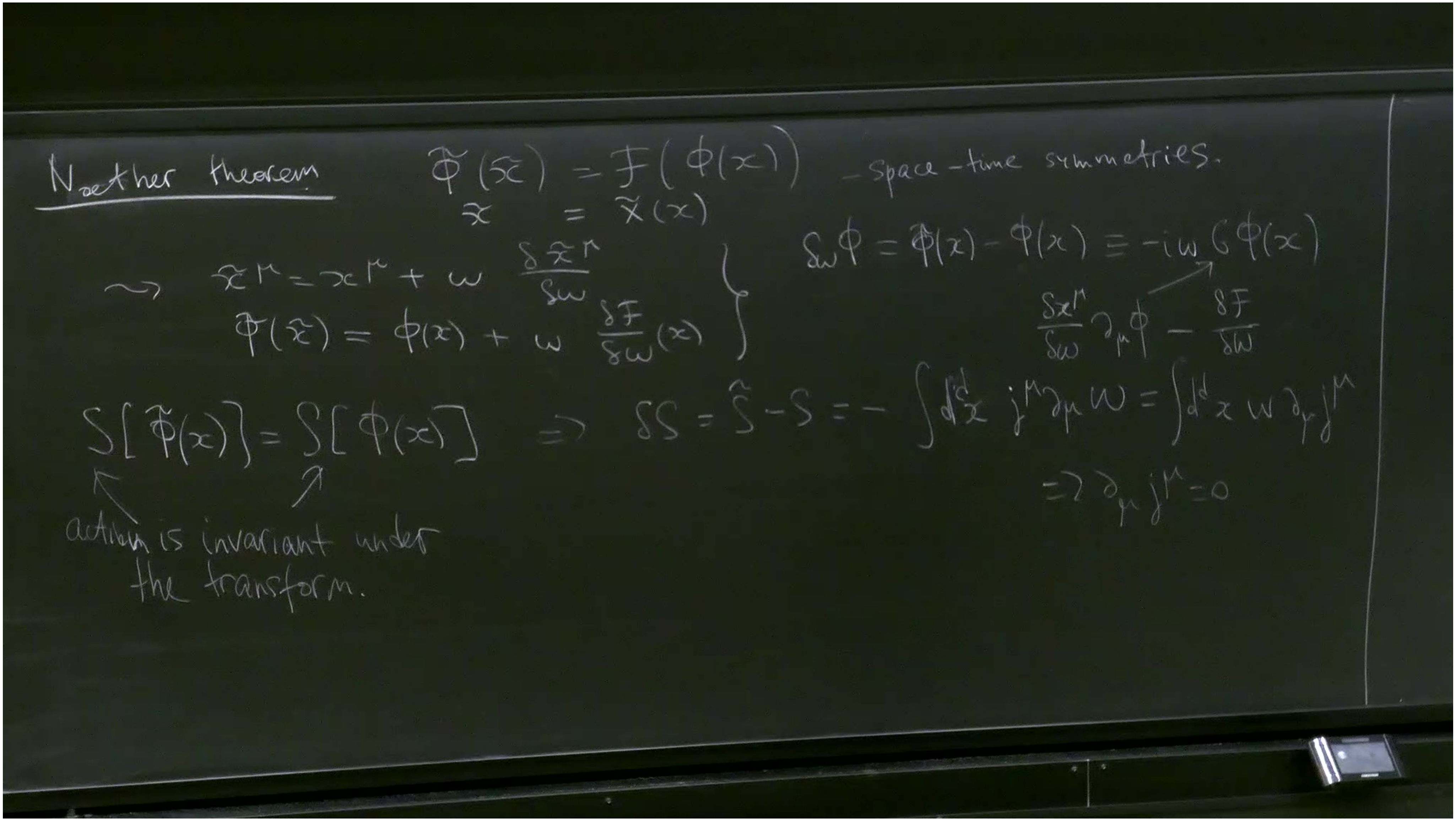
$h = \frac{1}{2}(\Delta + s)$, $\bar{h} = \frac{1}{2}(\Delta - s)$ - conformal weights (dimensions) of field

Primary fields: $\quad \tilde{\phi}(w(z), \bar{w}(\bar{z})) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z})$

<u>Remark</u> Primary fields are those which transform like quasi primary, but "locally". $\left|\frac{dw}{dz}\right| = e^{\sigma}$ (local dilatation) $\quad \arg \frac{dw}{dz} = \varphi$ (local rotation)

$w = z + \epsilon$
$\bar{w} = \bar{z} + \bar{\epsilon}$

Infinitesimally. $\quad \delta_{\epsilon, \bar{\epsilon}} \phi(z) = \tilde{\phi}(z) - \phi(z) = -(h \partial_z \epsilon \cdot \phi + \epsilon \partial_z \phi) - (\bar{h} \partial_{\bar{z}} \bar{\epsilon} \phi + \bar{\epsilon} \partial_{\bar{z}} \phi)$


Noether theorem
$\tilde{\Phi}(\tilde{x}) = \mathcal{F}(\phi(x))$ – space-time symmetries.
$\tilde{x} = \tilde{x}(x)$
$\rightsquigarrow \tilde{x}^\mu = x^\mu + \omega \frac{\delta \tilde{x}^\mu}{\delta \omega}$
$\tilde{\Phi}(\tilde{x}) = \phi(x) + \omega \frac{\delta \mathcal{F}}{\delta \omega}(x)$
$\delta_\omega \phi = \tilde{\Phi}(x) - \phi(x) \equiv -i\omega G \phi(x)$
$\frac{\delta x^\mu}{\delta \omega} \partial_\mu \phi - \frac{\delta \mathcal{F}}{\delta \omega}$
$S[\tilde{\Phi}(x)] = S[\phi(x)]$
action is invariant under the transform.
$\Rightarrow \delta S = \hat{S} - S = -\int d^d x \, j^\mu \partial_\mu \omega = \int d^d x \, \omega \partial_\mu j^\mu$
$\Rightarrow \partial_\mu j^\mu = 0$