Title: Lecture - Mathematical Physics, PHYS 777

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This happens if $F(A) = \star F(A)$,

MATH

What is a complex manifold?

A real manifold M, charts U,

$f: U \hookrightarrow \mathbb{R}^n$

transition fns are smooth

ADHM constructions

Sol^{n}s to SDYM eq^{n}s $\Longleftrightarrow$ Algebraic data

Proof goes by twistor theory + complex geometry

1^{st} part of course: talk about complex geometry

... complex geometry

We can also consider

$\mathrm{tr}\, F(A) \wedge F(A)$

If we vary A, this changes by a total derivative

$\int \mathrm{tr}\, F(A) \wedge F(A)$ is topological

Fact: $\frac{1}{8\pi^2} \int \mathrm{tr}\, F(A) \wedge F(A) \in \mathbb{Z}$, is the instanton number

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cf. $\frac{1}{8\pi^2} \int \mathrm{tr}\, F(A) \wedge F(A) \in \mathbb{Z}$, is the instanton number.

$$\int \operatorname{tr} F(A) \wedge * F(A) \geq \left| \int \operatorname{tr} F(A) \wedge F(A) \right|$$

Equality holds if and only if

$$F(A) = \pm * F(A)$$

An <u>instanton</u> is a gauge field A

such that:

1) $F(A) \to 0$ at ∞ on $\mathbb{R}^4$

2) $F(A) = *F(A)$

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such that:

1) $F(A) \to 0$ at ∞ on $\mathbb{R}^4$

2) $F(A) = *F(A)$

2) $\Rightarrow$ EOM: $d_A F = 0$ so $d_A * F = 0$

ADHM: Classified instantons

Why is this important?

When we do the path integral

$$\int_A e^{-\frac{1}{g^2}S(A)}$$

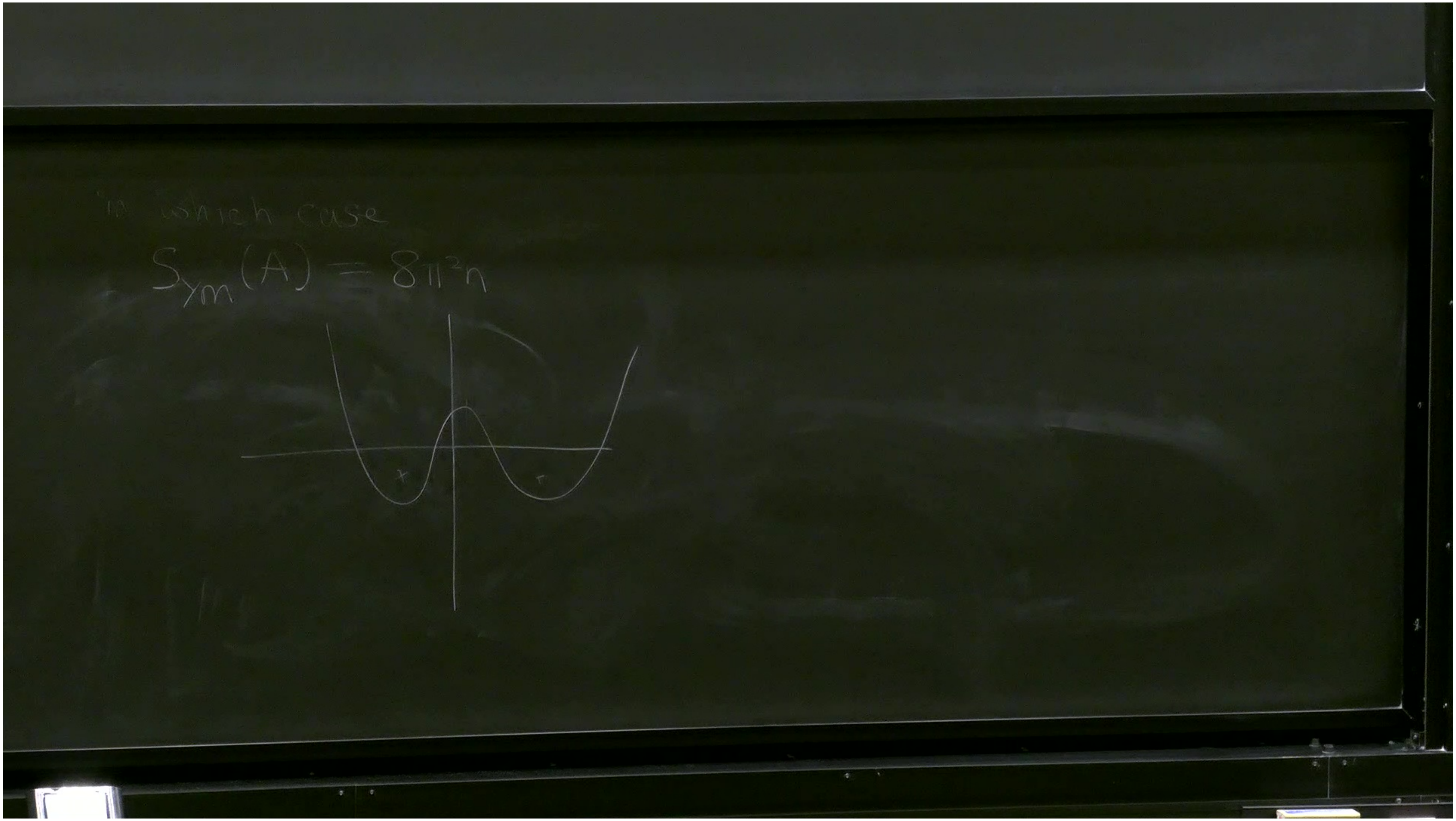
we should include those A which are top' non-trivial.

$$\int_A e^{-\frac{1}{g^2}S_{YM}(A)} = \sum_n \int_{A,\ \frac{1}{8\pi^2}\int F\wedge F = n} e^{-\frac{1}{g^2}S_{YM}(A)}$$

Saddle point analysis:

If $n>0$, dominant contribution is near A with $S_{YM}(A)$ minimized

This happens if $F(A) = *F(A)$,


in which case
$S_{YM}(A) = 8\pi^2 n$

A complex manifold has local coords on each chart which live in $\mathbb{C}^n$

Local coords might be $z_1, \ldots, z_n \to$ on U

If $w_1, \ldots, w_n$ are local coordinates on V

then on $U \cap V$, we can write $w_i = f_i(z_1 \ldots z_n)$

We require $\dfrac{\partial f_i}{\partial \bar{z}_j} = 0$

If $x_i = \mathrm{Re}\, z_i$
$y_i = \mathrm{Im}\, z_i$

$$\frac{\partial}{\partial \bar{z}_i} = \frac{\partial}{\partial x_i} + \sqrt{-1}\,\frac{\partial}{\partial y_i}$$

Example

$\mathbb{CP}^1$

Has two patches each of which is $\mathbb{C}$

Coord. z on one patch
w on other patch

They intersect on the locus $z \neq 0, w \neq 0$
and here $z = 1/w$

$$\mathbb{CP}^1 = S^2$$

z = a coord on $S^2 \setminus (1,0,0) = \mathbb{R}^2$ by stereographic projection

w " " $S^2 \setminus (-1,0,0)$

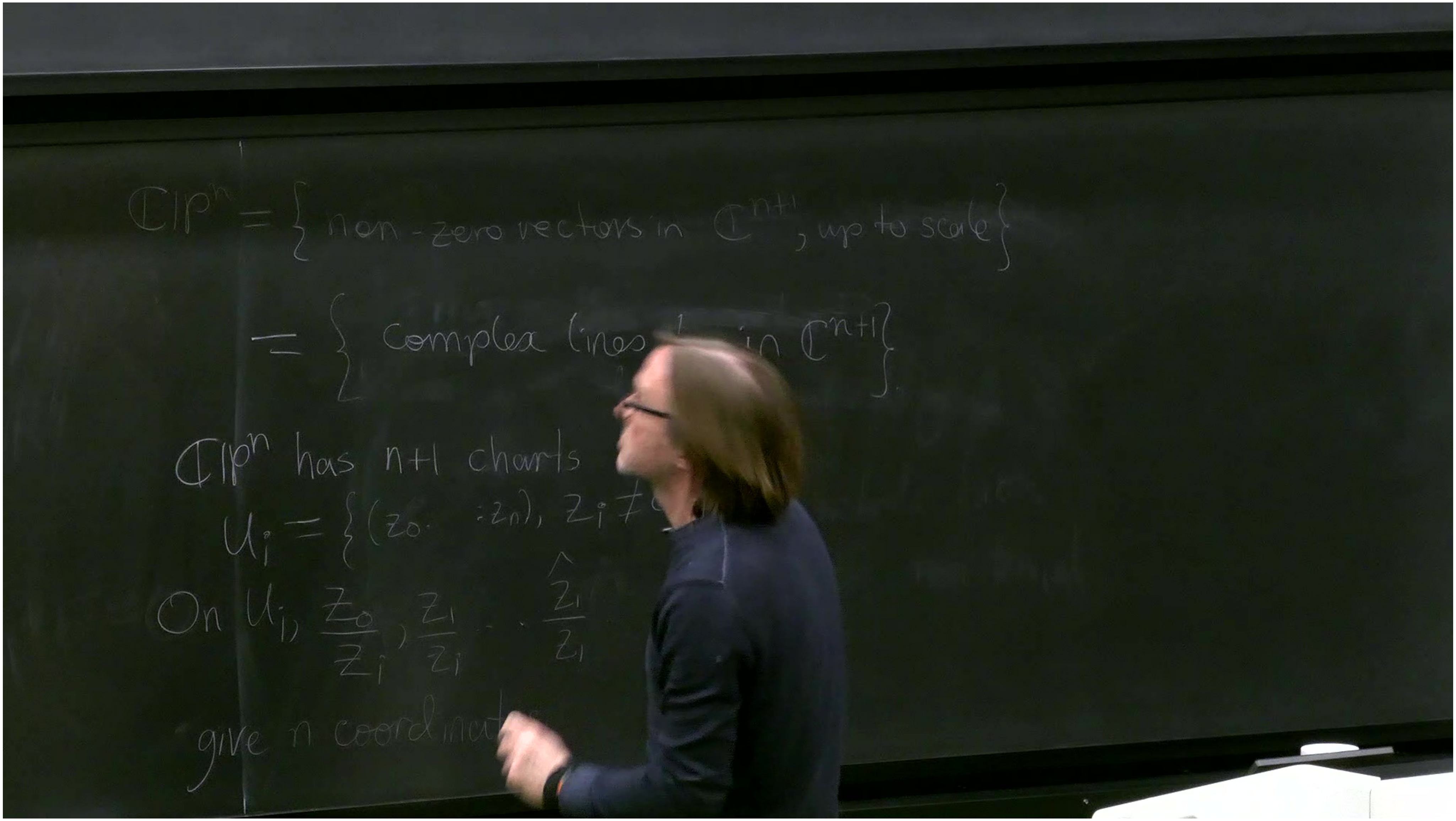
$z = \infty,$
$w = 0$

$$\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$$

Projective Space $\mathbb{CP}^n$

A point in $\mathbb{CP}^n$ is a collection of complex numbers $(z_0 : \cdots : z_n)$ where:

1) They are not all 0
2) If $\lambda \in \mathbb{C}$, $\lambda \neq 0$, we identify
$$(z_0 : \cdots : z_n) = (\lambda z_0 : \cdots : \lambda z_n)$$


$\mathbb{CP}^n = \{$ non-zero vectors in $\mathbb{C}^{n+1}$, up to scale $\}$
$= \{$ complex lines
in $\mathbb{C}^{n+1} \}$.
$\mathbb{CP}^n$ has $n+1$ charts
$U_i = \{(z_0 \cdot$
$: z_n), z_i \neq$
On U_i, $\frac{z_0}{z_i}, \frac{z_1}{z_i} \cdots \frac{\hat{z}_i}{z_i}$
give n coordinat

On $U_i \cap U_j$, have coords

$$u_k = \frac{z_k}{z_i} \quad \text{and} \quad v_k = \frac{z_k}{z_j}$$

The transition function is holomorphic

$$u_k = \frac{z_j}{z_i} v_k = u_j v_k$$

No $\bar{u}$ or $\bar{v}$ involved $\Rightarrow$ holomorphic

If $x_i = \mathrm{Re}\, z_i$

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Example

$\mathbb{CP}^1$

Has two patches each of which is $\mathbb{C}$

Coord. z on one patch

w on other patch

If M is a $2n$ dim'l real manifold,

an almost complex structure

on M is an operator

$J : TM \circlearrowleft$

so that $J^2 = -Id$

Then $TM \underset{\mathbb{R}}{\otimes} \mathbb{C}$ decomposes

into $+i$ and $-i$ eigenspaces

$+i$ eigenspace: vectors like $\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j} = \frac{\partial}{\partial z_j}$

$-i$ " "

$\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j} = \frac{\partial}{\partial \bar{z}_j}$

A fn f is holomorphic if

$\forall$ v. fields X, with $JX = -iX$

$Xf = 0$

Need: Locally $\exists$ n independent hol. fns

$\Longleftrightarrow$ If X, Y are in $-i$ eigenspace, so is $[X, Y]$

Then $TM \otimes \mathbb{C}$ decomposes

into $+i$ and $-i$ eigenspaces.

$+1$ eigenspace: vectors like $\dfrac{\partial}{\partial x_j} - \sqrt{-1}\,\dfrac{\partial}{\partial y_j} = \dfrac{\partial}{\partial z_j}$

$-i$ " " $\dfrac{\partial}{\partial x_j} + \sqrt{-1}\,\dfrac{\partial}{\partial y_j} = \dfrac{\partial}{\partial z_j}$

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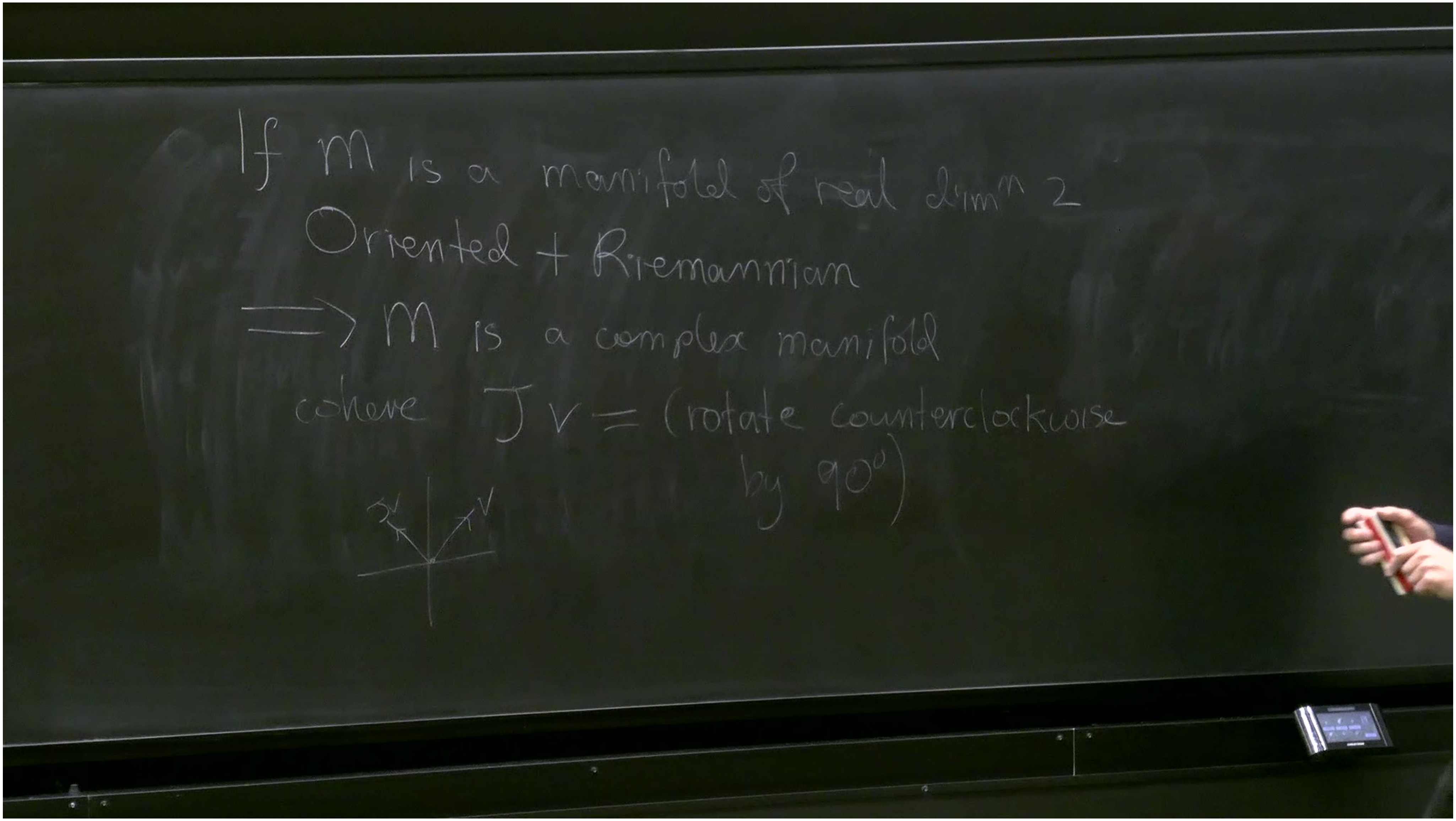
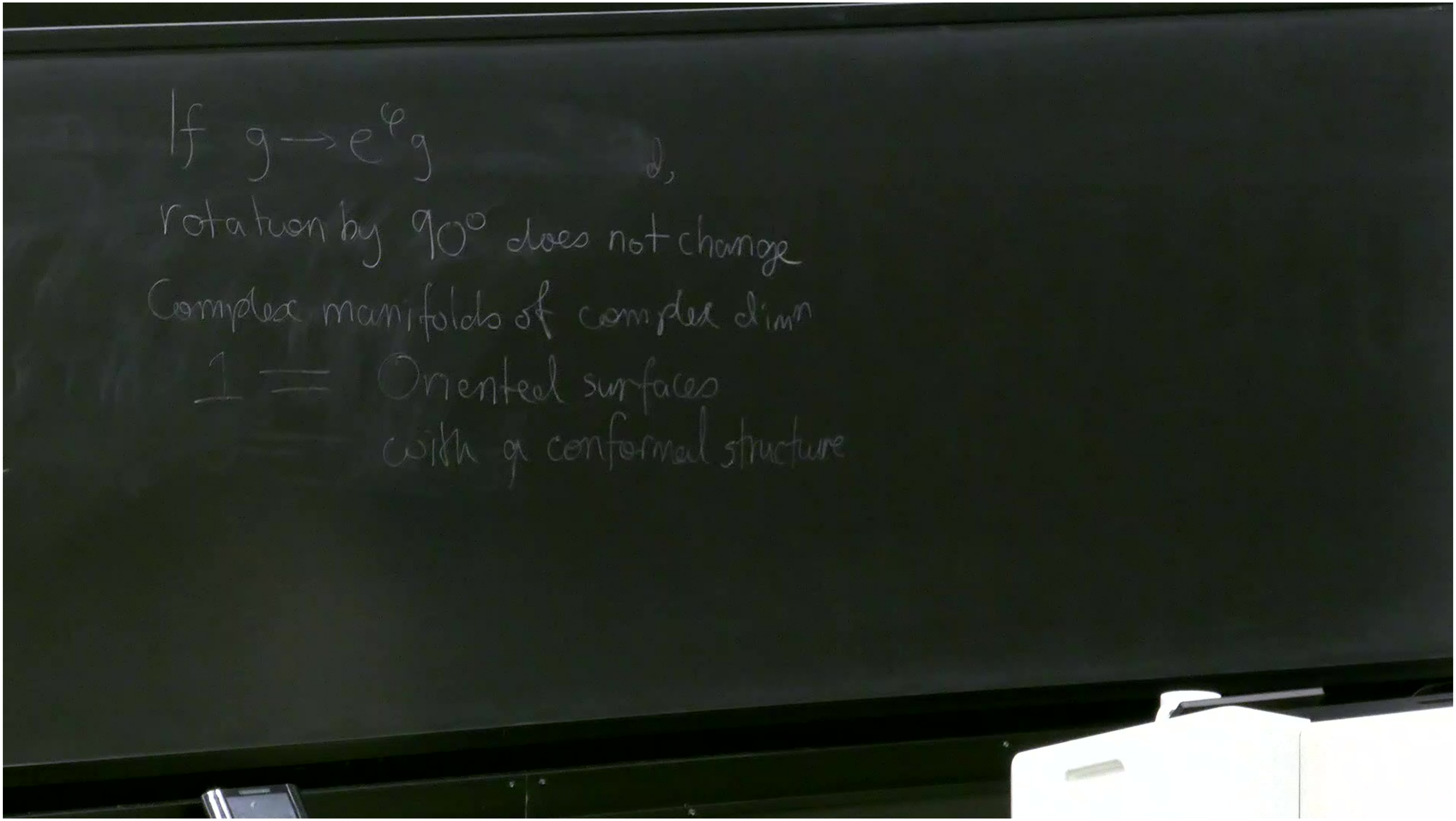
V real vector

$V \pm iJV$ are eigenvectors under J


If M is a manifold of real dim^n 2
Oriented + Riemannian
⟹ M is a complex manifold
where Jv = (rotate counterclockwise
by 90°)
Jv
v


If $g \to e^{\varphi} g$ 2,
rotation by 90° does not change
Complex manifolds of complex dimn
1 = Oriented surfaces
with a conformal structure