Title: Lecture - Causal Inference, PHYS 777

Speakers: Robert Spekkens

Collection/Series: Causal Inference (Elective), PHYS 777, March 31 - May 2, 2025

Subject: Quantum Foundations

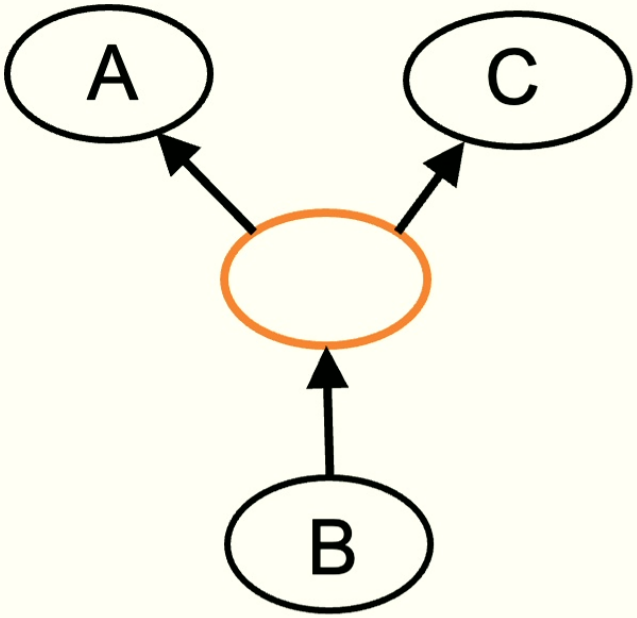
Date: April 14, 2025 - 10:15 AM

URL: https://pirsa.org/25030003

The observational and interventional dominance orders of causal structures, part 2

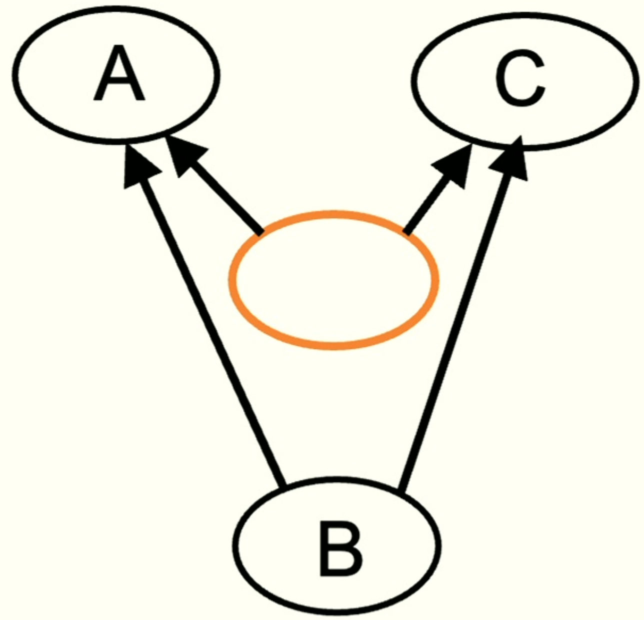
arXiv:2502.07891
arxiv:2407.01686

Exogenization rule in classical causal models

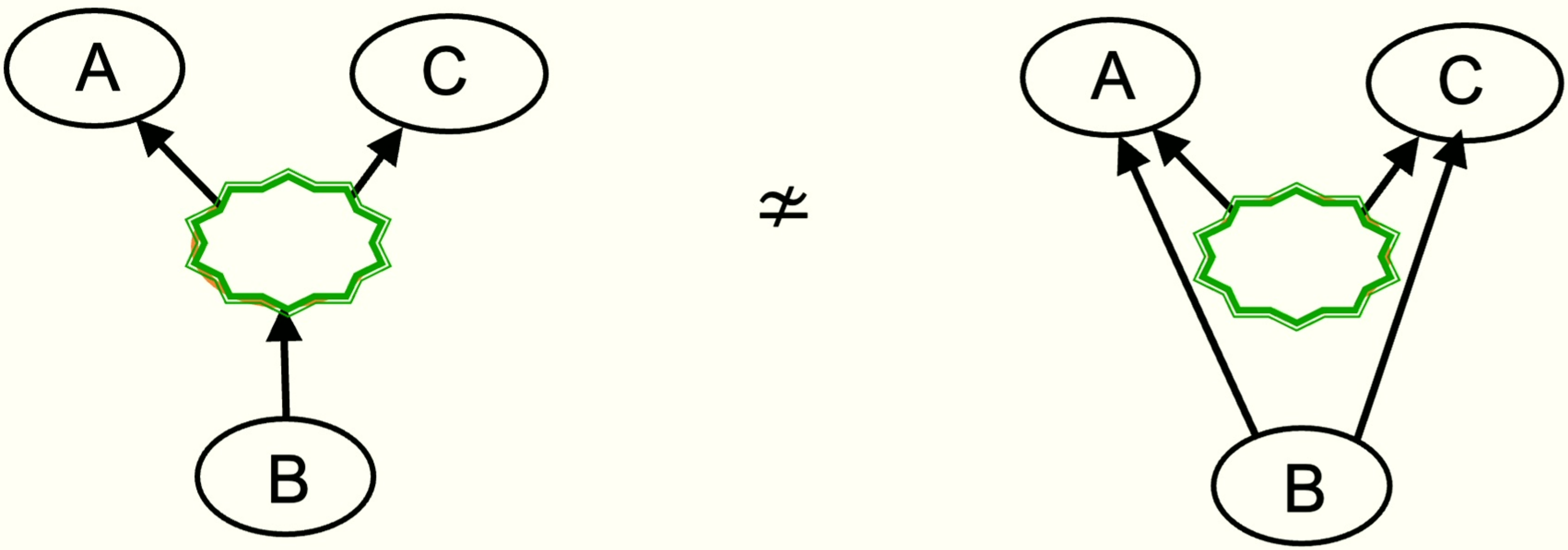


$\simeq$

interventionally equivalent



Exogenization rule does not hold quantumly



Wolfe et al, Phys. Rev. X 11, 021043 (2021)

Observational dominance order

For general mDAGs

Observational dominance of mDAGs $\not\Rightarrow$ Structural dominance of mDAGs

Observational dominance of mDAGs $\Leftarrow$ Structural dominance of mDAGs

$G \succeq_{obs} G'$

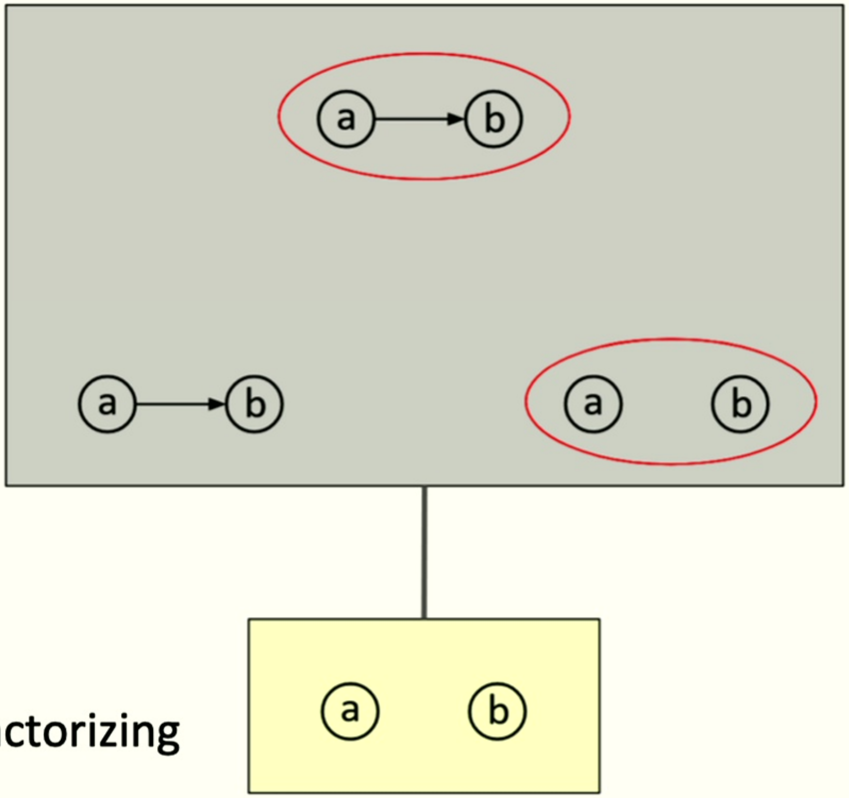
$G \succeq_{struct} G'$

In particular, observational equivalence does not imply equivalence of mDAGs

Different mDAGs can be observationally equivalent

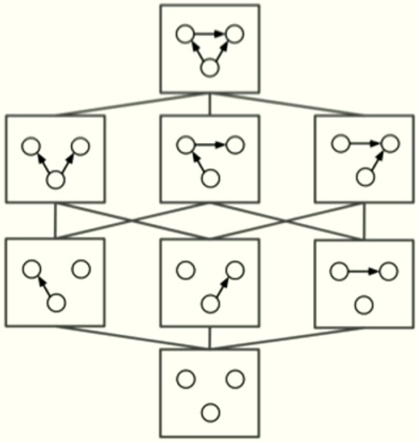
Observational order of 2-node mDAGs

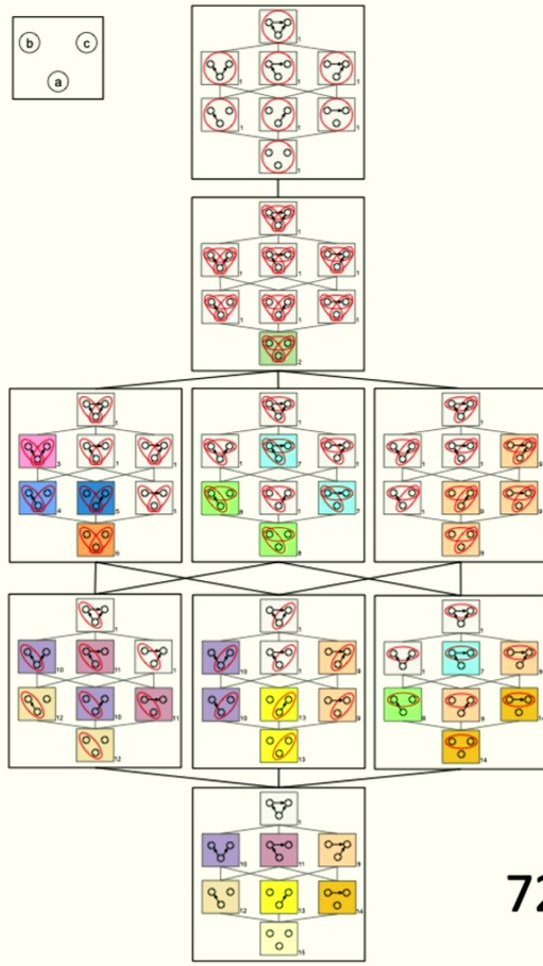
Saturating

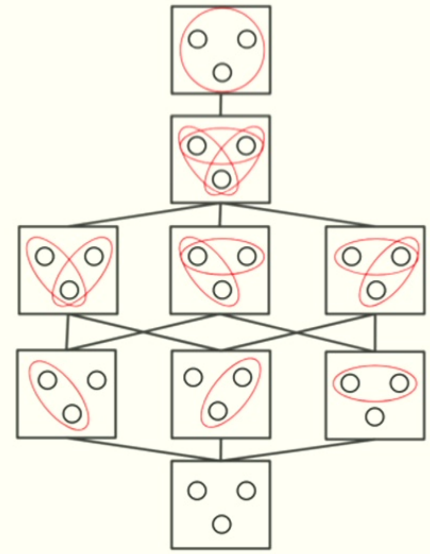


Factorizing

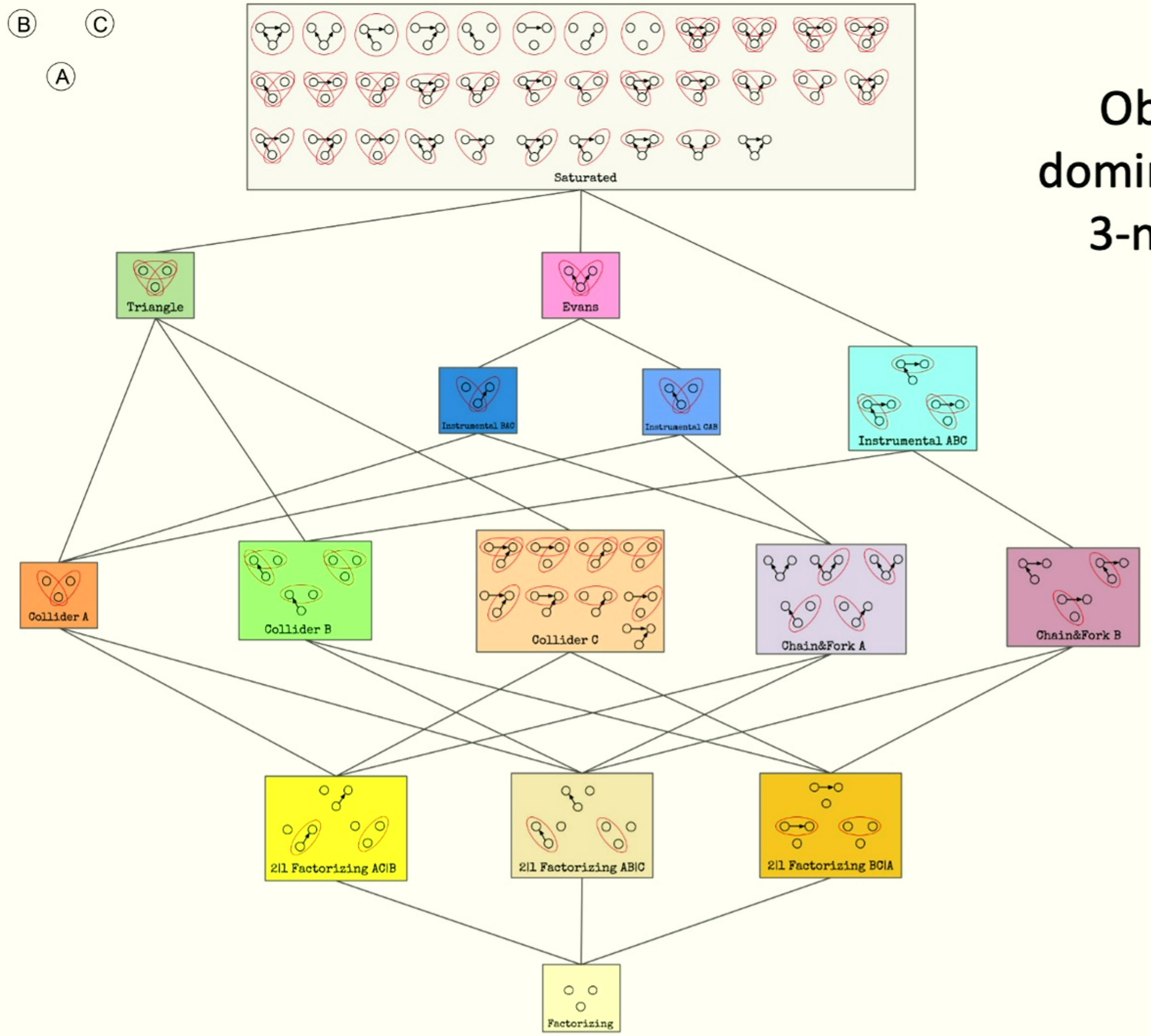
O&D dominance order of 3-node mDAGs



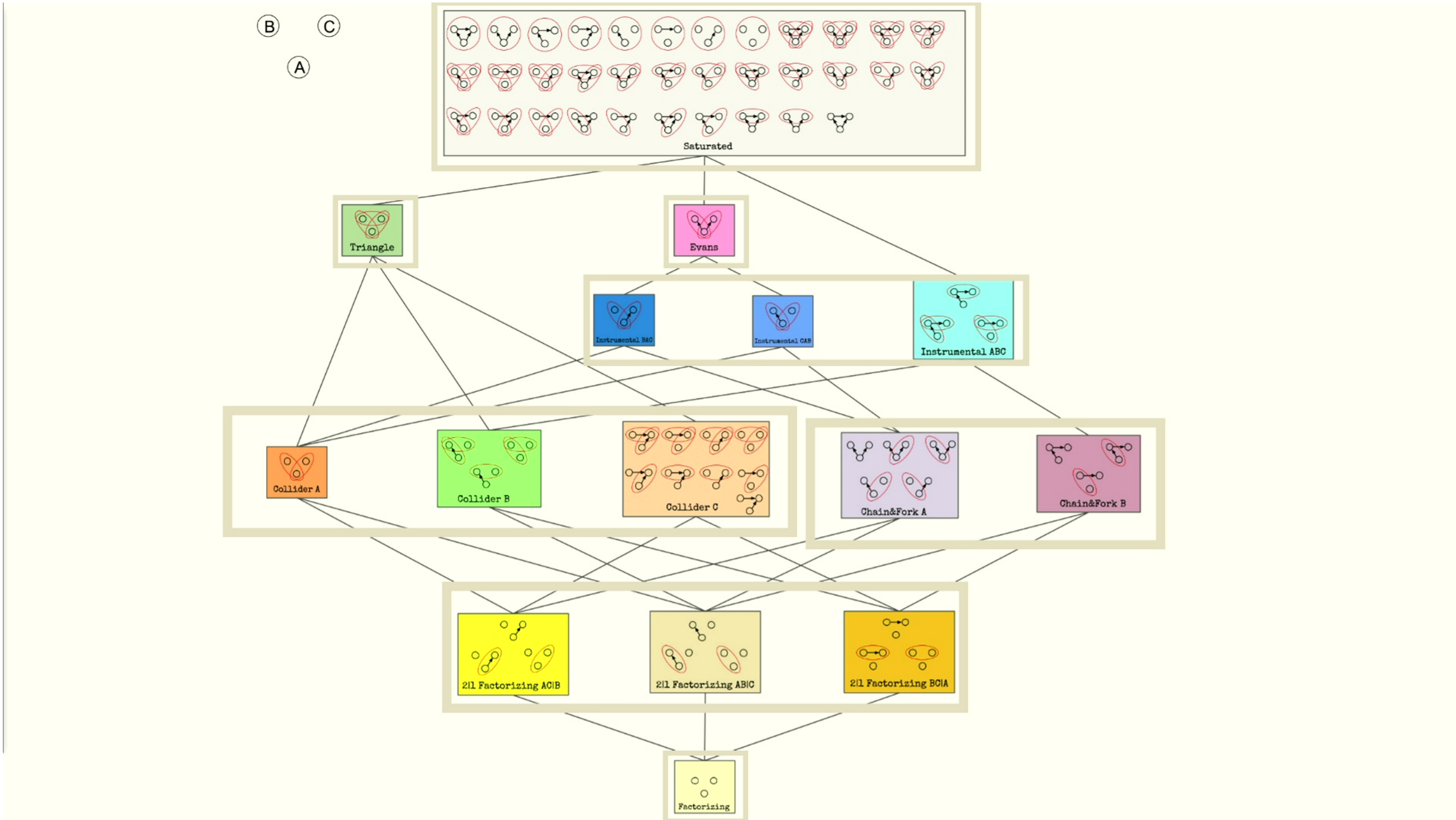


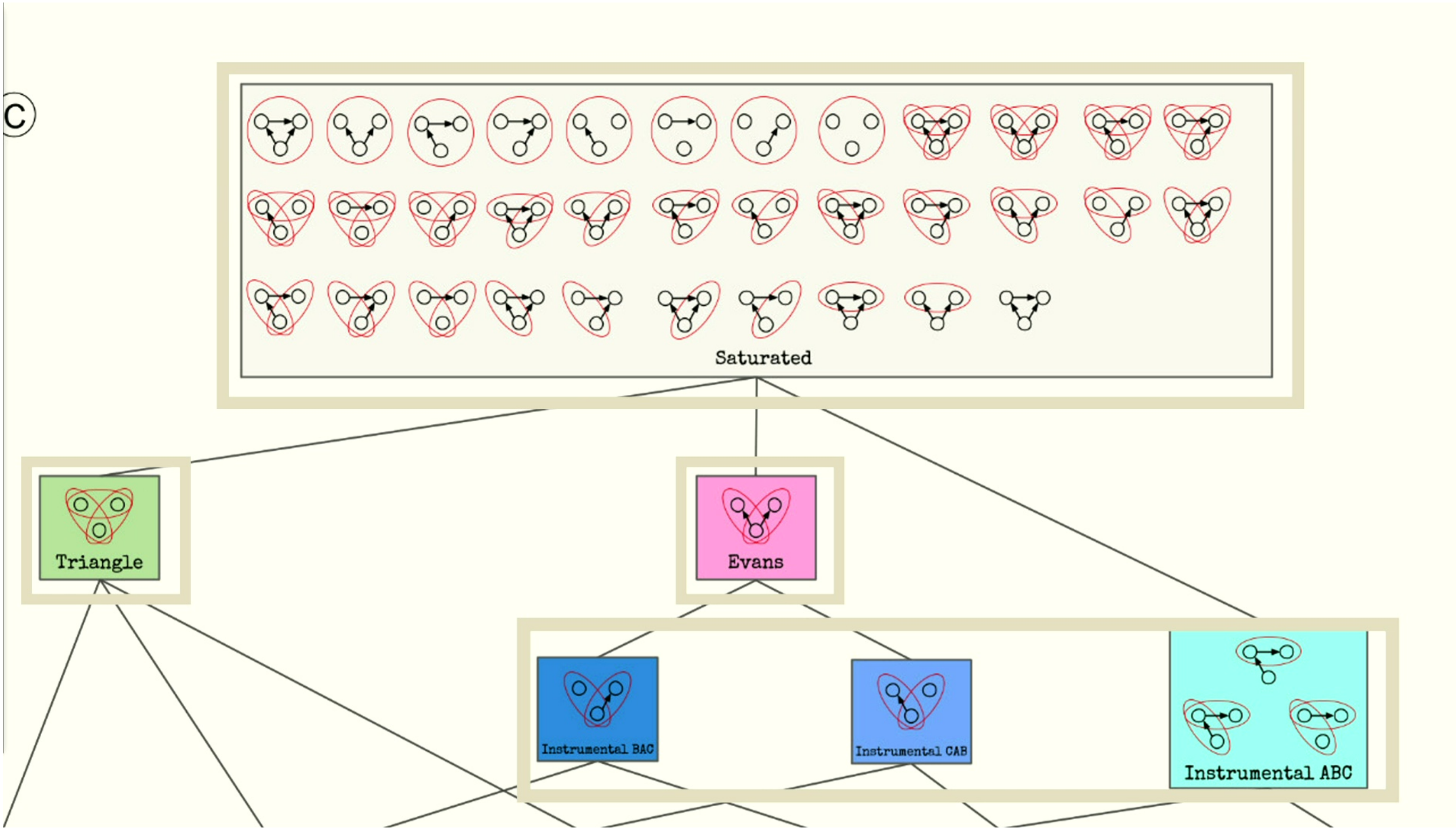


72 mDAGs

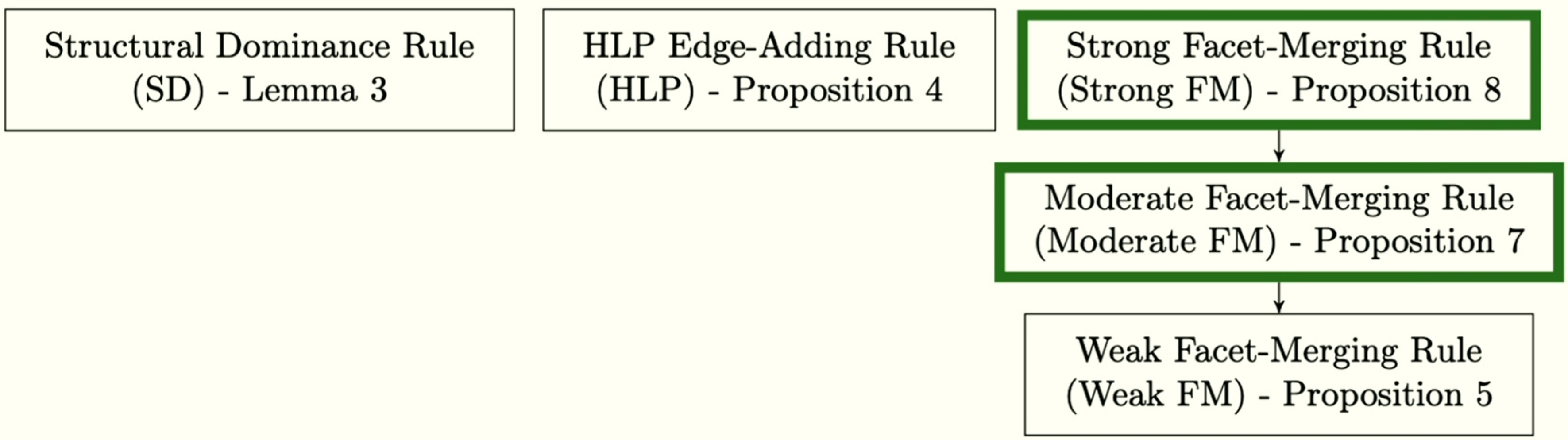


Observational dominance order of 3-node mDAGs

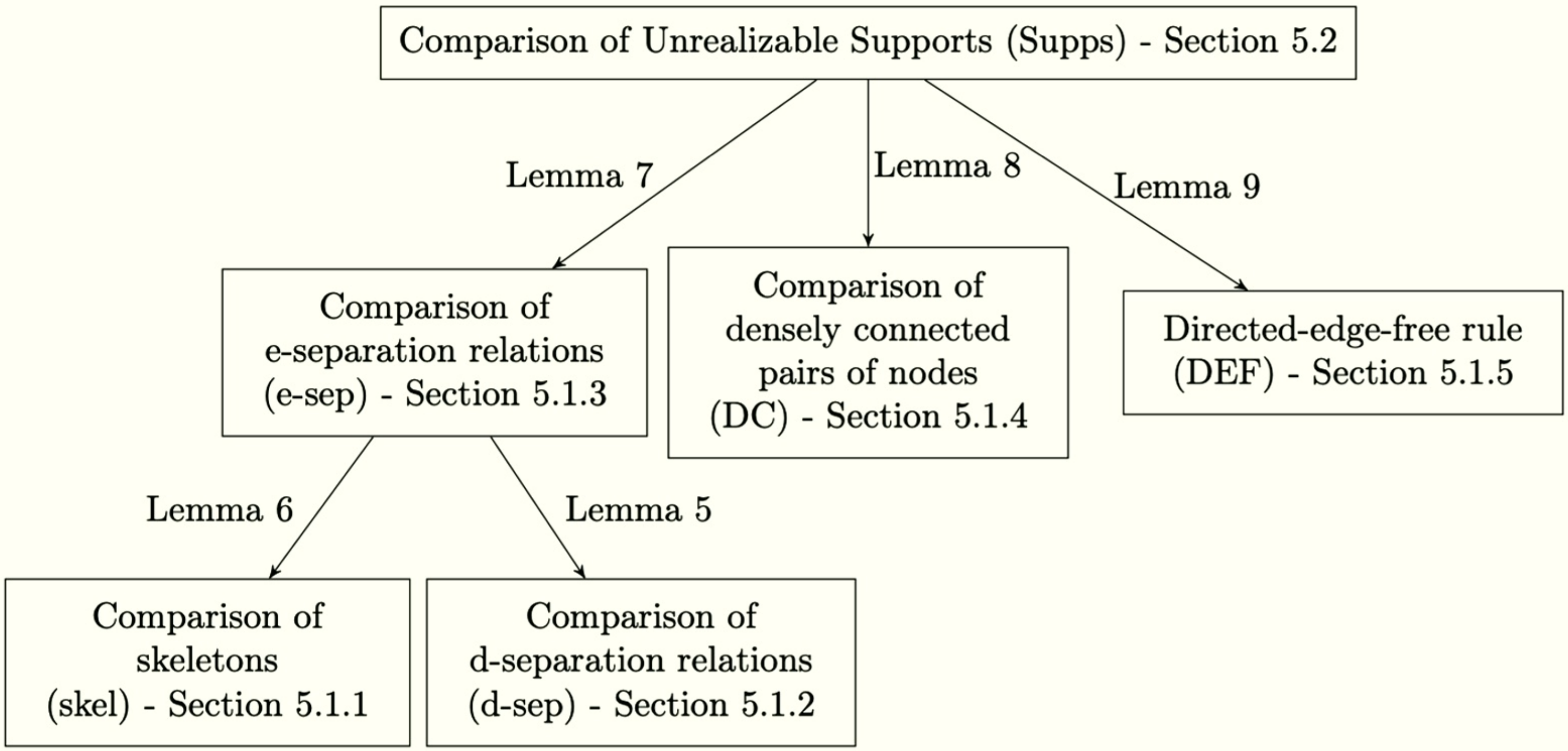

B
C
A
Saturated
Triangle
Evans
Instrumental BAC
Instrumental CAB
Instrumental ABC
Collider A
Collider B
Collider C
Chain&Fork A
Chain&Fork B
2|1 Factorizing AC|B
2|1 Factorizing AB|C
2|1 Factorizing BC|A
Factorizing

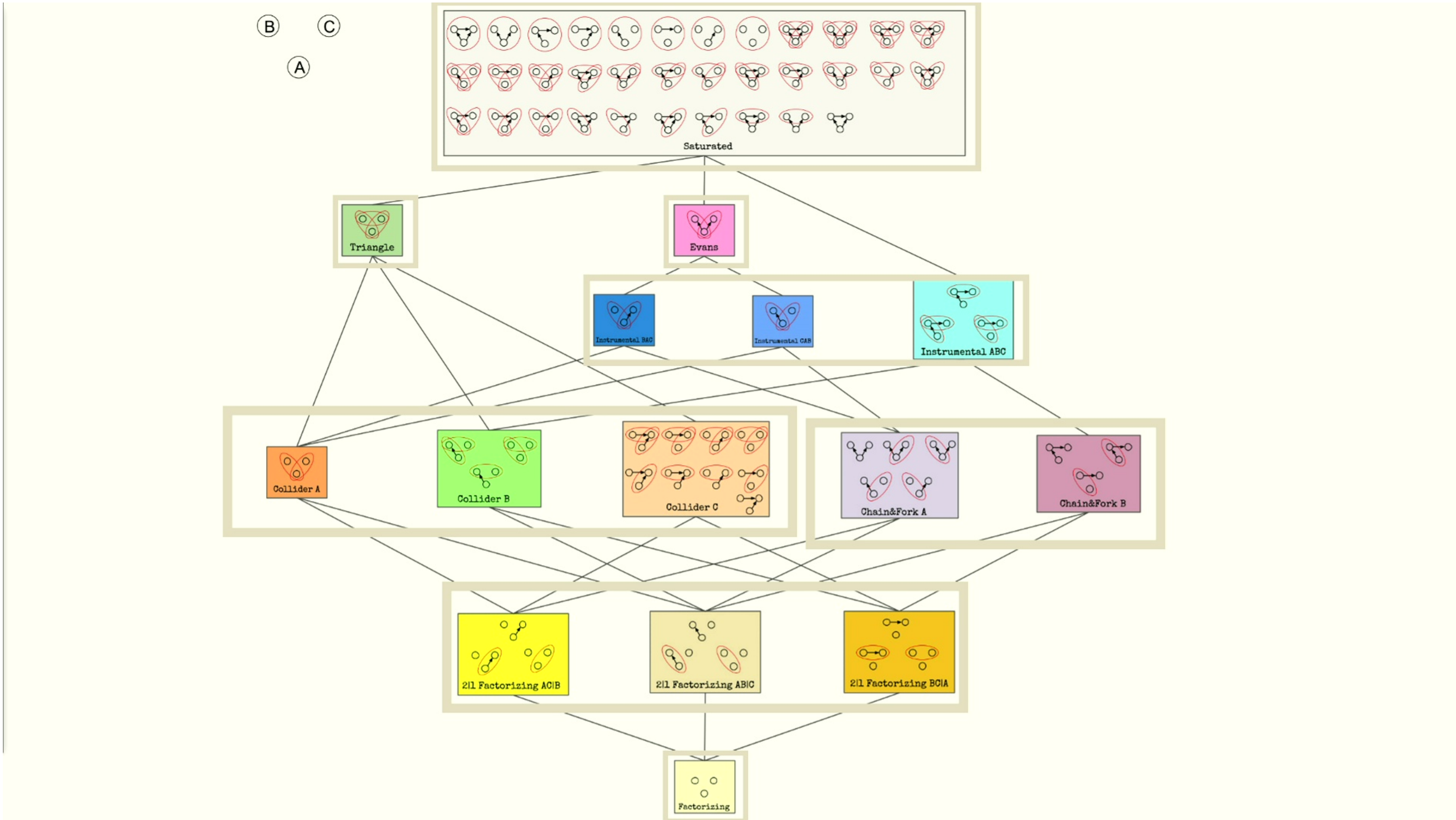

c
Saturated
Triangle
Evans
Instrumental BAC
Instrumental CAB
Instrumental ABC

Some dominance-proving rules and relations among them



Some nondominance-proving rules and relations among them

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2|1 Factorizing AC|B
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Factorizing

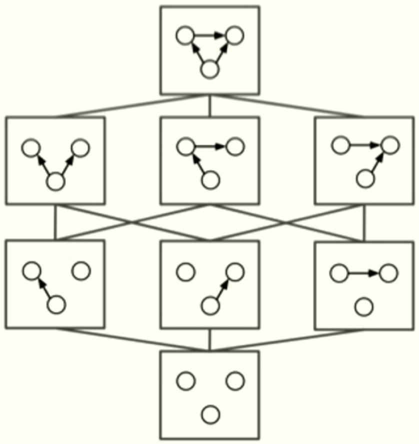
Structural dominance rule

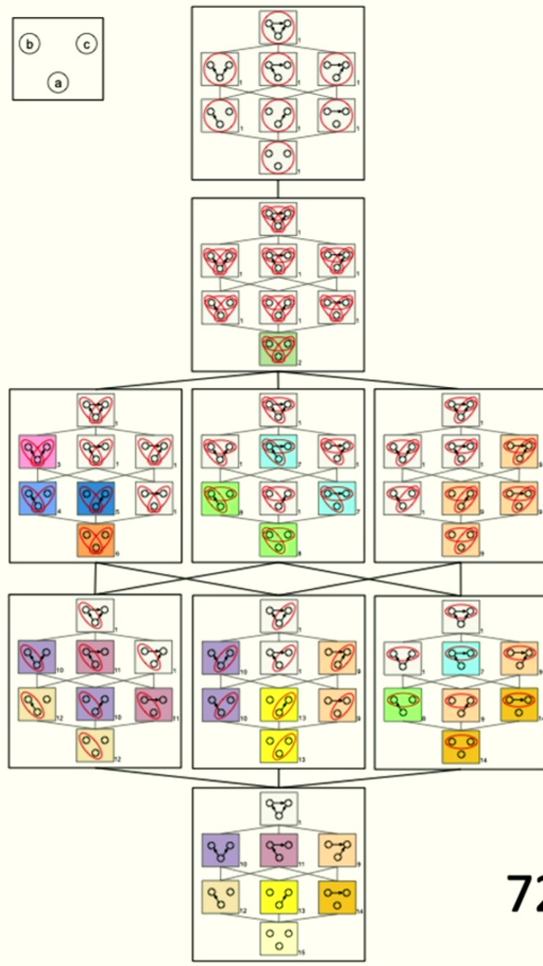
Observational dominance of mDAGs $\Longleftarrow$ Structural dominance of mDAGs

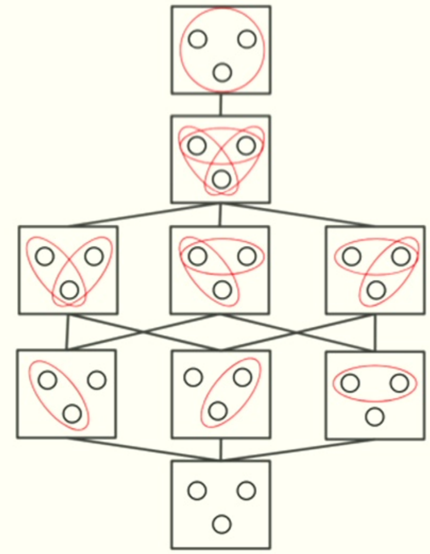
$G \succeq_{obs} G'$ $\qquad$ $G \succeq_{struct} G'$

Follows from fact that presence of a directed edge or face includes possibility of it not being used.
If one mDAGs is higher in the structural order than another, it can observationally realize all the distributions of the other

Structural dominance order of 3-node mDAGs







72 mDAGs

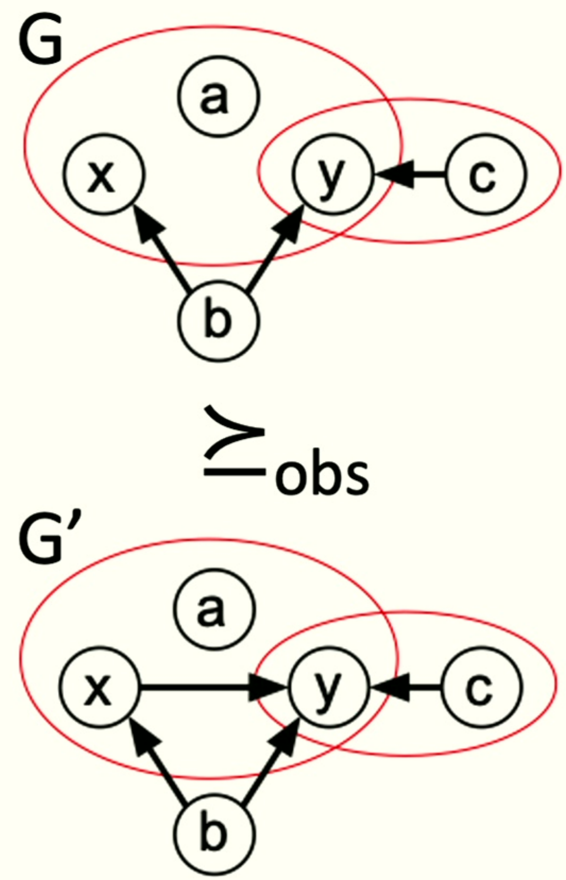
HLP Edge-adding rule

Proposition 4 (HLP Edge-Adding Rule (HLP)). *Let* $\mathfrak{G} = \{\mathcal{D}, \mathcal{B}\}$ *be an mDAG, and let* x *and* y *be two of its nodes. Let* $\mathfrak{G}'$ *be the mDAG obtained from* $\mathfrak{G}$ *by adding a directed edge* $x \to y$.

Suppose that:

1. $\mathtt{pa}_{\mathcal{D}}(x) \subseteq \mathtt{pa}_{\mathcal{D}}(y)$,
2. *Whenever* $x \in B$ *for a facet* $B \in \mathcal{B}$, *then also* $y \in B$.

In this case, $\mathfrak{G}$ *observationally dominates* $\mathfrak{G}'$, *i.e.,* $\mathfrak{G} \succeq \mathfrak{G}'$.



Henson,.Lal and Pusey, New Journal of Physics 16, 113043 (2014)

HLP Edge-adding rule

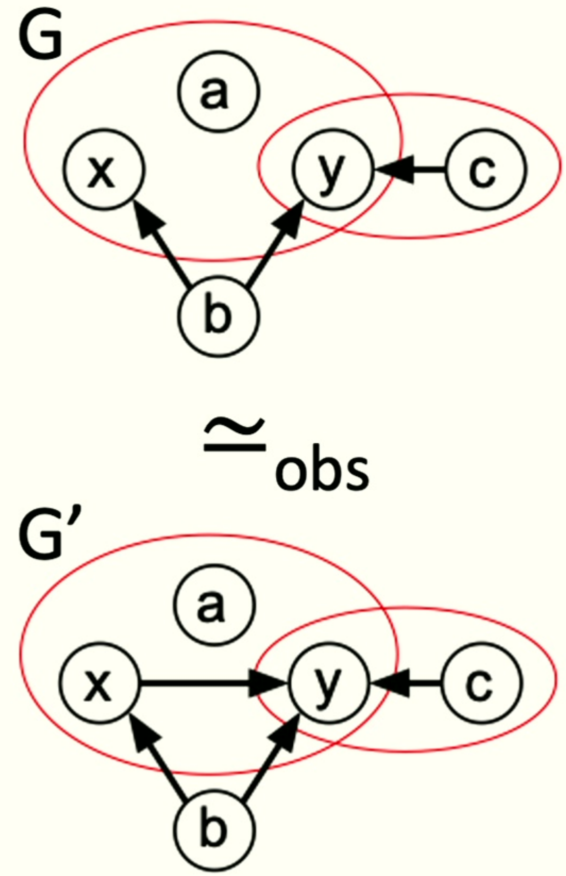
Proposition 4 (HLP Edge-Adding Rule (HLP)). *Let $\mathfrak{G} = \{\mathcal{D}, \mathcal{B}\}$ be an mDAG, and let x and y be two of its nodes. Let $\mathfrak{G}'$ be the mDAG obtained from $\mathfrak{G}$ by adding a directed edge $x \to y$.*

Suppose that:

1. $\mathtt{pa}_{\mathcal{D}}(x) \subseteq \mathtt{pa}_{\mathcal{D}}(y)$,
2. *Whenever $x \in B$ for a facet $B \in \mathcal{B}$, then also $y \in B$.*

In this case, $\mathfrak{G}$ observationally dominates $\mathfrak{G}'$, i.e., $\mathfrak{G} \succeq \mathfrak{G}'$.

Note that, since $\mathfrak{G}'$ structurally dominates $\mathfrak{G}$, by Lemma 3 we know that $\mathfrak{G}'$ observationally dominates $\mathfrak{G}$, i.e., $\mathfrak{G}' \succeq \mathfrak{G}$. Therefore, $\mathfrak{G}$ and $\mathfrak{G}$ are observationally equivalent, i.e., $\mathfrak{G} \cong \mathfrak{G}'$.



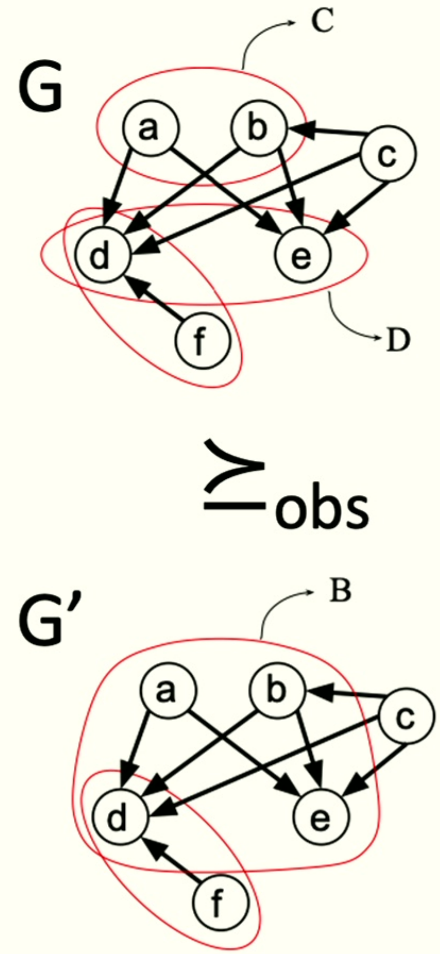
Weak facet-merging rule

Proposition 5 (Weak Facet-Merging (Weak FM)). *Let* $\mathfrak{G} = \{\mathcal{D}, \mathcal{B}\}$ *be an mDAG whose simplicial complex* $\mathcal{B}$ *contains two disjoint facets* C *and* D. *Let* $\mathfrak{G}'$ *be the mDAG obtained by starting from* $\mathfrak{G}$ *and adding a facet* $B = C \cup D$ *and all of the faces contained in* B *to its simplicial complex.*

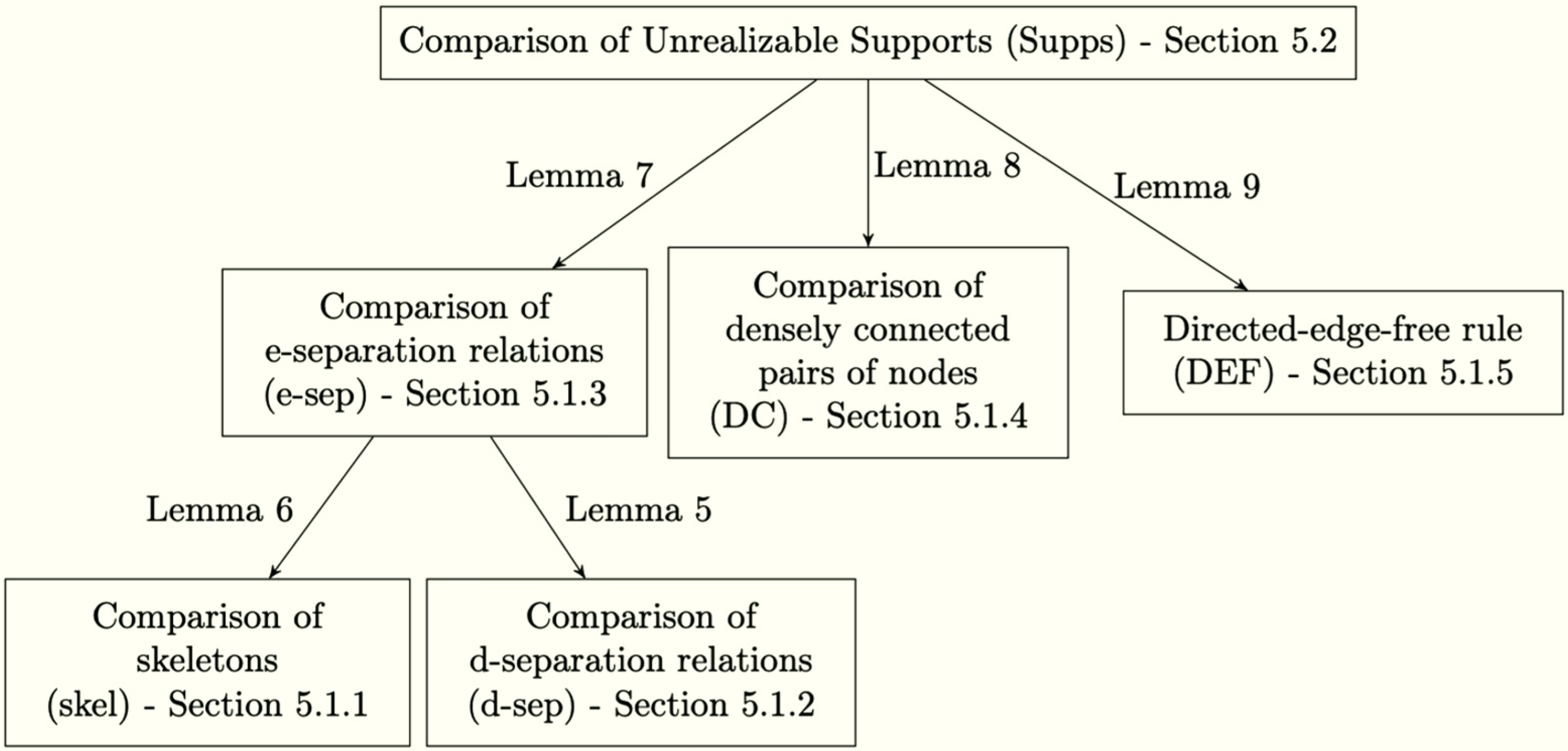
Suppose that:

1. $\mathtt{pa}_{\mathcal{D}}(C) \cup C \subseteq \mathtt{pa}_{\mathcal{D}}(d)$ *for each* $d \in D$,
2. *For every* $c \in C$, C *is the only facet that contains* c.

In this case, $\mathfrak{G}$ *observationally dominates* $\mathfrak{G}'$, *i.e.,* $\mathfrak{G} \succeq \mathfrak{G}'$.



Relations among nondominance-proving rules



Comparison of d-separation relations rule

Proposition 10 (Comparison of d-separation relations). *Let* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *be two mDAGs such that* $\mathtt{nodes}(\mathfrak{G}) = \mathtt{nodes}(\mathfrak{G}')$. *If there is a d-separation relation that is presented by* $\mathfrak{G}$ *but not by* $\mathfrak{G}'$*, then* $\mathfrak{G}$ does not *observationally dominate* $\mathfrak{G}'$*, i.e.,* $\mathfrak{G} \not\succeq \mathfrak{G}'$.

Extension of d-separation theorem to latent-permitting causal models:

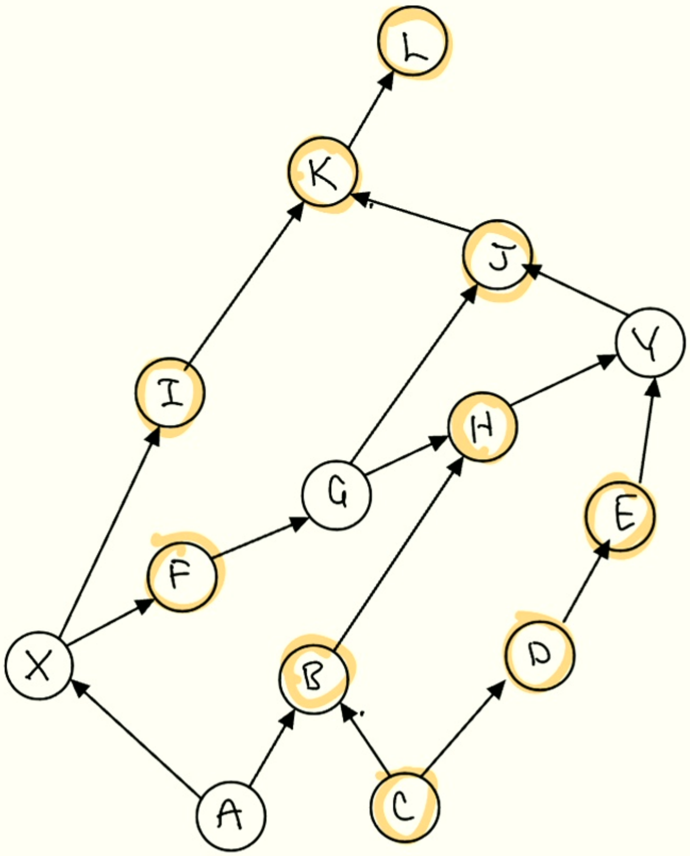
Consider a latent-permitting causal structure G and three disjoint subsets of observed variables **X**, **Y** and **Z**.

Soundness

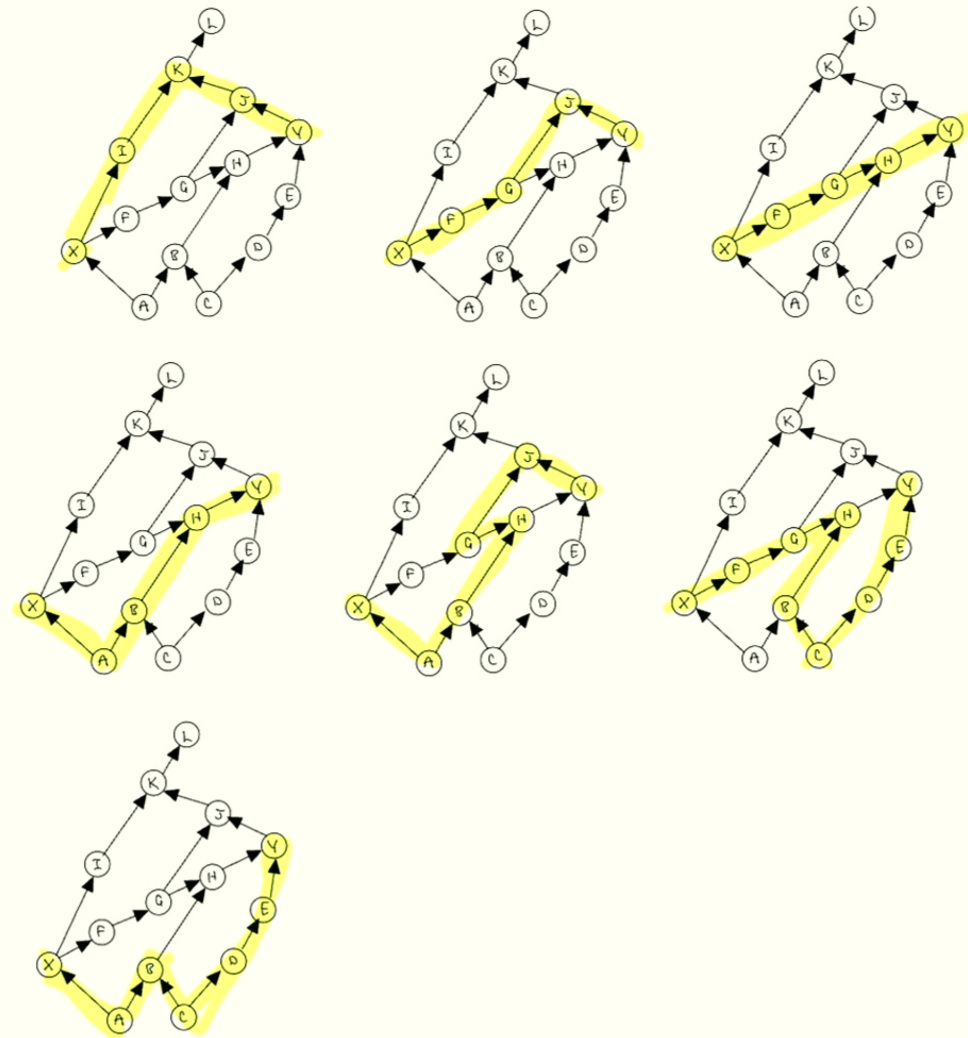
$$\mathbf{X} \perp_d \mathbf{Y}|\mathbf{Z} \text{ in } G \quad \Longrightarrow \quad \forall P \in \text{Comp}_G : \mathbf{X} \perp \mathbf{Y}|\mathbf{Z} \text{ in } P$$

Completeness

$$\forall P \in \text{Comp}_G : \mathbf{X} \perp \mathbf{Y}|\mathbf{Z} \text{ in } P \quad \Longrightarrow \quad \mathbf{X} \perp_d \mathbf{Y}|\mathbf{Z} \text{ in } G$$



$$X \perp Y | GA$$

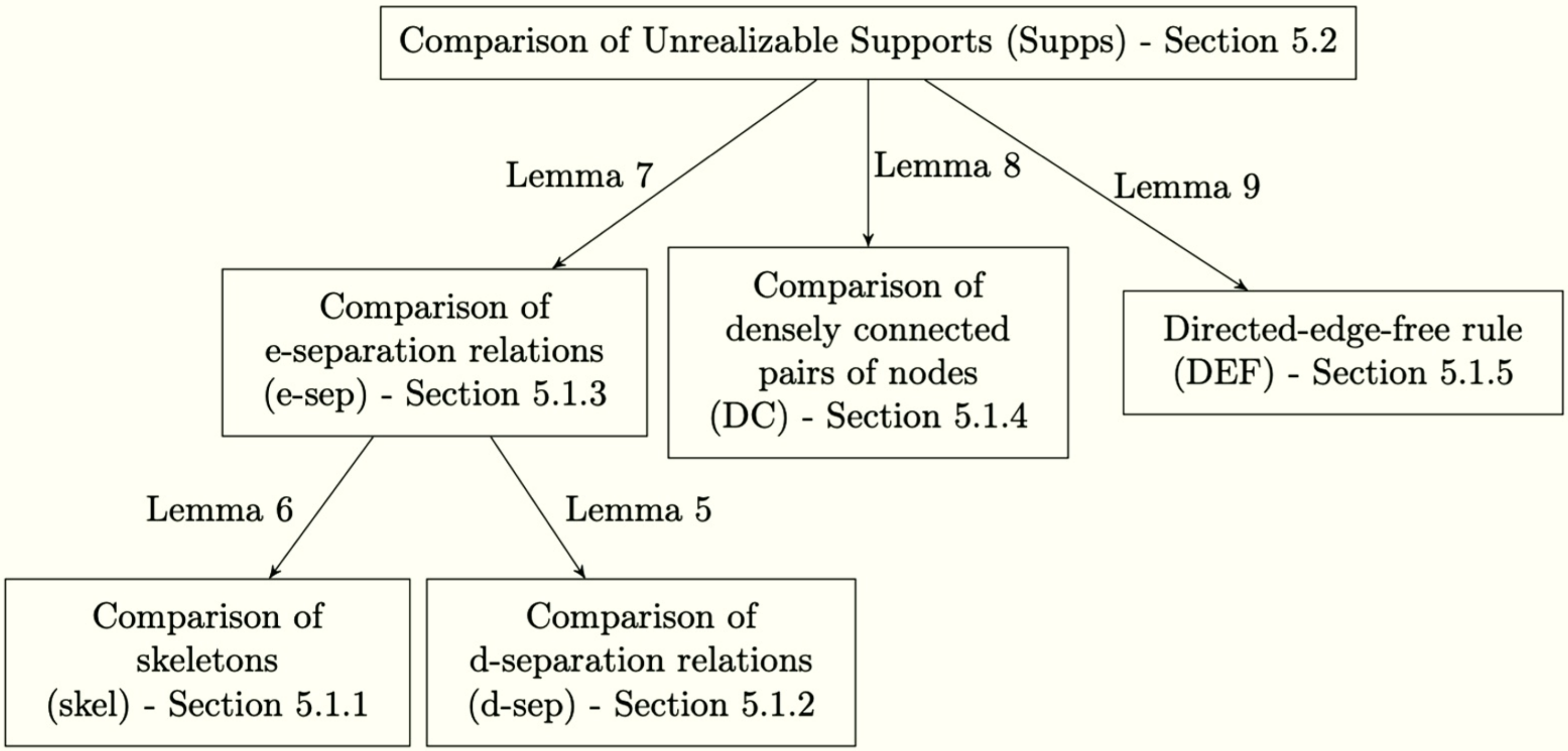


IC* algorithm and PC algorithm

Set of conditional independence relations on observed variables ⇨ Find latent-permitting DAGs that have the right d-separation relations, but these DAGs might still fail to be compatible with the full distribution

Example: CI relations of quantum-realizable Bell correlations yield classical Bell model

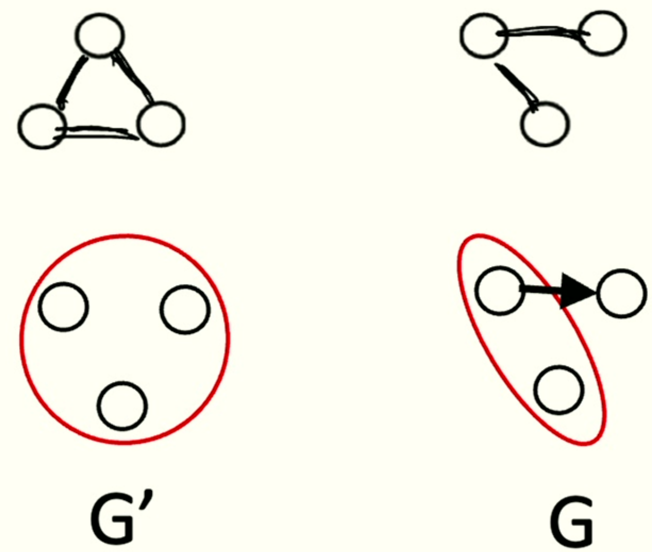
Relations among nondominance-proving rules



Comparison of skeletons rule

Definition 13 (Skeleton)**.** *Let* $\mathfrak{G} = (\mathcal{D}, \mathcal{B})$ *be an mDAG. We define the* skeleton *of* $\mathfrak{G}$ *by the undirected graph with the same nodes as* $\mathcal{D}$ *and with an edge between nodes* u *and* w *whenever there is a directed edge between them in* $\mathcal{D}$ *or when* $u, w \in B$ *for some* $B \in \mathcal{B}$*.*

Proposition 9. *(Comparison of skeletons) Let* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *be two mDAGs such that* $\texttt{nodes}(\mathfrak{G}) = \texttt{nodes}(\mathfrak{G}')$. *If there exist nodes* $x, y \in \texttt{nodes}(\mathfrak{G})$ *such that the undirected edge between* x *and* y *is present in the skeleton of* $\mathfrak{G}'$ *but not in the skeleton of* $\mathfrak{G}$*, then* $\mathfrak{G}$ does not *observationally dominate* $\mathfrak{G}'$*, i.e.,* $\mathfrak{G} \not\succeq \mathfrak{G}'$*.*



R.J. Evans: Graphs for Margins of Bayesian Networks (2016)

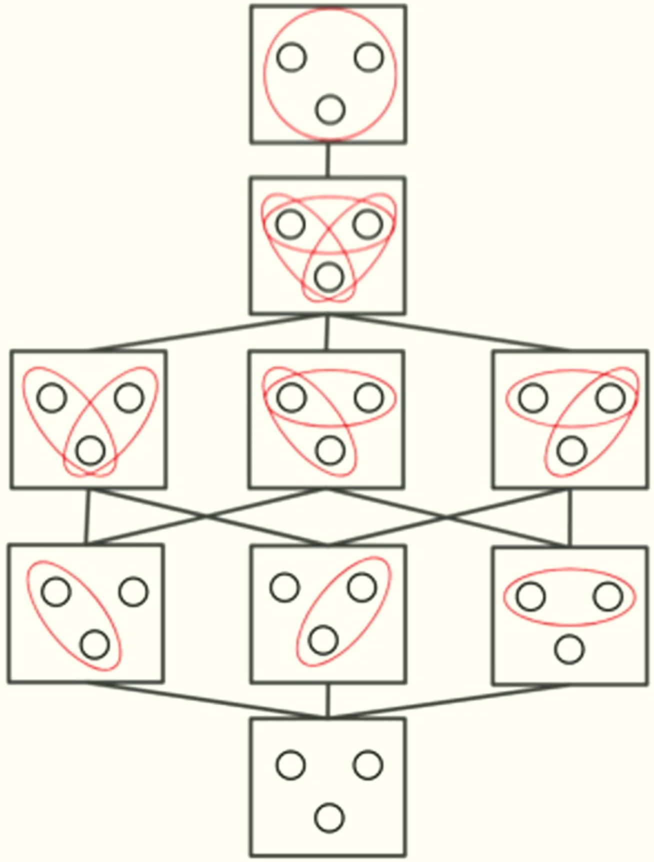
Recall: for directed-edge-free mDAGs

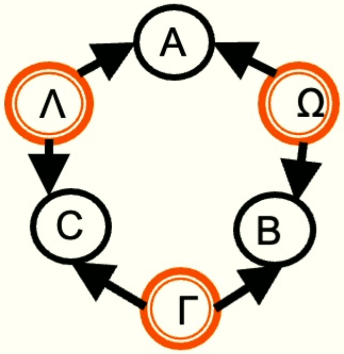
Observational dominance of mDAGs ⇐ Structural dominance of mDAGs

$G \succeq_{obs} G'$ ⇐ $G \succeq_{struct} G'$

Observational nondominance of mDAGs ⇐ Structural nondominance of mDAGs

$G \not\succeq_{obs} G'$ ⇐ $G \not\succeq_{struct} G'$





Compatible set does not include:

$$P^{\text{corr}}_{ABC} = w[000] + (1-w)[111] \qquad \text{where } w > 0$$

Proof: The marginals of the target state are:

$$P^{\text{corr}}_{AB} = P^{\text{corr}}_{AC} = P^{\text{corr}}_{BC} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$

$$P_{ABC} = \sum_{\Lambda\Omega\Gamma} \mathrm{P}_{A|\Lambda\Omega}\mathrm{P}_{B|\Omega\Gamma}\mathrm{P}_{C|\Gamma\Lambda}\mathrm{P}_{\Lambda}\mathrm{P}_{\Omega}\mathrm{P}_{\Gamma}$$

$$P_{AB} = \sum_{\Lambda\Omega\Gamma} \mathrm{P}_{A|\Lambda\Omega}\mathrm{P}_{B|\Omega\Gamma}\mathrm{P}_{\Lambda}\mathrm{P}_{\Omega}\mathrm{P}_{\Gamma}$$

Require:

$$\mathrm{P}_{A|\Lambda\Omega} = \delta_{A,\Omega}$$

$$\mathrm{P}_{B|\Gamma\Omega} = \delta_{B,\Omega}$$

$$P_{\Omega} = w[0] + (1-w)[1]$$

So that

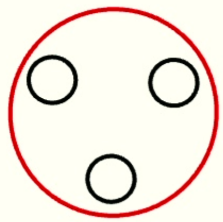
$$P_{AB} = \left(\sum_{\Omega} \delta_{A,\Omega}\delta_{B,\Omega}\mathrm{P}_{\Omega}\right)\sum_{\Lambda}\mathrm{P}_{\Lambda}\sum_{\Gamma}\mathrm{P}_{\Gamma}$$

$$= w[00] + (1-w)[11]$$

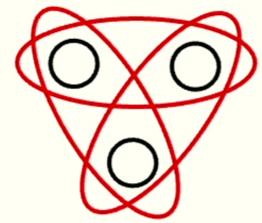
But then:

$$P_{AC} = \sum_{\Lambda\Omega\Gamma} \mathrm{P}_{A|\Lambda\Omega}\mathrm{P}_{C|\Lambda\Gamma}\mathrm{P}_{\Lambda}\mathrm{P}_{\Omega}\mathrm{P}_{\Gamma}$$

$$= \left(\sum_{\Omega}\delta_{A,\Omega}\mathrm{P}_{\Omega}\right)\left(\sum_{\Lambda\Gamma}\mathrm{P}_{C|\Lambda\Gamma}\mathrm{P}_{\Lambda}\mathrm{P}_{\Gamma}\right)$$



$\not\preceq_{\text{obs}}$



Unrealizable supports:

none

Unrealizable supports:

$\{\{0,0,0\},\{1,1,1\}\}$

$\{\{1,0,0\},\{0,1,0\},\{0,0,1\}\}$

...

2-node mDAGs