Title: Spacetime Entropy of Free Fields, and its Relevance to Black Holes

Speakers: Joshua Jones

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Abstract:

Motivated by the frame independence of the Bekenstein-Hawking black hole entropy, and the possibility of it being fundamentally entanglement entropy, I will present a spacetime definition for the entanglement entropy of free quantum fields. This definition is a spectral formulation, conducive to frame independent regularization. I will then go on to show an example calculation in Rindler spacetime, and comment on more general applications.

Spacetime Entropy of Free Fields
and its relevance to black holes

Joshua Y. L. Jones

Dublin Institute for Advanced Studies

Perimeter Institute (Feb 2025)

Overview

1. Entropy and Black Holes
2. Spacetime Entropy
3. An Example: Rindler

Entropy

Entropy has its origins in classical thermodynamics, quantifying reversibility

Given a macrostate, all accessible microstates are equally likely (Boltzmann 1877)

$$\Omega := N_{\text{microstates}} \quad (1)$$

$$S := \ln(\Omega) \quad (2)$$

Ω, and hence S is non-decreasing upon system composition and equilibration (starting states are a subset of possible states)

Entropy goes up

Black Holes

Black holes have a horizon (that is all we will need)

They are characterised by a few conserved charges

No Hair "Theorem" conjectured by Bekenstein (Israel '67,'68, Chase '70, Carter '71, Bekenstein '72 . . .)

Wheeler : "What happens if I throw a cup of tea into the black hole?"

Bekenstein : Black holes must have (a kind of generalised) entropy

Black Hole Entropy

By considering various physical situations (mergers and aquisitions), and the Area Theorem (Hawking '71, Chruściel, Delay, Galloway, Howard 2001)

Bekenstein ('72) got to

$$S = \eta \frac{A}{l_p^2}, \qquad \eta \approx \frac{1}{2}\ln(2) \tag{3}$$

$$= \eta \times 16\pi M^2 \tag{4}$$

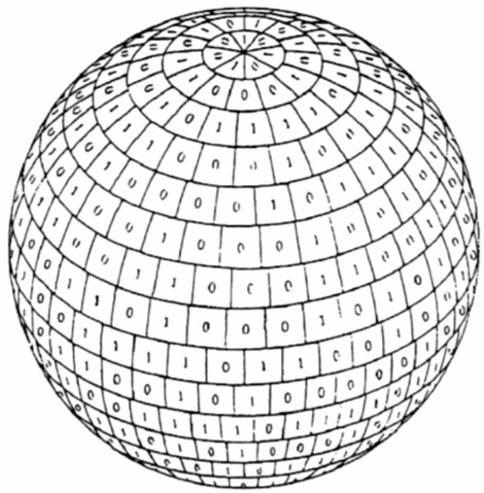
Hawking set out to disprove this result, via a more careful consideration of quantum fields around a black hole and found that they are thermally excited ('74)

$$T = \frac{\kappa}{2\pi}, \qquad \kappa = \frac{1}{4M} \tag{5}$$

$$dE = TdS, \; E = M, \; T = \frac{1}{8\pi M} \implies S = 4\pi M^2 = \frac{1}{4}A \tag{6}$$

Black Hole Entropy

Many took this to mean that the black hole horizon had some discrete degrees of freedom, each occupying a Planck area, upon the horizon



A Journey into Gravity and Spacetime - Wheeler '90

Horizon Entropy

In fact all horizons are thought to be thermodynamic and have an entropy that follows the same area law (Gibbons and Hawking '77, Laflamme '87, Massar and Parentani '99, Jacobson and Parentani 2003)

$$S = \frac{A}{4} \tag{7}$$

We must thus seek a universal culprit

Entropy beyond Thermodynamics

Let's now allow states of differing probabilities (Gibbs '02, Shannon '49)

We say entropy is the expected information/surprisal a system holds

$S = -\sum_j p_j \ln p_j$

In quantum systems, we have a Hilbert space. Pure states are rays ($S = 0$)

The density operator specifies a quantum state $\rho = \sum_j p_j |\text{pure}_j\rangle\langle\text{pure}_j|$

ρ s.t. $\text{Tr}(\rho A) = \sum_j p_j \langle\text{pure}_j|A|\text{pure}_j\rangle = \langle A\rangle$

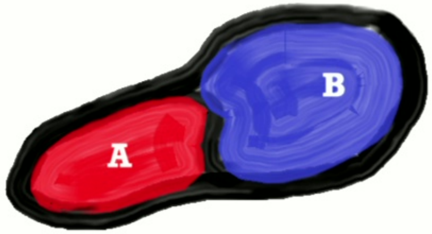
The density operator can be diagonalised, and the spectrum is interpreted as probabilities.

We can thus generalise the notion of entropy (von Neumann '32)
$\implies S = -\text{Tr}(\rho \ln \rho)$

Entanglement Entropy (Sorkin '83, '86)

Quantum states can be entangled

Start with a pure state ρ, with $S = 0$ in a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$



Remove (trace out) some physical degrees of freedom (states in the Hilbert space $\mathcal{H}_B$)

$$\rho_A = \mathrm{Tr}_B(\rho) \tag{8}$$

An entangled state, when the von Neumann entropy of ρ_A is calculated, has

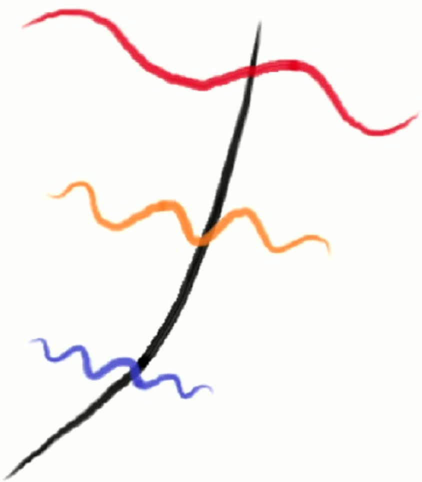
$$S_A = \mathrm{Tr}(\rho_A \ln \rho_A) \neq 0 \tag{9}$$

Entanglement Entropy in QFT

In QFT, one calculates entanglement entropy the same way, using the Hilbert space defined on a hypersurface

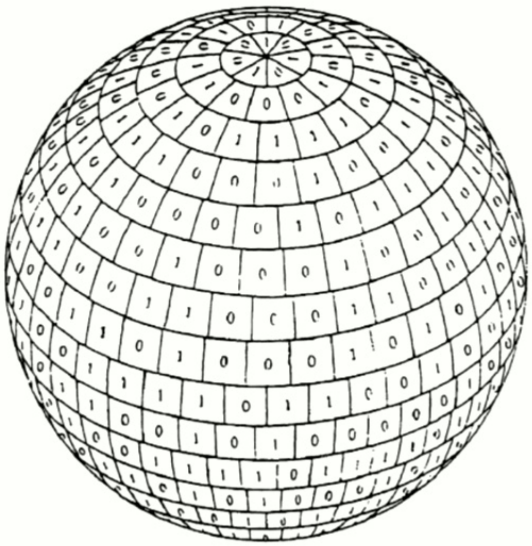
Entanglement entropy (in the vacuum state) follows an area law
$S \propto A$
One can think of entanglement happening modewise

The entanglement entropy must then count the number of modes split by the horizon



Entanglement Entropy in QFT

This is very reminiscent of Wheelers picture!



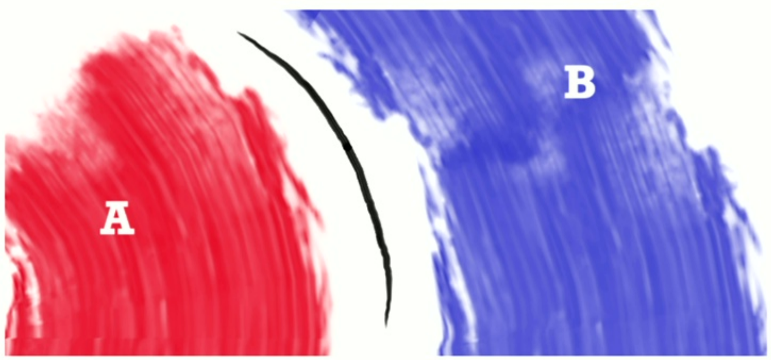
One problem: Entanglement entropy is divergent

Regularisation

Entanglement entropy has an ultraviolet divergence, modes of infinitely high frequency contribute

We must posit some new physics (gravity) to cure this divergence

This is the introduction of a cutoff (conventionally on a hypersurface)
A classic one is 't Hooft's brick wall ('84)



One problem: Spatial cutoffs are not covariant

Regularisation

The Bekenstein-Hawking entropy, on account of being proportional to the Schwarzschild area, is covariant

The size of spatial cutoffs are not

This means that cut off entanglement entropies are frame dependent, not what we want

We must formulate entropy (& hence entanglement entropy) in a hypersurface independent way (Sorkin 2012)

Background to the Calculation

We work in the regime of quantum field theory in curved spacetime (or something like that)

We have a Hilbert space for a quantum field on some background

We work with the Wightman function, and specify a free scalar field theory

We will work with countable spacetime indices for simplicity, one can think of a lattice field theory before the continuum limit is taken

Def: Wightman '56

$W^{i,j} := \langle\Omega|\, \phi^i \phi^j \,|\Omega\rangle$

This is vacuum 2-point correlation function. This is a matrix in spacetime indices.

Decoupling the Wightman Function

The Wightman matrix can be decomposed into two matrices, the real and imaginary part (also symmetric and antisymmetric).

$$W = \langle\Omega|\,\phi^i\phi^j\,|\Omega\rangle = \frac{1}{2}\left(\langle\Omega|\,\{\phi^i,\phi^j\}\,|\Omega\rangle + \langle\Omega|\,[\phi^i,\phi^j]\,|\Omega\rangle\right) =: \frac{1}{2}(H + i\Delta)$$

We can now go to the "quadrature basis", forming something like the conventional field operator, and its derivative q, p, due to having the unequal time two point function.

We only have one nontrivial commutation relation, $[p^i, q^j] = i\delta^{ij}$. Writing the quadrature basis vector as $(p^1, \ldots, p^n, q^1, \ldots, q^n)$, we have

$$\Delta = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}.$$

We see Δ is mode decoupled.

Decoupling the Wightman Function

Thm (Williamson '36):

For all symmetric positive definite real matrices, M, there exists some $S \in \mathrm{Sp}(2n)$, i.e. $S^T \Delta S = \Delta$ such that

$$S^T M S = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$$

Here Λ is a positive definite diagonal matrix, with entries λ, where $\pm i\lambda$ are the eigenvalues of $\Delta^{-1}M$. These λs are called symplectic eigenvalues.

By way of this theorem, we can simultaneously decouple both H and Δ, and so we have successfully decoupled W.

Single Mode Entropy

We have W for a single mode

$$W_m = \frac{1}{2}\left(H_m + i\Delta_m\right) = \frac{1}{2}\begin{pmatrix} \lambda & i \\ -i & \lambda \end{pmatrix}.$$

Using the fact that our theory is free, and calculating a few expectations, we find the position representation for a density matrix for a single mode,

$$\rho(q,q') = \sqrt{\frac{\lambda^{-1}}{\pi}}\exp\left(-\frac{\lambda^{-1}}{2}(q^2+q'^2) - \frac{\lambda-\lambda^{-1}}{4}(q-q')^2\right)$$

This is of the same form as that of a pair of coupled harmonic oscillators, with one traced out.

Single Mode Entropy

Thm (Bombelli, Koul, Lee, Sorkin '86):

$$\rho(q,q') = \sqrt{\tfrac{\lambda-1}{\pi}}\exp\left(-\tfrac{\lambda-1}{2}(q^2+q'^2) - \tfrac{\lambda-\lambda^{-1}}{4}(q-q')^2\right)$$

$$\implies \quad S = \tfrac{\lambda+1}{2}\ln\tfrac{\lambda+1}{2} - \tfrac{\lambda-1}{2}\ln\tfrac{\lambda-1}{2}$$

$\because W_m = \frac{1}{2}(H_m + i\Delta_m) = \frac{1}{2}\begin{pmatrix}\lambda & i\\ -i & \lambda\end{pmatrix}$, we see that

$$R := \mathrm{spec}(-i\Delta_m^{-1}W_m) = \left\{\frac{1+\lambda}{2}, \frac{1-\lambda}{2}\right\},$$

$$\therefore S = \sum_{r\in R} r\ln|r|, \quad \because \lambda \geq 1 \because W \geq 0.$$

The Full Entropy

The full entropy must just be the sum over the entropy of the modes.

We have

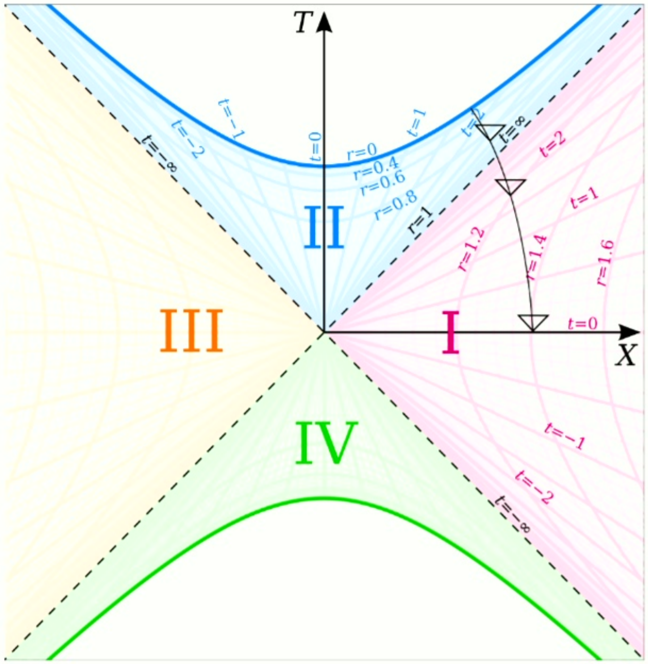
$$L = -i\Delta^{-1}W,$$

$$S = \sum_{l \in spec(L)} l \ln |l|.$$

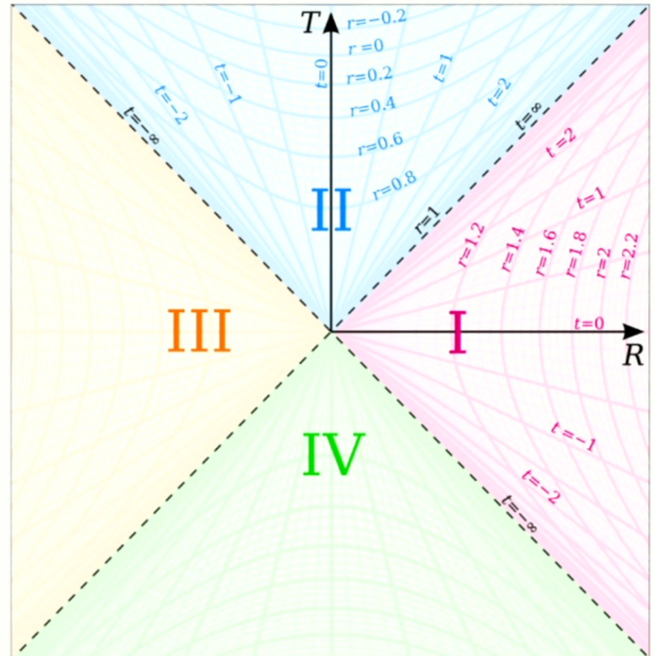
or

$$S = \sum_l l \ln |l|, \qquad Wv = il\Delta v.$$

Rindler and Schwarzschild



Schwarzschild



Rindler

The Entropy of a Rindler Wedge

For simplicity, we will treat the (1+1) dimensional case, and use the Klein-Gordon inner product, regularising via a brick wall spatially

Although entanglement entropy scales with an area law, in 2 dimensions, this scaling becomes a logarithm

With an ultraviolet and infrared cutoff, we expect (Calabrese and Cardy 2004):

$$S = \frac{1}{6} \ln \left(\frac{L_{\mathrm{IR}}}{L_{\mathrm{UV}}} \right) \tag{10}$$

Minkowski Mode expansion

We begin by writing the equation of motion in light cone coordinates, $u = -t + x$, $v = -t - x$, so it has the form

$$\partial_u \partial_v \phi = 0. \tag{11}$$

This has solutions of the form

$$\phi(u, v) = f(u) + g(v), \tag{12}$$

We have the complete orthonormal basis

$$f_k^{(rm)}(x) \propto (-t + x)^{\frac{ik}{a}}, \tag{13}$$

$$f_k^{(lm)}(x) \propto (-t - x)^{\frac{ik}{a}}. \tag{14}$$

Transforming to Rindler

We make the transformation

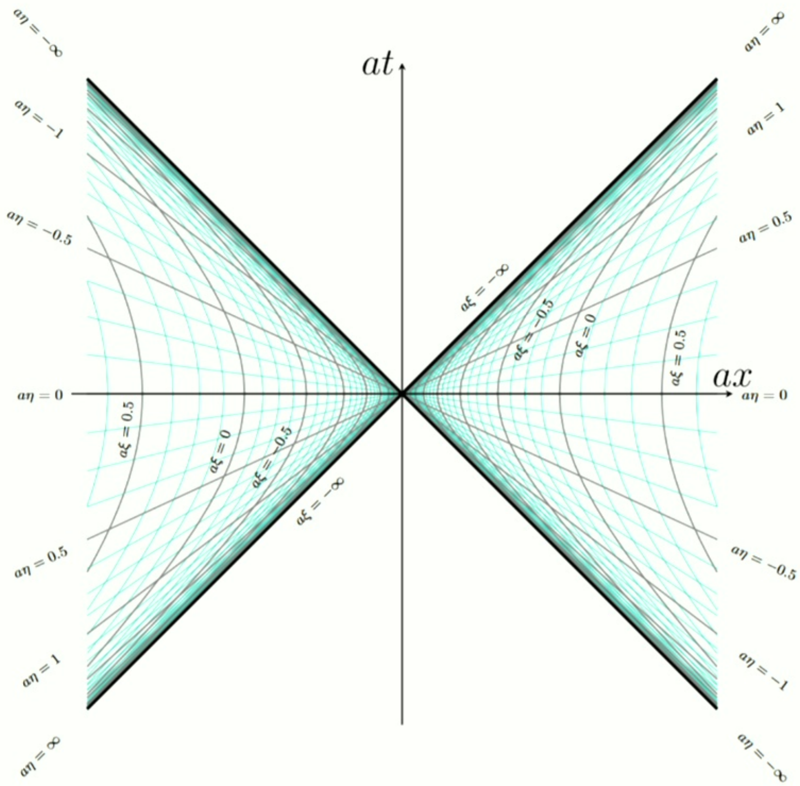
$$t = \pm \frac{e^{a\xi}}{a} \sinh(a\eta), \qquad (15)$$

$$x = \pm \frac{e^{a\xi}}{a} \cosh(a\eta), \qquad (16)$$

$$\eta, \xi \in \mathbb{R}$$

to go to the "Lass" ('63) coordinates

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2) \quad (17)$$



Rindler Mode Expansion

As the metric is conformal to Minkowski, we have Rindler mode functions

$$g_k^{(L/R)} \propto e^{-i\omega_k \eta + ik\xi} \tag{18}$$

with some work... we find

$$f_k^{(rm)} = \begin{cases} \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega_k}{a}\right)}} \left(e^{\frac{\pi\omega_k}{2a}} g_k^{(R)} + e^{-\frac{\pi\omega_k}{2a}} g_{-k}^{*(L)} \right) & k > 0, \\ \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega_k}{a}\right)}} \left(e^{\frac{\pi\omega_k}{2a}} g_{-k}^{(L)} + e^{-\frac{\pi\omega_k}{2a}} g_{-k}^{*(R)} \right) & k < 0, \end{cases}$$

$$f_k^{(lm)} = \begin{cases} \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega_k}{a}\right)}} \left(e^{\frac{\pi\omega_k}{2a}} g_{-k}^{(L)} + e^{-\frac{\pi\omega_k}{2a}} g_{-k}^{*(R)} \right) & k > 0, \\ \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega_k}{a}\right)}} \left(e^{\frac{\pi\omega_k}{2a}} g_k^{(R)} + e^{-\frac{\pi\omega_k}{2a}} g_{-k}^{*(L)} \right) & k < 0. \end{cases}$$

Restricting to the Right Wedge

We can write the two point function and commutator in the Minkowski mode expansion, and then apply the restriction

$$\langle 0_M|\phi(x)\phi(x')|0_M\rangle = \int_{-\infty}^{\infty} dk \left(f_k^{(rm)}(x) f_k^{(rm)*}(x') + f_k^{(lm)}(x) f_k^{(lm)*}(x') \right) \tag{19}$$

$$\begin{aligned} [\phi(x), \phi(x')] = \int_{-\infty}^{\infty} dk \Big(& f_k^{(rm)}(x) f_k^{(rm)*}(x') + f_k^{(lm)}(x) f_k^{(lm)*}(x') \\ & - f_k^{(rm)*}(x) f_k^{(rm)}(x') - f_k^{(lm)*}(x) f_k^{(lm)}(x') \Big). \end{aligned} \tag{20}$$

$$\begin{aligned} f_k^{(rm)} &\to \begin{cases} \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega_k}{a}\right)}} e^{\frac{\pi\omega_k}{2a}} g_k^{(R)} & k > 0, \\ \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega_k}{a}\right)}} e^{-\frac{\pi\omega_k}{2a}} g_{-k}^{(R)*} & k < 0, \end{cases} \\ f_k^{(lm)} &\to \begin{cases} \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega_k}{a}\right)}} e^{-\frac{\pi\omega_k}{2a}} g_{-k}^{(R)*} & k > 0, \\ \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega_k}{a}\right)}} e^{\frac{\pi\omega_k}{2a}} g_k^{(R)} & k < 0, \end{cases} \end{aligned} \tag{21}$$

Restricting to the Right Wedge

And we finally arrive at

$$\langle 0_M|\phi(\eta,\xi)\phi(\eta',\xi')|0_M\rangle = \int_{-\infty}^{\infty} dk \frac{1}{2\sinh\left(\frac{\pi\omega_k}{a}\right)}\Big(e^{\frac{\pi\omega_k}{a}} g_k(\eta,\xi)g_k^*(\eta',\xi') + e^{-\frac{\pi\omega_k}{a}} g_{-k}^*(\eta,\xi)g_{-k}(\eta',\xi')\Big) \tag{22}$$

$$[\phi(\eta,\xi),\phi(\eta',\xi')] = \int_{-\infty}^{\infty} dk\Big(g_k(\eta,\xi)g_k^*(\eta',\xi') - g_{-k}^*(\eta,\xi)g_{-k}(\eta',\xi')\Big). \tag{23}$$

Schematically, we want to solve

$$\langle 0_M|\phi^i\phi_j|0_M\rangle f^j = i\lambda[\phi^i,\phi_j]f^j \tag{24}$$

Eigenvalue Equation

We find

$$\lambda_k^+ = \frac{e^{\frac{\pi\omega_k}{a}}}{2\sinh\left(\frac{\pi\omega_k}{a}\right)}, \qquad \lambda_k^- = -\frac{e^{-\frac{\pi\omega_k}{a}}}{2\sinh\left(\frac{\pi\omega_k}{a}\right)}, \tag{25}$$

hence

$$\begin{aligned} S &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left(\frac{e^{\frac{\pi\omega_k}{a}}}{2\sinh\left(\frac{\pi\omega_k}{a}\right)} \ln\left(\frac{e^{\frac{\pi\omega_k}{a}}}{2\sinh\left(\frac{\pi\omega_k}{a}\right)} \right) - \frac{e^{-\frac{\pi\omega_k}{a}}}{2\sinh\left(\frac{\pi\omega_k}{a}\right)} \ln\left(\frac{e^{-\frac{\pi\omega_k}{a}}}{2\sinh\left(\frac{\pi\omega_k}{a}\right)} \right) \right), \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left(\frac{2\pi\omega_k}{a} \frac{1}{e^{\frac{2\pi\omega}{a}} - 1} - \ln\left(1 - e^{-\frac{2\pi\omega_k}{a}}\right) \right), \\ &= \frac{a}{6} \qquad \because \text{massless} \end{aligned} \tag{26}$$

This is the local entropy density for our observer.

The Entropy

We do the integral

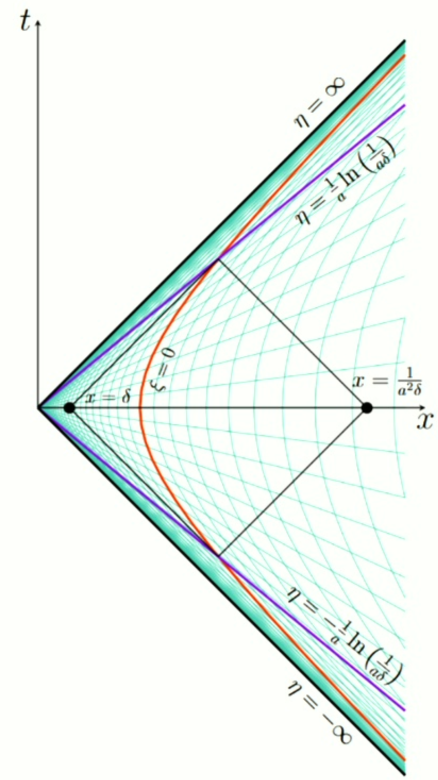
$$S = \frac{a}{6}\int_{-\frac{1}{a}\ln(\frac{1}{a\delta})}^{\frac{1}{a}\ln(\frac{1}{a\delta})} d\eta \qquad (27)$$

$$= \frac{1}{6}\ln\left(\frac{1}{a^2\delta^2}\right) \qquad (28)$$

agreeing with

$$S = \frac{1}{6}\ln\left(\frac{L_{\mathrm{IR}}}{L_{\mathrm{UV}}}\right) \qquad (29)$$

This was an entirely spatial calculation



Spacetime Regularised Rindler

Work has been done on spacetime regularisation of Rindler, although these results have required numerics

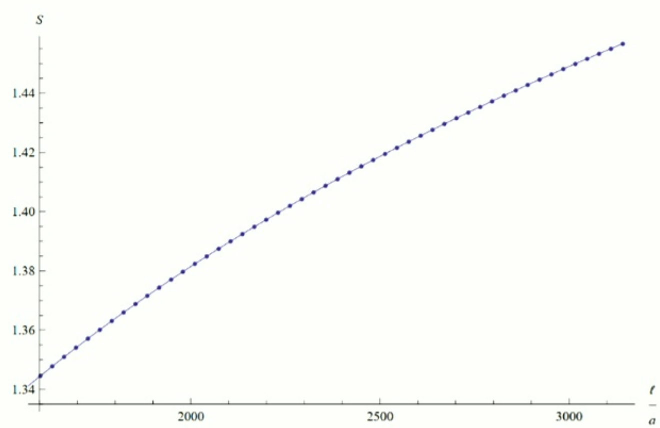


Figure 4. The entanglement entropy S versus the UV cutoff a for the case of the halfspace.

Saravani, Sorkin, Yazdi 2013

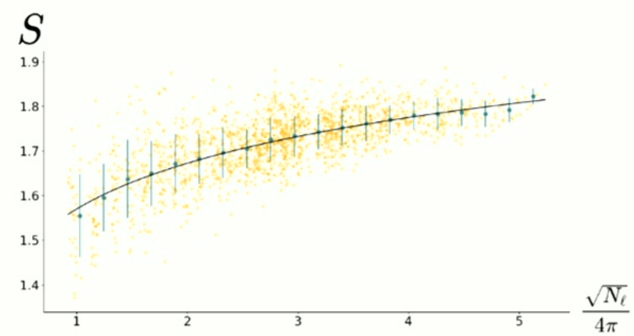


Figure 3. The scaling of entanglement entropy with respect to the UV cutoff, for the corner diamond configuration. The raw data has been plotted (orange), as well as averaged data points where binning has been performed (blue). The error bars give the standard deviation of the bins (blue). A function of the form $\alpha \ln(x) + \beta$, shown in black, was fit to the binned points with coefficient $\alpha = 0.165 \pm 0.0195$, and $\beta = 1.553 \pm 0.0278$.

Duffy, Jones, Yazdi 2022

Conclusions

Black holes have an entropy, yet the underlying microstates remain poorly understood

Entanglement entropy is a promising candidate for the source of this entropy, but is divergent, and must be regulated

Whilst the physical origins of this regulation remain unknown, the covariance of the Bekenstein-Hawking entropy suggests a covariant treatment

Such a covariant formalism for entanglement entropy can be expressed spectrally in terms of the eigenvalues of the two point function and spacetime commutator

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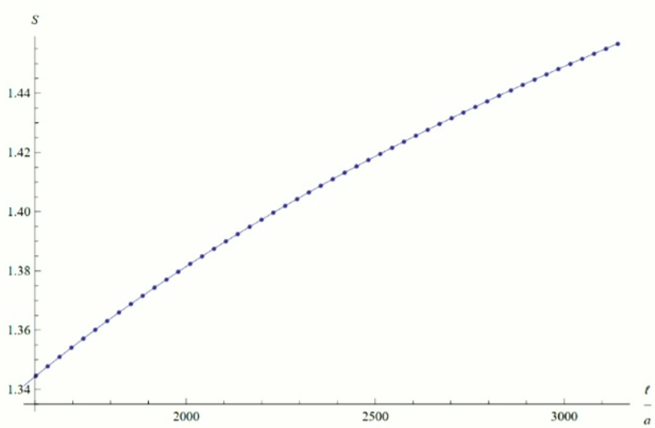

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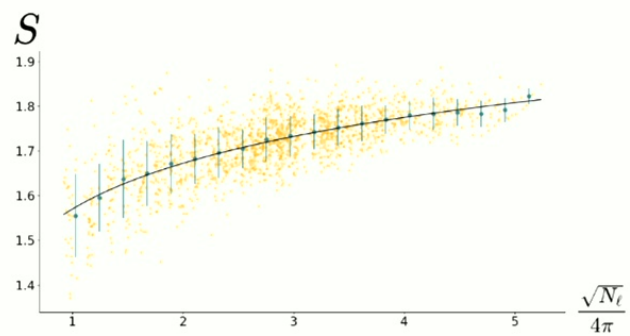

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Saravani, Sorkin, Yazdi 2013

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