Title: Graphs, curves, and their moduli spaces (Part 1 of 2)

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Collection/Series: Mathematical Physics

Subject: Mathematical physics

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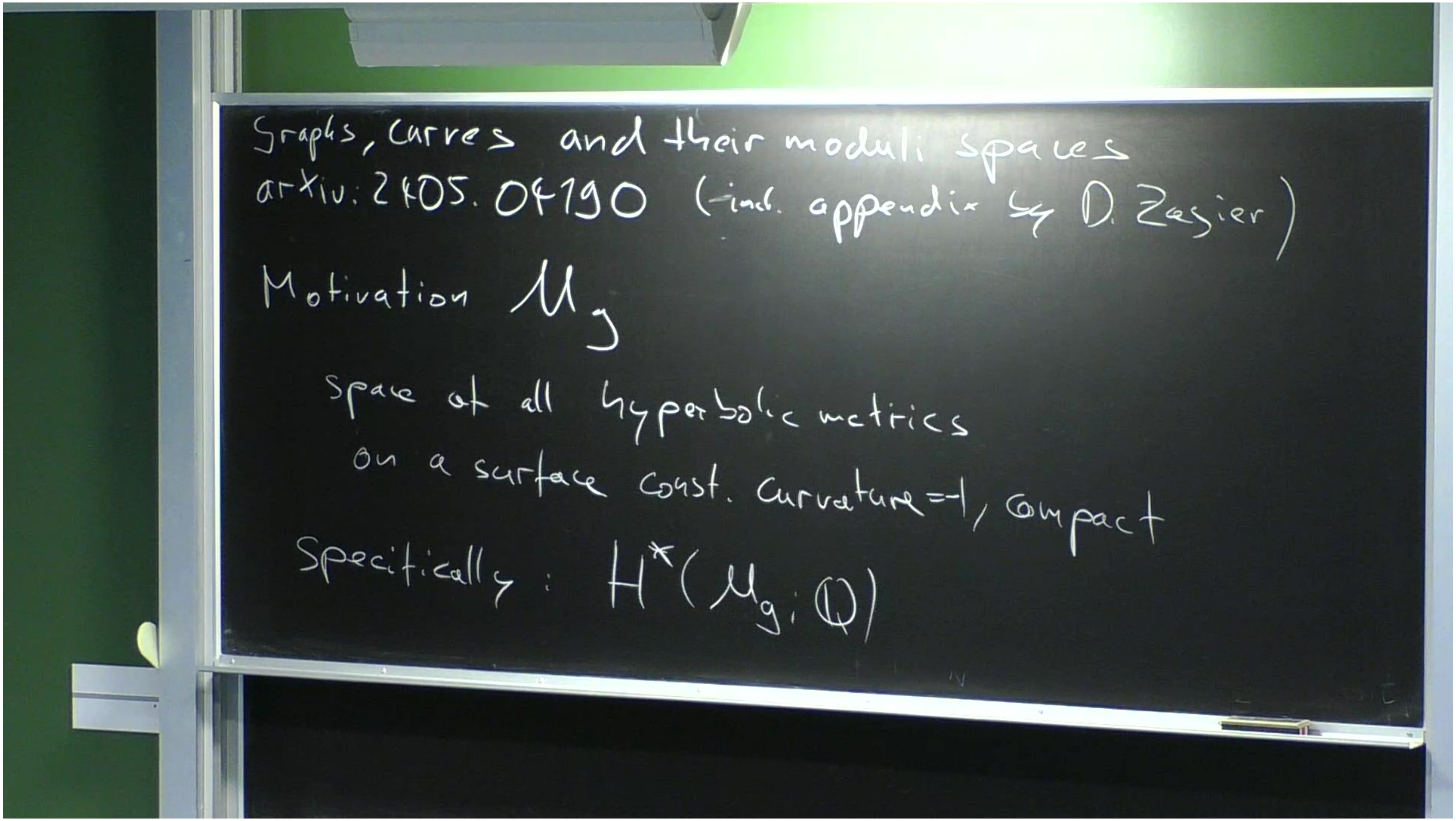
Abstract:

I will give a gentle introduction to the moduli space of graphs and its fine moduli space cousin known as Outer Space. This moduli space of graphs has many applications to various branches of mathematical physics, algebraic geometry, and geometric group theory. It is a natural object to consider while studying Feynman amplitudes in parametric space, and it can be seen as the configuration space of one-dimensional quantum gravity. I will explain how this moduli space of graphs recently became the largest provider of information on the homology of the moduli space of curves of genus g and how associated graph complexes can be used to shed light on the 'dark-matter problems' of these moduli space's cohomology.


Graphs, curves and their moduli spaces
arXiv: 2405.04190 (incl. appendix by D. Zagier)
Motivation $\mathcal{M}_g$
space of all hyperbolic metrics
on a surface const. curvature $=-1$, compact
Specifically: $H^*(\mathcal{M}_g; \mathbb{Q})$

$\dots : H^*(\mathcal{M}_g; \mathbb{Q})$

Known about $H^*(\mathcal{M}_g)$?

$*$ support $0 \le * \le 4g-5$

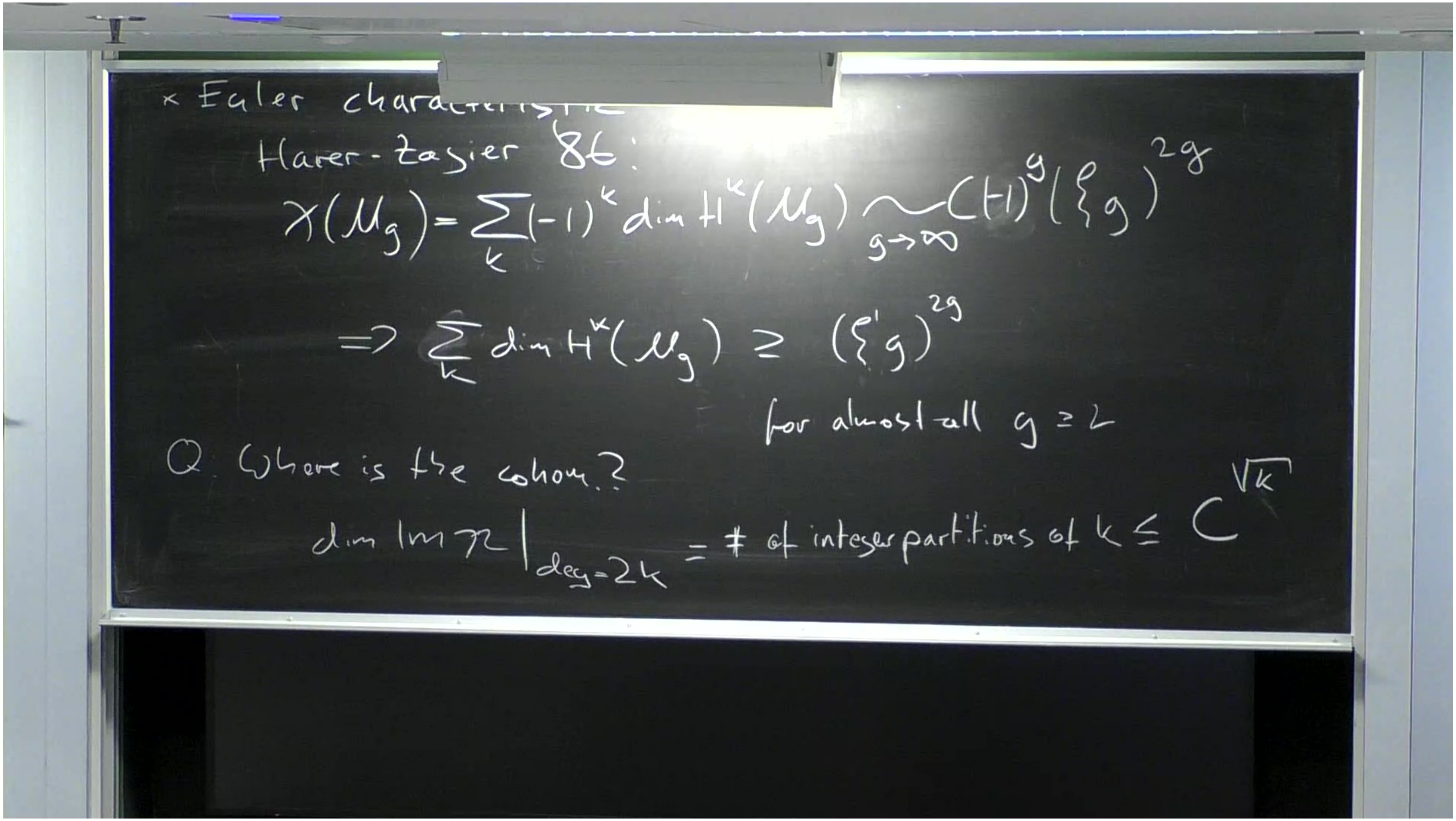
tautological classes

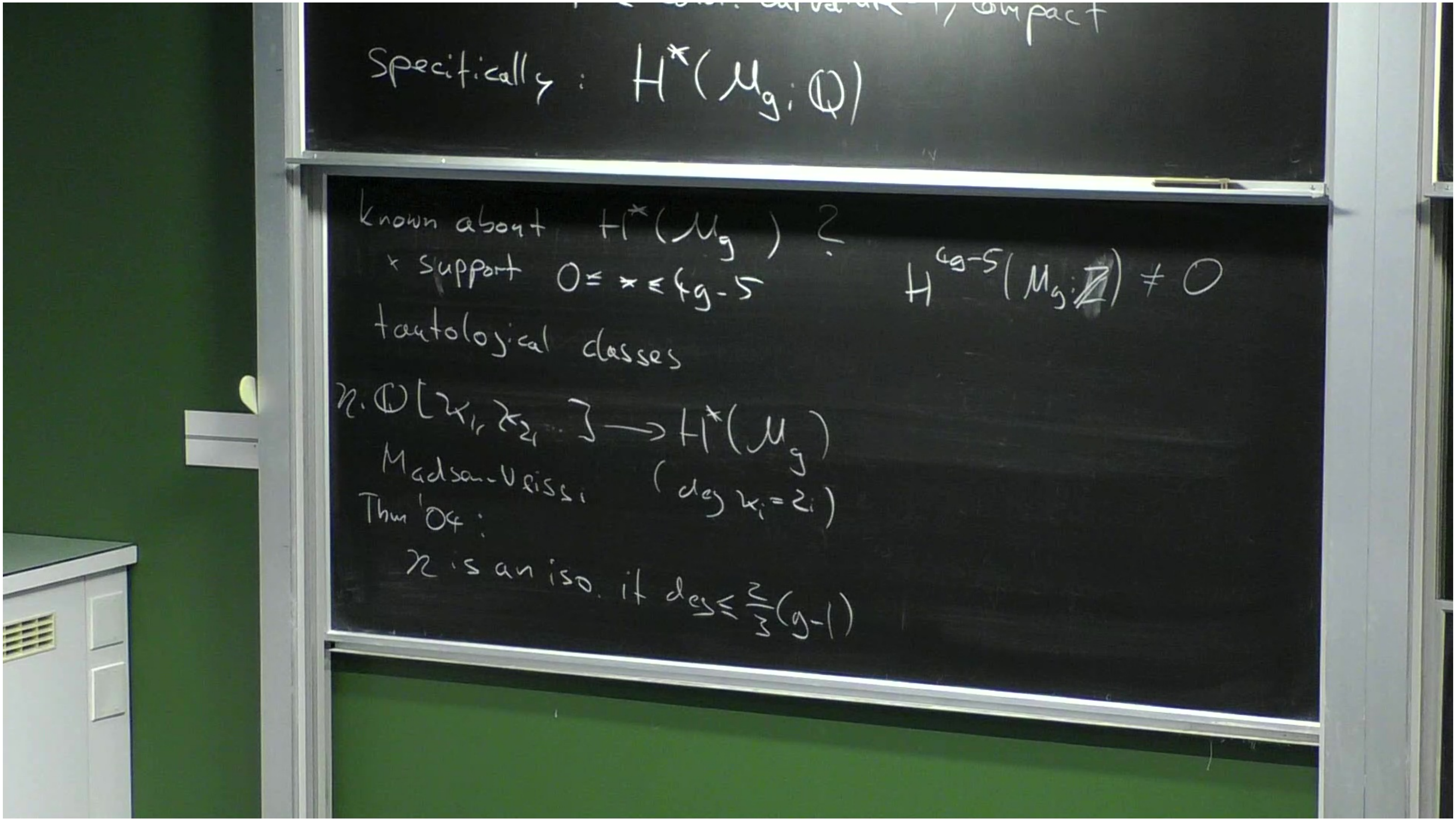
$\eta : \mathbb{Q}[\kappa_1, \kappa_2, \dots] \longrightarrow H^*(\mathcal{M}_g)$

$(\deg \kappa_i = 2i)$

Madsen–Weiss Thm '04:

η is an iso. if $\deg \le \frac{2}{3}(g-1)$


× Euler charac
Harer-Zagier 86:
$\chi(\mathcal{M}_g) = \sum_k (-1)^k \dim H^k(\mathcal{M}_g) \underset{g\to\infty}{\sim} (-1)^g (\xi_g)^{2g}$
$\Rightarrow \sum_k \dim H^k(\mathcal{M}_g) \geq (\xi'_g)^{2g}$
for almost all $g \geq 2$
Q. Where is the cohom.?
$\dim \operatorname{Im} \kappa \big|_{\deg = 2k} = \#$ of integer partitions of $k \leq C^{\sqrt{k}}$


Specifically : $H^*(\mathcal{M}_g;\mathbb{Q})$
known about $H^*(\mathcal{M}_g)$?
* support $0 \le * \le 4g-5$
$H^{4g-5}(\mathcal{M}_g;\mathbb{Z}) \neq 0$
tautological classes
$\kappa: \mathbb{Q}[\kappa_1, \kappa_2, \dots] \longrightarrow H^*(\mathcal{M}_g)$
Madsen-Weiss
$(\deg \kappa_i = 2i)$
Thm '04:
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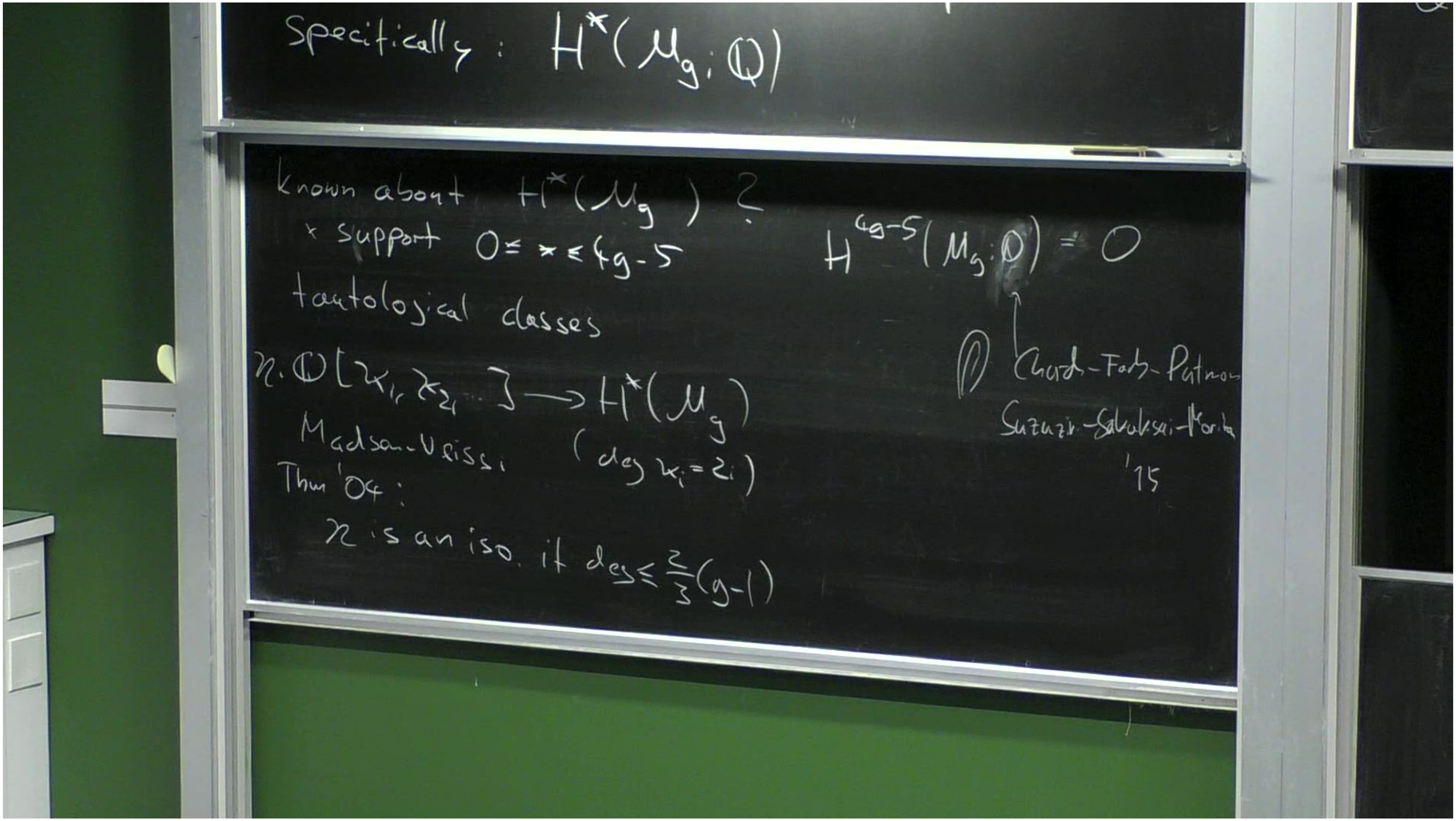
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Thm '04:
κ is an iso. if $\deg \le \frac{2}{3}(g-1)$
$H^{4g-5}(\mathcal{M}_g; \mathbb{Q}) = 0$
Church-Farb-Putman
Suzuki-Sakasai-Morita
'15

$\dim \ldots \big|_{\deg = 2k} = \#$ of integer partitions of k ...

Chan-Galatius-Payne '21:

$$H_k(GC^{(g)}) \hookrightarrow H^{6g-6-k}(\mathcal{M}_g) \quad \text{for all } k \geq 0$$

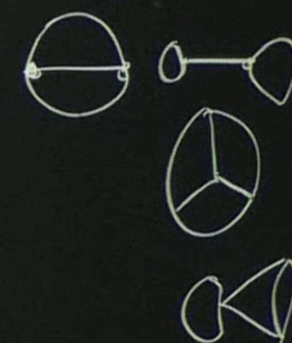
Kontsevich's commutative graph complex

Image: Top-weight cohomology

Commutative graph complex

Consider all pairs (G, σ) (up to iso)

* G is a graph cntd, g loops $(h_1(G) = g)$
with vertex degree ≥ 3 Ex.

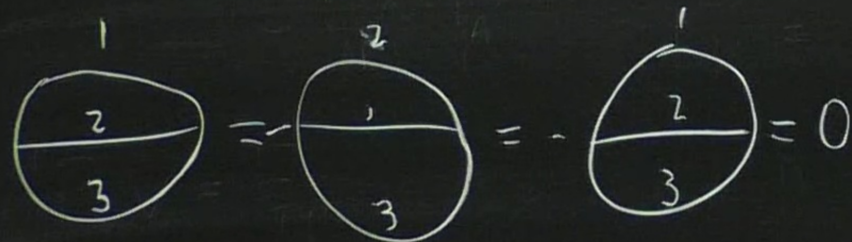


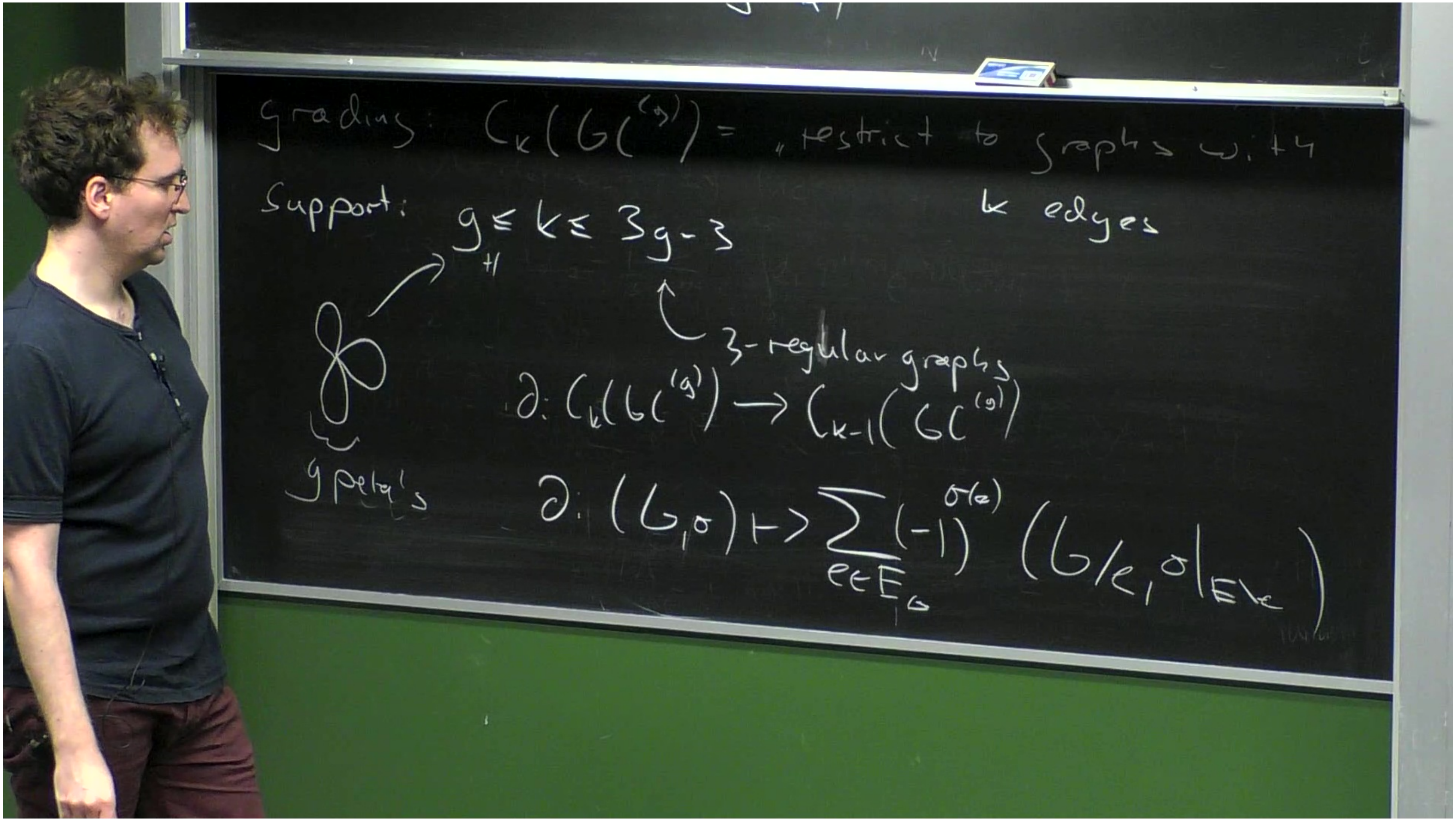
* $\sigma : E_G \longrightarrow \{1, \ldots, |E_G|\}$
(ordering of edges)

$GC^{(g)}$ = $\mathbb{Q}$-vector space gen by such pairs.

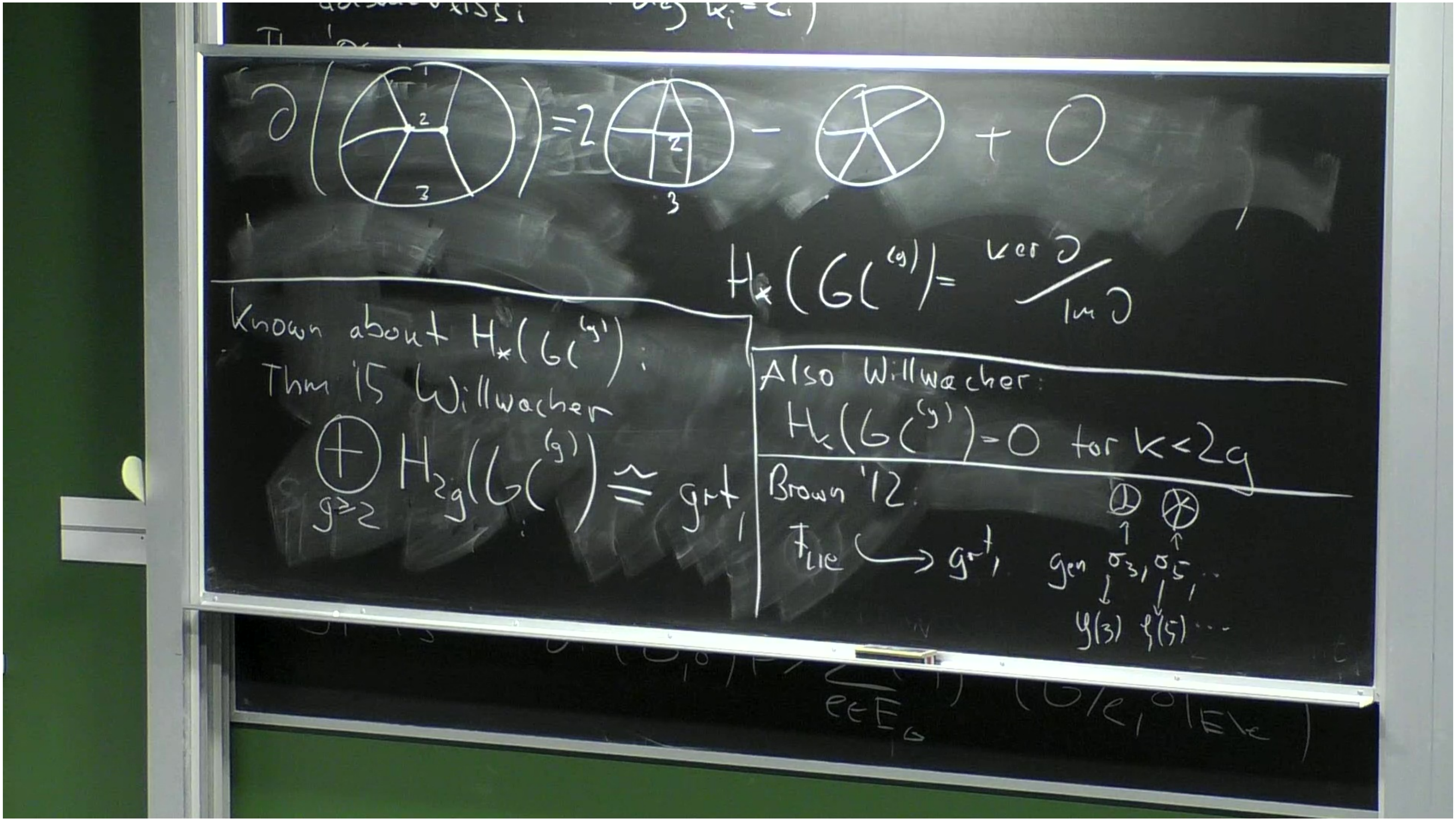
mod relations for G, $\pi \in S_{|E_G|}$

$$(G, \pi\circ\sigma) = \mathrm{sign}(\pi)\,(G,\sigma)$$





grading: $C_k(GC^{(g)})$ = "restrict to graphs with k edges"
Support: $g \leq k \leq 3g-3$
+1
3-regular graphs
g petal's
$\partial: C_k(GC^{(g)}) \longrightarrow C_{k-1}(GC^{(g)})$
$\partial: (G,\sigma) \mapsto \sum_{e \in E_G} (-1)^{\sigma(e)} (G/e, \sigma|_{E \setminus e})$


$= 2$
$- \quad + \; 0$
$H_*(GC^{(g)}) = \ker\partial / \operatorname{im}\partial$
Known about $H_*(GC^{(g)})$:
Thm '15 Willwacher
$\bigoplus_{g\geq 2} H_{2g}(GC^{(g)}) \cong \mathfrak{grt}_1$
Also Willwacher:
$H_k(GC^{(g)}) = 0$ for $k<2g$
Brown '12:
$\mathbb{F}_{Lie} \hookrightarrow \mathfrak{grt}_1$ gen $\sigma_3, \sigma_5, \ldots$
$\zeta(3)\ \zeta(5) \ldots$

$$\dim \operatorname{Im} \pi \Big|_{\deg = 2k} = \# \text{ of integer partitions of } k \leq C$$

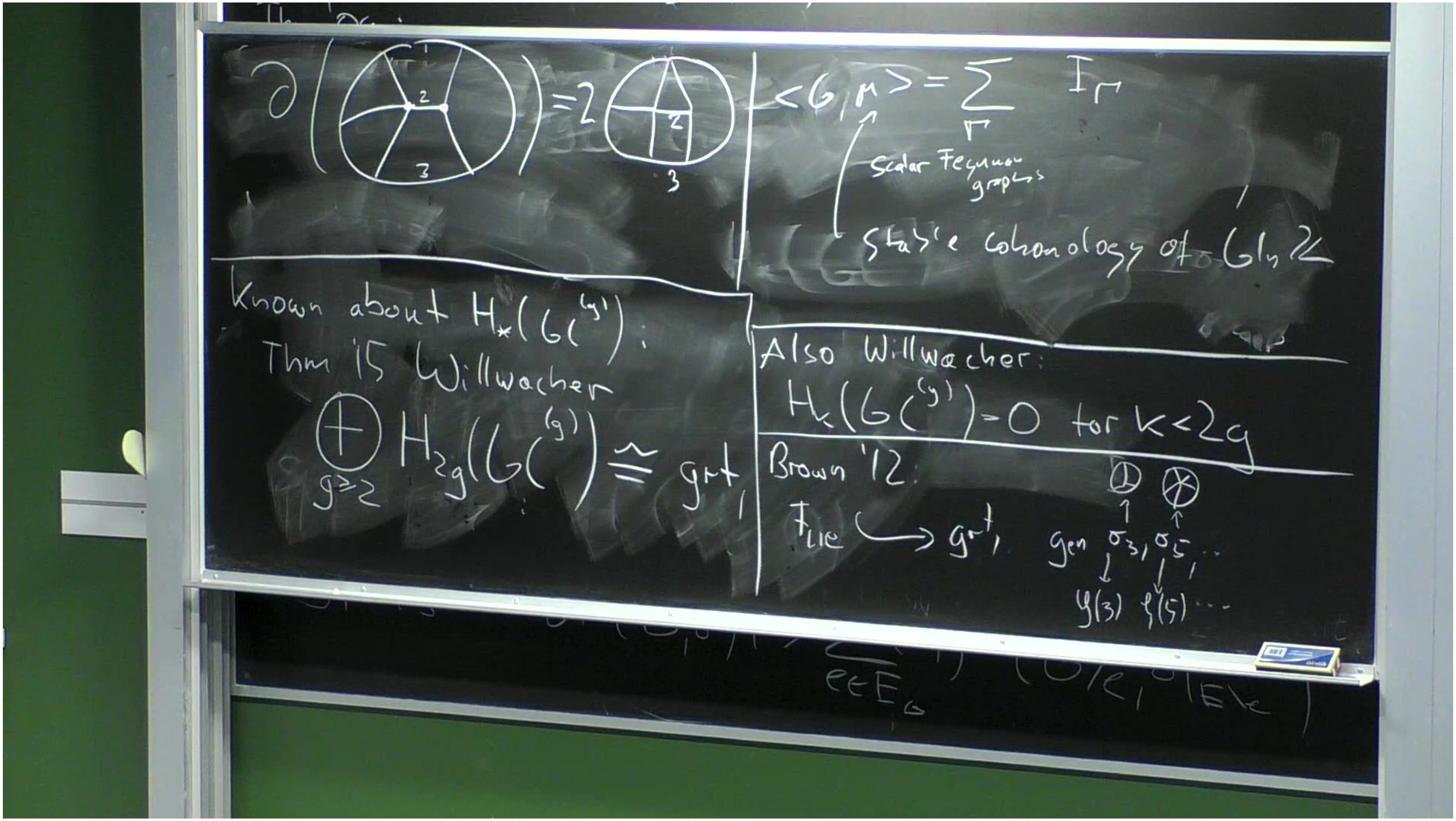
Chan-Galatius-Payne '21:

$$H_k(GC^{(g)}) \hookrightarrow H^{6g-6-k}(\mathcal{M}_g) \quad \text{for all } k \geq 0$$

Kontsevich's commutative graph complex

Image: Top-weight cohomology

$$\dim H^{4g-6}(\mathcal{M}_g) \geq \dim F_{Lie} \geq C^g$$


=2
3
<G,μ> = Σ I_Γ
Γ
Scalar Feynman graphs
stable cohomology of GL_n Z
Known about H_*(G^(g)):
Thm is Willwacher
⊕ H_2g(G^(g)) ≅ grt_1
g≥2
Also Willwacher:
H_k(G^(g)) = 0 for k<2g
Brown '12:
F_Lie ↪ grt_1 gen σ_3, σ_5, ...
ζ(3) ζ(5) ...