Title: Heun operator and Bethe ansatz

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Collection/Series: Mathematical Physics

Subject: Mathematical physics

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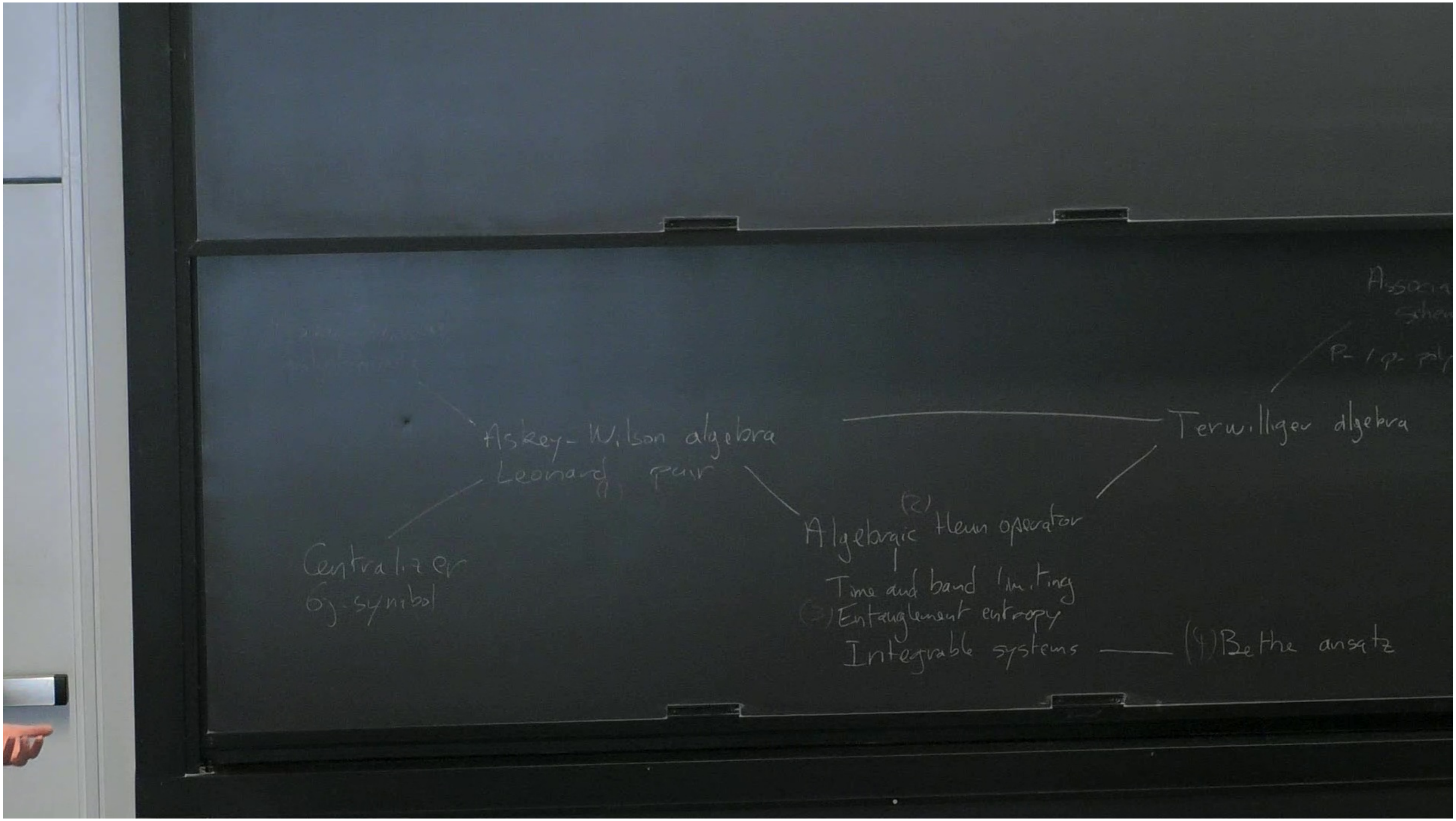
Abstract:

After an introduction to the notion of Leonard pairs, I explain their different uses. Then, I provide the definition of the associated Heun operator and how it allows us to simplify the computation of the quantum entanglement entropy. Finally, I show, in the simplest example, how the Bethe ansatz can be used to diagonalize the Heun operator.


Askey-Wilson algebra
Leonard pair
(1)
Terwilliger algebra
Centralizer
6j-symbol
(2)
Algebraic Heun operator
Time and band limiting
(3) Entanglement entropy
Integrable systems
(4) Bethe ansatz

Heun operator and Bethe ansatz

1) Leonard pair:

$(H, X) \in \mathrm{End}(V)$ $\quad$ V is finite

(i) $\exists\, \mathcal{B}$, H is diagonal and X is tridiagonal irreducible

(ii) $\exists\, \mathcal{B}^*$, X ______ and H ______

Example Lie algebra $su(2)$: J_1, J_2, J_3

$$[J_1, J_2] = i J_3, \ldots$$

spin s ($s \in \mathbb{N}/2$)

$$J_3 = \begin{pmatrix} +s & & & \\ & +s-1 & & \\ & & \ddots & \\ & & & -s \end{pmatrix}, \quad J_1 = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{(2s)1} & & & \\ \sqrt{(2s)1} & 0 & \sqrt{(2s-1)2} & & \\ & \sqrt{(2s-1)2} & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{2s} \\ & & & \sqrt{2s} & 0 \end{pmatrix}, \quad J_2$$

..., H is diagonal and X is tridiagonal irreducible

(ii) $\exists\, \mathcal{B}^*$, X ______ and H ______

2) Algebraic Heun operator;

(H, X)

$$T = r_0 \mathbb{1} + r_1 X + r_2 H + r_3 \{H, X\} + r_4 [H, X]$$

$$X = \hat{x}$$

$$H = x(x-1)\partial_x^2 + (x-a)\partial_x$$

$k \in \{0, \dots N\}$

H	X
k	k
k	k^2
k^2	k
k^2	k^2
q^k	q^k
$q^k + q^{-k}$	q^k
q^k	$q^k + q^{-k}$
$q^k + q^{-k}$	$q^k + q^{-k}$
$k + (-1)^k k$	

..., H is diagonal and X is tridiagonal irreduc...

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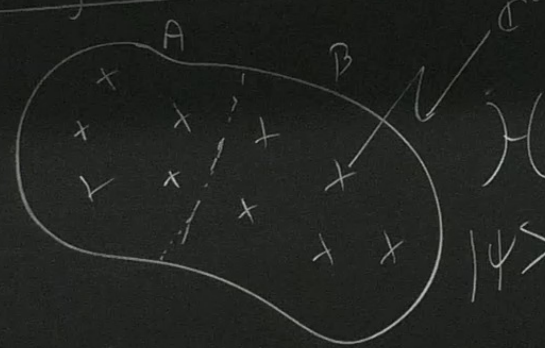
Ex: $T = \{J_1, J_3\}$ $\quad s = \frac{7}{2}$ $\quad$ spectrum $= \left\{ \pm\sqrt{63 \pm 12\sqrt{21}} \right\}$

$-s$

$\sqrt{2s}$

$\sqrt{2s}$ 0

3) Entanglement entropy



$\mathcal{H} = (\mathbb{C}^2)^{\otimes N+1}$

$|\psi\rangle \subset \mathcal{H}$

$\rho = \mathrm{tr}_B |\psi\rangle\langle\psi|$

$S = -\mathrm{tr}_A\, \rho \ln \rho$

$c_n^\dagger \;,\; c_n \;;\; \{c_n^\dagger, c_m\} = \delta_{nm}$

$\{c_n, c_m\} = 0$

$\{c_n^\dagger, c_m^\dagger\} = 0$

$\mathcal{H} = \frac{1}{2}\sum_{n=0}^{N-1} \sqrt{(2s-n)n}\,\left(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n\right)$, $|\psi\rangle$ GS

to compute $S \iff$ to compute the eigenvalues of C

$C_{nm} = \langle\psi| c_n^\dagger c_m |\psi\rangle$

$0 \le n, m \le \ell$

$[T, C] = 0$

$J_3 = \begin{pmatrix} \ddots & & \\ & \ddots & \\ & & -s \end{pmatrix}$, J_1 ...

$\begin{pmatrix} \ddots & & \\ & & \sqrt{2s} \\ & \sqrt{2s} & 0 \end{pmatrix}$

4) Bethe ansatz

$$C = J_3^2 + \frac{1}{2}\{S_+, S_-\} = s(s+1)\cdot \mathbb{1}$$

a) idea: $\begin{cases} J_3 e_0 = s e_0, & (J_-)^n e_0 = e_n \\ J_+ e_0 = 0 & J_3 e_n = (s-n) e_n \end{cases}$

$J_\pm = J_1 \pm i J_2$

b) B.A for $T = \{J_1, J_3\}$

$$\begin{cases} S_3(x) := \tan(2x) J_3 \\ S_\pm(x,n) := \frac{1}{\cos(2x)} J_\pm + J_3 - n + \frac{1}{2} \end{cases}$$

$$\Rightarrow \begin{cases} S_-(x,n) S_-(y,n+1) = S_-(y,n) S_-(x,n+1) \\ [S_3(x), S_-(y,n)] = \frac{\sin(2x)}{2\sin(x+y)\sin(x-y)} \begin{pmatrix} S_-(y,n) \\ - S_-(x,n) \end{pmatrix} \\ \dots \end{cases}$$

$$t(x,n) = \cos(2x)\left(S_3(x)^2 + \frac{1}{2} S_+(x,-n+1) S_-(x,-n+1) + \frac{1}{2} S_-(x,n) S_+(x,n) - \frac{1}{4} \right) \Rightarrow \begin{cases} [t(x,n), t(y,n)] = 0 \\ t(x,0) = \{J_1, J_3\} + \frac{1}{\cos(2x)} C \end{cases}$$

(ii) $\exists\, \mathcal{B}^*, X$ ______ is tridiagonal irreducible and H ______

$$V(\{z_1 \ldots z_{2s+1}\}) = S_-(z_1,1)\, S_-(z_2,2) \ldots S_-(z_{2s+1}, 2s+1)\, e_0$$

$\ldots$

$$t(x,0)\, V(\{z_1 \ldots z_{2s+1}\}) = \lambda(x,\{z\})\, V(\{z\}) + \sum_{k=1}^{2s+1} f_k(x)\, \underbrace{U_k(\{z\})}_{=\,0}\, S_-(z_1,1) \ldots S(\overset{z_k}{x}, k) \ldots S_-(z_{2s+1}, 2s+1)\, e_0$$

Bethe equations

(ii) $\exists\, \mathcal{B}^*$ ... is tridiagonal irreducible

, X ______ and H ______

Multivariate AW polynomials

?

Askey-Wilson polynomials

Higher rank AW algebra

Askey-Wilson algebra

Leonard pair (1)

Terwilliger algebra

?

multivariate

Centralizer

6j-symbol

3nj-symbol

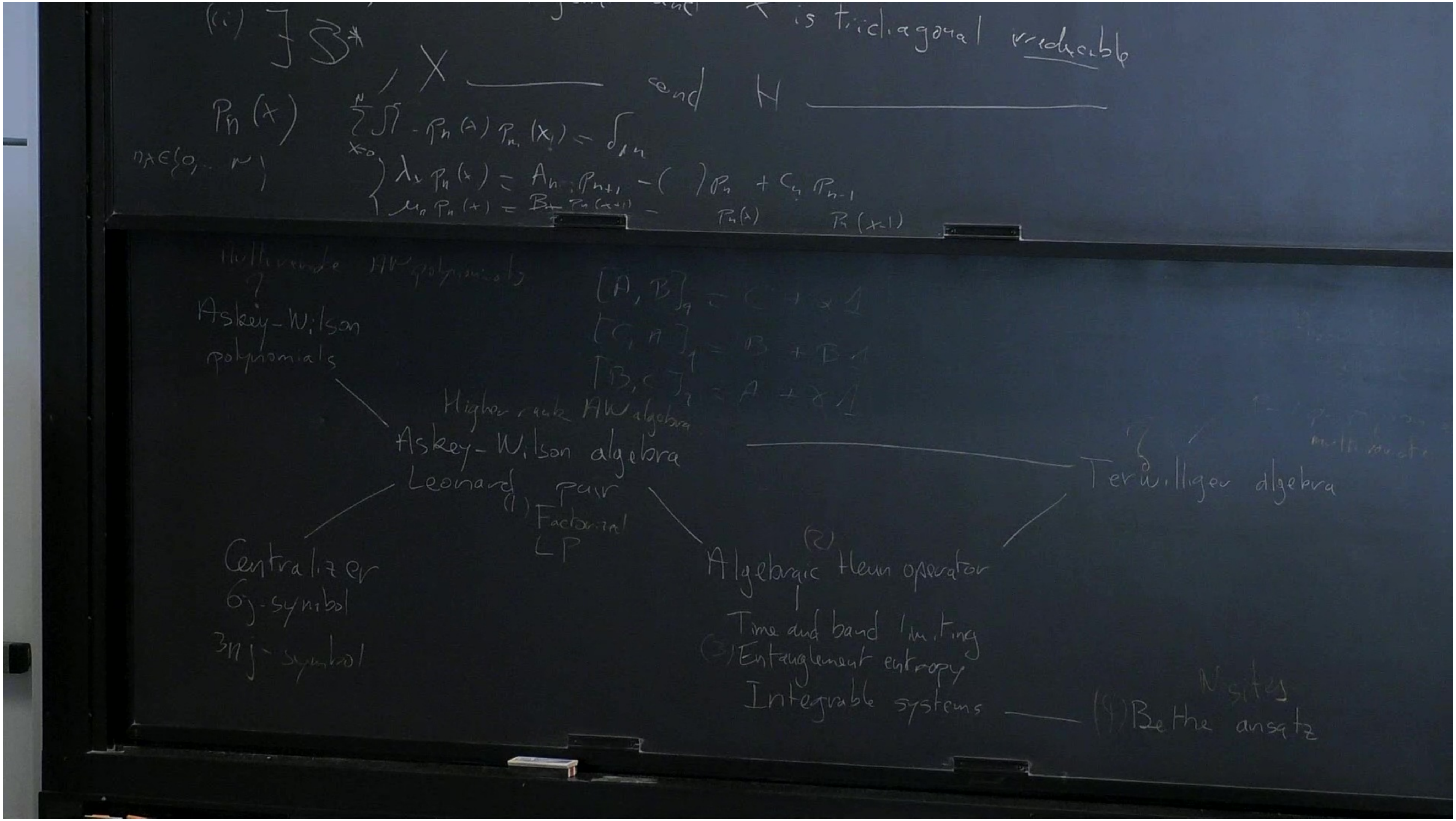
(2) Algebraic Heun operator

Time and band limiting

(3) Entanglement entropy

Integrable systems ______ (4) Bethe ansatz

N sites



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and H
Askey-Wilson
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