Title: Dynamical Yangians of cotangent Lie algebras over moduli spaces of G-bundles

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Abstract:

In this talk, I will present the construction of the Yangian of a cotangent Lie algebra from the geometry of the equivariant affine Grassmannian. Furthermore, I will discuss how this quantum group can be dynamically twisted to a quantum groupoid over a neighborhood in the moduli space of G-bundles over a compact Riemann surface. These constructions are motivated by relations between a certain holomorphic-topological 4d gauge field theory and the geometric Langlands correspondence. Representations of the Yangian are pertubative line operators of said theory, while the dynamical twist of the Yangian controls the action of these operators via Hecke modifications in this setting. This talk is based on the two joint works arXiv:2405.19906 and arXiv:2411.05068 with Wenjun Niu.

Dynamical Yangians of cotangent Lie algebras from moduli spaces of G-bundles

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(based on arXiv:2405.19906 and arXiv:2411.05068 with Wenjun Niu)

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Motivation

- Kapustin-Witten (2006): 4d $\mathcal{N} = 4$ super Yang-Mills $\rightsquigarrow$ geometric Langlands correspondence. In particular, line operators act by Hecke modification.

- Kapustin (2006): holomorphic-topological twist for 4d $\mathcal{N} = 2$ QFT associated to a complex reductive group G $\rightsquigarrow$ coherent Hecke modification.

- Costello-Witten-Yamazaki (2018): pertubative line operators in 4d Chern-Simons type theory over simple G = modules of the Yangian over $\mathfrak{g}$.

- Elliott-Safranov-Williams (2020): perturbative Kapustin's theory = perturbative 4d Chern-Simons for $T^*\mathfrak{g}$.

- Goal: Realize perturbative line operators in Kapustin's theory and their Hecke action via a Yangian to $T^*\mathfrak{g}$.

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- Recall: Yangian $Y_\hbar(\mathfrak{g})$ of a simple complex Lie algebra $\mathfrak{g}$

  = unique (t)-graded quantization of the Lie bialgebra structure on $\mathfrak{g}[t]$

  $$\mathfrak{g}[t] = \text{Polynomials in } t \text{ with coefficients in } \mathfrak{g}$$

  (i.e. co-Poisson structure on $U(\mathfrak{g}[t])$) given by

  $$\delta(x) = [x(t_1) \otimes 1 + 1 \otimes x(t_2), \gamma_{\mathfrak{g}}(t_1, t_2)],$$

  where $\gamma_{\mathfrak{g}}(t_1, t_2) = \frac{C}{t_1 - t_2}$ for the Casimir element $C \in \mathfrak{g} \otimes \mathfrak{g}$

  = (t)-graded Hopf algebra s.t. $Y_\hbar(\mathfrak{g})|_{\hbar=0} = U(\mathfrak{g}[t])$ and $\frac{\Delta_\hbar - \Delta_\hbar^{\text{op}}}{\hbar} \overset{\hbar=0}{\mapsto} \delta$.

- $Y_\hbar(\mathfrak{g})$ is pseudotriangular: $\Delta_\hbar$ and $\Delta_\hbar^{\text{op}}$are, up to $t \mapsto t + z$, intertwined by a solution of the spectral quantum Yang-Baxter equation.

- Can be completed to a quantization of

  $$\mathfrak{g}[\![t]\!] = \text{Formal power series in } t \text{ with coefficients in } \mathfrak{g}$$

Consider $\mathfrak{d} := T^*\mathfrak{g} \cong \mathfrak{g} \ltimes \mathfrak{g}^* \cong \mathfrak{g}[\epsilon]/\epsilon^2\mathfrak{g}[\epsilon]$

$$\Longrightarrow \gamma_{\mathfrak{d}}(t_1, t_2) = (1 \otimes \epsilon)\gamma_{\mathfrak{g}}(t_1, t_2) - (\epsilon \otimes 1)\gamma_{\mathfrak{g}}^{(21)}(t_2, t_1)$$
$$= \frac{\sum_{a=1}^{\dim(\mathfrak{g})}(I_a \otimes I^a + I^a \otimes I_a)}{t_1 - t_2}$$

defines a Lie bialgebra structure δ on $\mathfrak{d}[\![t]\!]$.

Theorem (A.-Niu, 2024)

The following results are true:

- There exists a unique (t, ϵ)-graded quantization $Y_\hbar(\mathfrak{d})$ of $(\mathfrak{d}[\![t]\!], \delta)$;
- $Y_\hbar(\mathfrak{d})|_{\hbar=1}$-modules = pertubative line operators of Kapustin's theory as monodial categories;
- $Y_\hbar(\mathfrak{d})$ is pseudotriangular.

Perturbative line operators

Pertubative line operators of Kapustin twisted theory

$= \widehat{\mathfrak{g}[\![t]\!]} \cong \widehat{G[\![t]\!]}$-equivariant coherent sheaves on $\widehat{G(\!(t)\!)} \, / \, \widehat{G[\![t]\!]} = \widehat{\mathsf{Gr}_G}$,
where $G[\![t]\!], G(\!(t)\!) =$ alg. functions $\mathsf{Spec}(\mathbb{C}[\![t]\!]), \mathsf{Spec}(\mathbb{C}(\!(t)\!)) \to G$.

$= Y := U(\mathfrak{g}[\![t]\!]) \# \mathbb{C}[\widehat{\mathsf{Gr}_G}]$-modules.

Monoidal structure: push-pull using

$$G[\![t]\!] \backslash (G(\!(t)\!) \times_{G[\![t]\!]} \mathsf{Gr}_G)$$

$$G[\![t]\!] \backslash \underbrace{\mathsf{Gr}_G}_{G(\!(t)\!)/G[\![t]\!]} \times G[\![t]\!] \backslash \mathsf{Gr}_G \longleftarrow G[\![t]\!] \backslash (G(\!(t)\!) \times_{G[\![t]\!]} \mathsf{Gr}_G) \longrightarrow G[\![t]\!] \backslash \mathsf{Gr}_G$$

Constructing the Yangian: multiplication

$\mathfrak{g}(\!(t)\!) = \mathfrak{g}[\![t]\!] \oplus W$ for $W := t^{-1}\mathfrak{g}[t^{-1}]$

$$\Longrightarrow \widehat{\mathrm{Gr}}_G \cong \widehat{W} \quad \text{and} \quad U(W)^* = \mathbb{C}[\widehat{W}] = \widehat{S}(\mathfrak{g}^*[\![t]\!])$$

and defines actions $\triangleright, \triangleleft$ of $\mathfrak{g}[\![t]\!]$, W on each other.

Passing to $\hbar$-adic language: $U(W)^*[\![\hbar]\!] = S(\hbar\mathfrak{g}^*[\![t]\!])[\![\hbar]\!]$ and

$$Y_\hbar(\mathfrak{d}) = U(\mathfrak{g}[\![t]\!])[\![\hbar]\!]\#_{\mathbb{C}[\![\hbar]\!]}S(\mathfrak{g}^*[\![t]\!])[\![\hbar]\!] \Longrightarrow Y_\hbar(\mathfrak{d})|_{\hbar=1} = Y,$$

where $\#_{\mathbb{C}[\![\hbar]\!]}$ uses the action $\triangleright$ of $\mathfrak{g}[\![t]\!]$ on W.

Constructing the Yangian: comultiplication

Coproduct $\Delta_\hbar$ defined by resembling

$$\widehat{\mathfrak{g}[\![t]\!]}\backslash\widehat{W}\times\widehat{\mathfrak{g}[\![t]\!]}\backslash\widehat{W}\leftarrow\widehat{\mathfrak{g}[\![t]\!]}\backslash\overbrace{(\widehat{W}\times\widehat{W})}^{\widehat{\mathfrak{g}((t))}\times_{\widehat{\mathfrak{g}[\![t]\!]}}\widehat{W}}\rightarrow\widehat{\mathfrak{g}[\![t]\!]}\backslash\widehat{W}.$$

- $\Delta_\hbar$ on $S(\mathfrak{g}^*[\![t]\!])$ dual to $U(W)$;
- $\widehat{\mathfrak{g}[\![t]\!]}$-action on $\widehat{W}\times\widehat{W}$: $g\cdot(w_1,w_2)=(g\cdot w_1,g^{w_1}\cdot w_2)$

$$\rightsquigarrow\Delta_\hbar(x)=x\otimes 1+\phi(x)\,,\qquad x\in\mathfrak{g}[\![t]\!],$$

where ϕ is defined by the action $\lhd$ of W on $\mathfrak{g}[\![t]\!]$. Using $\mathcal{E}:=e^{-\hbar(\epsilon\otimes 1)\gamma_{\mathfrak{g}}}$, we can rewrite this as

$$\Delta(x)=\mathcal{E}^{-1}(x\otimes 1+1\otimes x)\mathcal{E}.$$

Constructing the Yangian: comultiplication

Coproduct $\Delta_\hbar$ defined by resembling

$$\widehat{\mathfrak{g}[\![t]\!]}\backslash\widehat{W} \times \widehat{\mathfrak{g}[\![t]\!]}\backslash\widehat{W} \leftarrow \widehat{\mathfrak{g}[\![t]\!]}\backslash\overbrace{(\widehat{W} \times \widehat{W})}^{\widehat{\mathfrak{g}((t))} \times_{\widehat{\mathfrak{g}[\![t]\!]}} \widehat{W}} \to \widehat{\mathfrak{g}[\![t]\!]}\backslash\widehat{W}.$$

- $\Delta_\hbar$ on $S(\mathfrak{g}^*[\![t]\!])$ dual to $U(W)$;
- $\widehat{\mathfrak{g}[\![t]\!]}$-action on $\widehat{W} \times \widehat{W}$: $g \cdot (w_1, w_2) = (g \cdot w_1, g^{w_1} \cdot w_2)$

$$\rightsquigarrow \Delta_\hbar(x) = x \otimes 1 + \phi(x)\,, \qquad x \in \mathfrak{g}[\![t]\!],$$

where ϕ is defined by the action $\lhd$ of W on $\mathfrak{g}[\![t]\!]$. Using $\mathcal{E} := e^{-\hbar(\epsilon\otimes 1)\gamma_{\mathfrak{g}}}$, we can rewrite this as

$$\Delta(x) = \mathcal{E}^{-1}(x \otimes 1 + 1 \otimes x)\mathcal{E}.$$

$\rightsquigarrow Y_\hbar(\mathfrak{d})$ is a quantization of $(\mathfrak{d}[\![t]\!], \delta_\rho)$.

Pseudotriangular structure

$Y_\hbar(\mathfrak{d})$ has a meromorphic comultiplication Δ_z defined:

- On $U(t^{-1}\mathfrak{g}[t^{-1}])^*[\![\hbar]\!] = S(\hbar\mathfrak{g}^*[\![t]\!])[\![\hbar]\!]$ dual to vertex algebra structure:

$$V_0(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}[\![t]\!]}^{\mathfrak{g}((t))}\mathbb{C} \cong U(W) \xrightarrow{\mathcal{Y}} \mathrm{End}(V_0(\mathfrak{g}))[\![z, z^{-1}]\!];$$

- On $U(\mathfrak{g}[\![t]\!])$ by:

$$\Delta_z(x) = \tau_z(x) \otimes 1 + 1 \otimes x\,, \qquad x \in \mathfrak{g}[\![t]\!],$$

where τ_z quantizes $t \longmapsto t + z$.

Pseudotriangular structure II

Consider the Taylor expansions $s_\pm(z) \in (\mathfrak{g}((t_1)) \otimes \mathfrak{g}((t_2)))[\![z^{\pm 1}]\!]$ of

$$-(\epsilon \otimes 1)\gamma_{\mathfrak{g}}(t_1 + z, t_2) = \frac{(\epsilon \otimes 1)C}{t_2 - z - t_1}$$

in $|t_1 + z| < |t_2|$ and $|t_1 - t_2| < |z|$.

Writing $R_s(z) = e^{\hbar s_-(z)}$ and $\mathcal{E}(z) = e^{\hbar s_+(z)} = (\tau_z \otimes 1)\mathcal{E}$, we have:

$$\mathcal{Y}(z) = R_s(z)\mathcal{E}(z) \implies R_s(z)^{-1}(\tau_z \otimes 1)\Delta_\hbar R_s(z) = \Delta_z.$$

Δ_z is weakly cocommutative

$$\implies R(z)(\tau_z \otimes 1)\Delta_\hbar R(z)^{-1} = (\tau_z \otimes 1)\Delta_\hbar^{\mathrm{op}},$$

for $R(z) = R_s(-z)^{21} R_s(z)^{-1}$.

$R(z)$ satisfies cocycle condition and QYBE $\rightsquigarrow$ pseudotriangular structure.

- Structure of $\widehat{\mathfrak{g}[\![t]\!]}\backslash\widehat{\mathfrak{g}(\!(t)\!)}/\widehat{\mathfrak{g}[\![t]\!]}$ does not depend on choice of W s.t. $\mathfrak{g}(\!(t)\!) = \mathfrak{g}[\![t]\!] \oplus W$.
- $W \longleftrightarrow$ not-necessarily skew-symmetric classical r-matrices with coefficients in $\mathfrak{g}$.
- $\rho(t_1, t_2) := (1 \otimes \epsilon)r(t_1, t_2) - (\epsilon \otimes 1)r^{21}(t_2, t_1)$ is skew-symmetric classical r-matrix with coefficients in $\mathfrak{d}$.
- ρ defines a Lie bialgebra structure δ_ρ on $\mathfrak{d}[\![t]\!]$.
- Quantization of $(\mathfrak{d}[\![t]\!], \delta_\rho)$ = quantum twist of $Y_\hbar(\mathfrak{d})$.

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X = irreducible smooth projective curve

Bun_G = G-bundles on X

Loop group uniformization

For every $p \in X$ with local coordinate t, we have an isomorphism of stacks

$$\mathrm{Bun}_G \cong G[\![t]\!] \backslash G(\!(t)\!) / G(X \setminus \{p\}),$$

given by transition function w.r.t. $X = (X \setminus \{p\}) \cup \mathrm{Spec}(\mathbb{C}[\![t]\!])$.

Trivializing Bun_G around P satisfying $\mathrm{Aut}(P)$ = center of G:

$$\sigma \colon B \to G(\!(t)\!) \text{ section of } G(\!(t)\!) \to \mathrm{Bun}_G,$$

B = formal neighborhood of P, $\mathbb{C}[B] = \mathbb{C}[\![\lambda]\!] = \mathbb{C}[\![\lambda_1, \dots, \lambda_N]\!]$

Geometric Langlands correspondence: equivalence of derived categories

$$\boxed{\text{Twisted } \mathcal{D}\text{-modules on } \mathrm{Bun}_G} \longleftrightarrow \boxed{\text{Coherent sheaves on } \mathrm{Loc}_{^LG}}$$

compatible with action of $\mathrm{Rep}(^LG)$ on both sides.

Action on the left by Hecke modification using geometric Satake equivalence:

$$\mathrm{Rep}(^LG) \cong \mathrm{Perv}(G[\![t]\!]\backslash G(\!(t)\!)/G[\![t]\!])$$

via push-pull using

$$\begin{array}{ccc} & G[\![t]\!]\backslash(G(\!(t)\!) \times_{G[\![t]\!]} G(\!(t)\!))/G(X \setminus \{p\}) & \\ \swarrow & & \searrow \\ G[\![t]\!]\backslash G(\!(t)\!)/G[\![t]\!] \times \underbrace{\mathrm{Bun}_G}_{G[\![t]\!]\backslash G(\!(t)\!)/G(X\setminus\{p\})} & & \mathrm{Bun}_G \end{array}$$

Here: perturbative line operators = coherent sheaves on $\widehat{\mathfrak{g}[\![t]\!]}\backslash\widehat{\mathfrak{g}(\!(t)\!)}/\widehat{\mathfrak{g}[\![t]\!]}$

$\rightsquigarrow$ coherent version of Hecke modification.

$\rightsquigarrow$ Locally using $\sigma\colon B \to G(\!(t)\!)$, Hecke modification is determined by a monoidal functor

$$Y_\hbar(\mathfrak{d})\text{-mod} \longrightarrow \mathrm{End}(\mathbb{C}[B]\text{-mod}).$$

Hopf algebroids (e.g. quantum groupoids) over $\mathbb{C}[B]$ have monoidal representation categories with monoidal forgetful functor to $\mathbb{C}[B]$-mod.

Idea: $Y_\hbar(\mathfrak{d}) \rightsquigarrow$ quantum groupoid $\Upsilon_\hbar(\mathfrak{d})$ over $\mathbb{C}[B]$ as algebraic structure controlling Hecke modifications.

We had: $\mathfrak{g}((t)) = \mathfrak{g}[\![t]\!] \oplus W \rightsquigarrow$ Lie bialgebra $\rightsquigarrow$ quantum group $Y_\hbar(\mathfrak{d})$.

Plan: Lie algebroid splitting $\rightsquigarrow$ Lie bialgebroid $\rightsquigarrow$ quantum groupoid $\Upsilon_\hbar(\mathfrak{d})$.

Lie algebroid over algebra R: R-module with Lie bracket and Lie alg. hom. $\phi\colon L \to \mathrm{Der}(R)$ such that

$$[x, ry] = \phi(x)(r)y + r[x,y]\,, \qquad x, y \in L, r \in R.$$

Lie bialgebroid: two Lie algebroids in duality.

Quantum groupoid: H with source target maps $s,t\colon R \to H$ and compatible unit, counit, multiplication, and comultiplication deforming the universal enveloping algebroid of a Lie bialgebroid.

- Lie algebroid over $R = \mathbb{C}[\![\lambda]\!]$: $\mathfrak{g}(\!(t)\!) \rightsquigarrow \mathsf{Der}(R) \ltimes \mathfrak{g}(\!(t)\!)[\![\lambda]\!] \xrightarrow{\phi} \mathsf{Der}(R)$.

- Lie algebroid splitting: $\sigma\colon B \to G(\!(t)\!) \rightsquigarrow \xi_\alpha = \sigma^{-1}\frac{\partial\sigma}{\partial\lambda_\alpha} \in \mathfrak{g}(\!(t)\!)[\![\lambda]\!]$

$$\rightsquigarrow \mathsf{Der}(R) \ltimes \mathfrak{g}(\!(t)\!)[\![\lambda]\!] = (\mathsf{Der}(R) \ltimes \mathfrak{g}[\![t;\lambda]\!]) \oplus \mathfrak{W},$$

where $\mathfrak{W} = \mathsf{Ad}(\sigma)(\mathfrak{g} \otimes \Gamma(X \setminus \{p\}, \mathcal{O}_X)) \oplus \mathsf{Span}_R\left\{\frac{\partial}{\partial\lambda_\alpha} + \xi_\alpha\right\}$.

- Lie bialgebroid structure: $\mathfrak{W}$ defines generalized classical dynamical r-matrix $r = r(t_1, t_2; \lambda)$ defining KZB connection (Felder 1996) and provides involutivity of pointed Hitchin system (A. 2024);

$\rightsquigarrow$ Classical dynamical r-matrix $\rho = (1 \otimes \epsilon)r - (\epsilon \otimes 1)r^{21}$;

$\rightsquigarrow$ Lie bialgebroid structure δ_ρ on $\mathsf{Der}(R) \ltimes \mathfrak{g}[\![t;\lambda]\!]$.

Repeating quantization procedure of $(\mathrm{Der}(R) \ltimes \mathfrak{g}[\![t;\lambda]\!], \delta_\rho)$ as before in algebroid language implies:

Theorem (A.-Niu 2024)

The following results are true:

- There exists an explicit quantization of $(\mathrm{Der}(R) \ltimes \mathfrak{g}[\![t;\lambda]\!], \delta_\rho)$ to a quantum groupoid $\Upsilon_\hbar(\mathfrak{d})$.

Repeating quantization procedure of $(\mathrm{Der}(R) \ltimes \mathfrak{g}[\![t;\lambda]\!], \delta_\rho)$ as before in algebroid language implies:

Theorem (A.-Niu 2024)

The following results are true:

- There exists an explicit quantization of $(\mathrm{Der}(R) \ltimes \mathfrak{g}[\![t;\lambda]\!], \delta_\rho)$ to a quantum groupoid $\Upsilon_\hbar(\mathfrak{d})$.
- $\Upsilon_\hbar(\mathfrak{d}) \cong \mathrm{Diff}(R) \otimes_R Y_\hbar(\mathfrak{d})[\![\lambda]\!]$ as algebras and are dynamically twist equivalent.
- There is an explicit quantization of ρ to a solution of the quantum Yang-Baxter equation defining the pseudotriangular structure.

Summary

1. Lie algebra splitting $\mathfrak{g}((t)) = \mathfrak{g}[\![t]\!] \oplus W$ (e.g. $W = t^{-1}\mathfrak{g}[t^{-1}]$)
   $\rightsquigarrow$ classical r-matrices over $\mathfrak{d} = T^*\mathfrak{g}$ (e.g. Yang's r-matrix $\gamma_{\mathfrak{d}}$ over $\mathfrak{d}$)
   $\rightsquigarrow$ Lie bialgebras structures on $\mathfrak{d}[\![t]\!]$
   $\rightsquigarrow$ pseudotriangular quantum groups over $\mathfrak{d}$ (e.g. the Yangian $Y_\hbar(\mathfrak{d})$ over $\mathfrak{d}$) which are all twist equivalent
   $\rightsquigarrow$ Realization of perturbative line operators for Kapustin's QFT via Yangian of $\mathfrak{d}$.

2. Formal neighborhoods of nice G-bundle in Bun_G
   $\rightsquigarrow$ Lie algebroid splitting $\mathrm{Der}(R) \ltimes \mathfrak{g}((t))[\![\lambda]\!] = (\mathrm{Der}(R) \ltimes \mathfrak{g}[\![t;\lambda]\!]) \oplus \mathfrak{W}$
   $\rightsquigarrow$ Classical dynamical r-matrices
   $\rightsquigarrow$ Lie bialgebroids structures on $\mathrm{Der}(R) \ltimes \mathfrak{g}[\![t;\lambda]\!]$
   $\rightsquigarrow$ Pseudotriangular quantum groupoids $\Upsilon_\hbar(\mathfrak{d})$ which are dynamical twists of $Y_\hbar(\mathfrak{d})$

Thank you for your attention!